Algebraic proof theory for LE-logics

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Starting point

N. Galatos, & P. Jipsen. (2013). “Residuated frames with applications to decidability". *Transactions of the American Mathematical Society* , 365 (3), 1219-1249.

- algebras: to present frames for arbitrary residuated lattices,
- proof theory: cut elimination, FMP, FEP,
- restricted to the signatures: $\cdot, \backslash, /$.

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Aim: generalize this approach to the lattices with normal expansions.

LE-logics

The logics algebraically captured by varieties of normal lattice expansions.

$$\phi ::= p \mid \bot \mid \top \mid \phi \wedge \phi \mid \phi \vee \phi \mid f(\overline{\phi}) \mid g(\overline{\phi})$$

where $p \in$ AtProp, $f \in \mathcal{F}$, $g \in \mathcal{G}$.

Normality

- Every $f \in \mathcal{F}$ is finitely join-preserving in positive coordinates and finitely meet-reversing in negative coordinates.
- Every $g \in \mathcal{G}$ is finitely meet-preserving in positive coordinates and finitely join-reversing in negative coordinates.

Examples: substructural, Lambek, Lambek-Grishin, Orthologic...

LE-frames

Definition

An $\mathcal{L}$-*frame* is a tuple $\mathbb{F} = (\mathbb{W}, \mathcal{R}_{\mathcal{F}}, \mathcal{R}_{\mathcal{G}})$ such that $\mathbb{W} = (W, U, N)$ is a polarity, $\mathcal{R}_{\mathcal{F}} = \{R_f \mid f \in \mathcal{F}\}$, and $\mathcal{R}_{\mathcal{G}} = \{R_g \mid g \in \mathcal{G}\}$ such that for each $f \in \mathcal{F}$ and $g \in \mathcal{G}$, the symbols R_f and R_g respectively denote $(n_f + 1)$-ary and $(n_g + 1)$-ary relations on $\mathbb{W}$,

$$R_f \subseteq U \times W^{\epsilon_f} \text{ and } R_g \subseteq W \times U^{\epsilon_g}, \tag{1}$$

In addition, we assume that the following sets are Galois-stable (from now on abbreviated as *stable*) for all $w_0 \in W$, $u_0 \in U$, $\overline{w} \in W^{\epsilon_f}$, and $\overline{u} \in U^{\epsilon_g}$:

$$R_f^{(0)}[\overline{w}] \text{ and } R_f^{(i)}[u_0, \overline{w}^i] \tag{2}$$

$$R_g^{(0)}[\overline{u}] \text{ and } R_g^{(i)}[w_0, \overline{u}^i] \tag{3}$$

Complex Algebras

The *complex algebra* of an LE-frame $\mathbb{F}$ is the algebra

$$\mathbb{F}^+ = (\mathbb{L}, \{f_{R_f} \mid f \in \mathcal{F}\}, \{g_{R_g} \mid g \in \mathcal{G}\}),$$

where $\mathbb{L} := (\gamma_N[\mathcal{P}(W)], \vee, \wedge, \top, \bot)$ is the lattice associated with the polarity $\mathbb{W}$, and for all $f \in \mathcal{F}$ and all $g \in \mathcal{G}$,

1. $f_{R_f} : \mathbb{L}^{n_f} \to \mathbb{L}$ is defined by the assignment $f_{R_f}(\overline{X}) = (R_f^{(0)}[\overline{X}^{\epsilon_f}])^{\downarrow}$
2. $g_{R_g} : \mathbb{L}^{n_g} \to \mathbb{L}$ is defined by the assignment $g_{R_g}(\overline{X}) = R_g^{(0)}[\overline{X}^{\epsilon_g^\partial}]$

Theorem

If $\mathbb{F}$ is an LE-frame, then $\mathbb{F}^+$ is an LE-algebra.

