# Proof theory and semantics for structural control


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# Overview

1. Typelogical grammars
2. The need of structural reasoning
3. Main problem: dealing with exceptions
4. The multi-type approach comes in handy
5. The broad picture

# Typelogical grammars

[Moot & Retoré]: book, [Moortgat 10]: Stanford Encyclopedia of Philosophy

Goal: develop a *compositional* and *modular* account of grammatical form and meaning in natural languages:

formal grammar is presented as a **logic**.

The **basic judgement**

$$x_1 : A_1, \ldots, x_n : A_n \vdash x : A$$

reads: the (structured configuration of) linguistic expressions $x_1$ of type $A_1, \ldots, x_n$ of type $A_n$ can be categorized as a well-formed expression $x$ of type $A$.

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- Form: residuated families of type-forming operations (**logical level**) + different means to control the grammatical resource management (**structural level**);
- Meaning: standard **computational** (via Curry-Howard), **algebraic**, **relational**, and **categorial semantics**.

# Parsing as deduction

[Ajdukiewicz 35, Bar-Hillel 64]: AB-grammars, [Lambek 58]: string of words, [Lambek 61]: bracketed strings (phrases)

- Parts of speech (noun, verb...) $\rightsquigarrow$ logical formulas - types.
- Grammaticality judgement $\rightsquigarrow$ logical deduction - computation.

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$$\underset{\text{time}}{np} \cdot \underset{\text{flies}}{(np\backslash s)} \cdot \underset{\text{like}}{(((np\backslash s)\backslash(np\backslash s))/np)} \cdot \underset{\text{an}}{(np/n)} \cdot \underset{\text{arrow}}{n} \vdash s$$

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Lexicon

- transitive verb 'love': $(np\backslash s)/np$
  - kids · (love · games)
- conjunction words 'and/but': *chameleon* word $(X\backslash X)/X$
  - $X = s$ : (kids like sweets)$_s$ but (parents prefer liquor)$_s$
  - $X = np\backslash s$: kids (like sweets)$_{np\backslash s}$ but (hate vegetables)$_{np\backslash s}$
- relative pronoun 'that': $(n\backslash n)/(s/np)$, i.e. it looks for a noun $n$ to its left and an *incomplete* sentence to its right ($s/np$: it misses a $np$, the *gap* at the right)

# Associativity √

$$
\dfrac{\dfrac{\text{alice}}{np} \qquad \dfrac{\dfrac{\text{found}}{(np\backslash s)/np} \qquad [\,\text{\textvisiblespace} \vdash np]^1}{\text{found} \cdot \text{\textvisiblespace} \vdash np\backslash s}\,/E}{\text{alice} \cdot (\text{found} \cdot \text{\textvisiblespace}\,) \vdash s}\,\backslash E
$$

$$
\dfrac{\dfrac{\text{alice} \cdot (\text{found} \cdot \text{\textvisiblespace}\,) \vdash s}{(\text{alice} \cdot \text{found}) \cdot \text{\textvisiblespace} \vdash s}\,A}{\text{alice} \cdot \text{found} \vdash s/np}\,/I^1
$$

$$
\dfrac{\dfrac{\text{key}}{n} \qquad \dfrac{\dfrac{\text{that}}{(n\backslash n)/(s/np)} \qquad \text{alice} \cdot \text{found} \vdash s/np}{\text{that} \cdot (\text{alice} \cdot \text{found}) \vdash n\backslash n}\,/E}{\text{key} \cdot (\text{that} \cdot (\text{alice} \cdot \text{found})) \vdash n}\,\backslash E
$$

# Mixed Commutativity √

$$\dfrac{\text{found}}{(np\backslash s)/np} \qquad [\,\text{\textvisiblespace} \vdash np\,]^1$$

$$\dfrac{(np\backslash s)/np \qquad [\,\text{\textvisiblespace} \vdash np\,]^1}{\text{found} \cdot \text{\textvisiblespace} \vdash np\backslash s}\ /E$$

$$\dfrac{\text{there}}{(np\backslash s)\backslash(np\backslash s)}$$

$$\dfrac{\text{found} \cdot \text{\textvisiblespace} \vdash np\backslash s \qquad (np\backslash s)\backslash(np\backslash s)}{(\text{found} \cdot \text{\textvisiblespace}) \cdot \text{there} \vdash np\backslash s}\ \backslash E$$

$$\dfrac{\dfrac{\text{alice}}{np} \qquad (\text{found} \cdot \text{\textvisiblespace}) \cdot \text{there} \vdash np\backslash s}{\text{alice} \cdot ((\text{found} \cdot \text{\textvisiblespace}) \cdot \text{there}) \vdash s}\ \backslash E$$

$$\dfrac{\text{alice} \cdot ((\text{found} \cdot \text{\textvisiblespace}) \cdot \text{there}) \vdash s}{\text{alice} \cdot ((\text{found} \cdot \text{there}) \cdot \text{\textvisiblespace}) \vdash s}\ MC$$

$$\dfrac{\text{alice} \cdot ((\text{found} \cdot \text{there}) \cdot \text{\textvisiblespace}) \vdash s}{(\text{alice} \cdot (\text{found} \cdot \text{there})) \cdot \text{\textvisiblespace} \vdash s}\ A$$

$$\dfrac{(\text{alice} \cdot (\text{found} \cdot \text{there})) \cdot \text{\