Display Calculi

- Natural generalization of Gentzen's sequent calculi;
- sequents $X \vdash Y$, where X and Y are **structures**:
  - formulas are **atomic structures**
  - built-up: **structural connectives** (generalizing meta-linguistic comma in sequents $\phi_1, \ldots, \phi_n \vdash \psi_1, \ldots, \psi_m$)
  - generation **trees** (generalizing sets, multisets, sequences)
- **Display property**:

$$\dfrac{\dfrac{\dfrac{Y \vdash X > Z}{X \,;\, Y \vdash Z}}{Y \,;\, X \vdash Z}}{X \vdash Y > Z}$$

display rules semantically justified by **adjunction/residuation**

- **Canonical proof of cut elimination (via metatheorem)**

The language of display calculus for LE-algebras

- Formulae

$$A ::= p \mid \bot \mid \top \mid A \wedge A \mid A \vee A \mid f(\overline{A}) \mid g(\overline{A})$$

- Structures

$$\begin{cases} X_f ::= A \mid \mathrm{F}\overline{X} \\ X_g ::= A \mid \mathrm{G}\overline{X} \end{cases}$$

Rules for the basic logic

$$p \vdash p \qquad \bot \vdash X \qquad X \vdash \top \qquad \frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\ \text{(Cut)}$$

$$\frac{A_1 \vdash X}{A_1 \wedge A_2 \vdash X} \qquad \frac{A_2 \vdash X}{A_1 \wedge A_2 \vdash X} \qquad \frac{X \vdash A_1}{X \vdash A_1 \vee A_2} \qquad \frac{X \vdash A_2}{X \vdash A_1 \vee A_2}$$

$$\frac{X \vdash A_1 \qquad X \vdash A_2}{X \vdash A_1 \wedge A_2} \qquad \frac{A_1 \vdash X \qquad A_2 \vdash X}{A_1 \vee A_2 \vdash X}$$

Introduction rules for $f \in \mathcal{F}$ and $g \in \mathcal{G}$

$$f_L \, \frac{F(A_1, \ldots, A_{n_f}) \vdash X}{f(A_1, \ldots, A_{n_f}) \vdash X} \qquad \frac{X \vdash G(A_1, \ldots, A_{n_g})}{X \vdash g(A_1, \ldots, A_{n_g})} \, g_R$$

$$f_R \, \frac{\left(X_i \vdash A_i \quad A_j \vdash X_j \quad \mid \quad \varepsilon_f(i) = 1 \quad \varepsilon_f(j) = \partial\right)}{F(X_1, \ldots, X_{n_f}) \vdash f(A_1, \ldots, A_{n_f})}$$

$$g_L \, \frac{\left(A_i \vdash X_i \quad X_j \vdash A_j \quad \mid \quad \varepsilon_g(i) = 1 \quad \varepsilon_g(j) = \partial\right)}{g(A_1, \ldots, A_{n_g}) \vdash G(X_1, \ldots, X_{n_g})}$$

Display postulates for $f \in \mathcal{F}$ and $g \in \mathcal{G}$

- If $\varepsilon_f(i) = \varepsilon_g(h) = 1$

$$\begin{array}{c} F\,(X_1, \ldots, X_i, \ldots, X_{n_f}) \vdash Y \\ \hline\hline X_i \vdash F_i^{\sharp}\,(X_1, \ldots, Y, \ldots, X_{n_f}) \end{array} \qquad \begin{array}{c} Y \vdash G\,(X_1 \ldots, X_h, \ldots X_{n_g}) \\ \hline\hline G_h^{\flat}\,(X_1, \ldots, Y, \ldots, X_{n_g}) \vdash X_h \end{array}$$

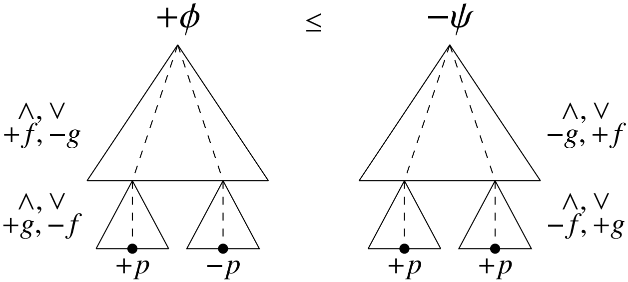
- If $\varepsilon_f(i) = \varepsilon_g(h) = \partial$

$$\begin{array}{c} F\,(X_1, \ldots, X_i, \ldots, X_{n_f}) \vdash Y \\ \hline\hline F_i^{\sharp}\,(X_1, \ldots, Y, \ldots, X_{n_f}) \vdash X_i \end{array} \qquad \begin{array}{c} Y \vdash G\,(X_1, \ldots, X_h, \ldots, X_{n_g}) \\ \hline\hline X_h \vdash G_h^{\flat}\,(X_1, \ldots, Y, \ldots, X_{n_g}) \end{array}$$

Which logics are properly displayable?

Complete characterization:

1. the logics of any **basic** normal (D)LE;
2. axiomatic extensions of these with **analytic inductive inequalities**: ⇝ unified correspondence



Fact: cut-elim., subfm. prop., sound-&-completeness, conservativity **guaranteed** by metatheorem + ALBA-technology.

Analytic Rules

- An analytic rule contains only structural connectives and each structural variable appears only once in the conclusion.

$$\frac{X ; Y \vdash Z}{Y ; X \vdash Z} \qquad \frac{W \vdash X > (Y ; Z)}{W \vdash (X > Y) ; Z}$$

$$\frac{X \vdash Y \qquad W \vdash Z}{\mathrm{I} \vdash (X > Z) ; (W > Y)}$$

Functional D-frames

Let D be a display calculus for a LE-logic $\mathcal{L}$. A *functional* D*-frame* is a structure $\mathbb{F}_{\mathrm{D}} := (W, U, N, \mathcal{R}_{\mathcal{F}}, \mathcal{R}_{\mathcal{G}})$, where

1. $W :=$ Str$_{\mathcal{F}}$ and $U :=$ Str$_{\mathcal{G}}$;
2. For every $f \in \mathcal{F}$ and $\overline{x} \in W^{\epsilon_f}$, $R_f(y, \overline{x})$ iff $\mathrm{F}_f(\overline{x})Ny$;
3. For every $g \in \mathcal{G}$ and $\overline{y} \in U^{\epsilon_g}$, $R_g(x, \overline{y})$ iff $xN\mathrm{G}_g(\overline{y})$;
4. If

$$\frac{x_1 \vdash y_1, \ldots, x_n \vdash y_n}{x \vdash y}$$

is a rule in D (including zero-ary rules), then

$$\frac{x_1 N y_1, \ldots, x_n N y_n}{x N y}$$

holds in $\mathbb{F}_{\mathrm{D}}$.

The complex lattice of functional D-frames

Let $h : \mathsf{AtProp} \to (\mathbb{F}_D)^+$. For every $S \in \mathsf{Str}_{\mathcal{F}}$ and $T \in \mathsf{Str}_{\mathcal{G}}$ we define $h\{S\} \subseteq W$ and $h\{T\} \subseteq U$ by simultaneous recursion as follows:

- $h\{\mathrm{F}_f(\overline{S})\} := \mathrm{F}_f[\overline{h\{S\}}] = \{\mathrm{F}_f(\overline{x})$ for some $\overline{x} \in \overline{h\{S\}}\}$;
- $h\{\mathrm{G}_g(\overline{T})\} := \mathrm{G}_g[\overline{h\{T\}}] = \{\mathrm{G}_g(\overline{y})$ for some $\overline{y} \in \overline{h\{T\}}\}$.

Theorem

For every $S \in \mathsf{Str}_{\mathcal{F}}$ and $T \in \mathsf{Str}_{\mathcal{F}}$ it holds that

$$\gamma_N(h\{S\}) = h(S) \qquad\qquad h\{T\}^{\downarrow} = h(T).$$

Corollary

The following are equivalent:

1. $h(S) \subseteq h(T)$;
2. sNt *for every* $s \in h\{S\}$ *and* $t \in h\{T\}$.

General Strategy for semantic cut-elimination

$\vdash_{\mathrm{D.LE}} X \vdash Y$ $\qquad\qquad$ $\vdash_{\mathrm{cfD.LE}} X \vdash Y$

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- for any $S \in W$ and $T \in U$,

$$S N_s T \quad \text{iff} \quad \vdash_D S \vdash T \text{ or } S \vdash T \notin (X \vdash Y)^{\leftarrow};$$

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- If $(X \vdash Y)^{\leftarrow}$ is finite or there are finite structures up to provable equivalence, the corresponding lattice is finite.

Conclusions

- Provided proof-theoretic semantics for a wide class of logics
- Obtained semantic proof of cut-elimination
- Some results in finite model property
- More to come in FMP, FEP, decidability....

Thank you for your attention!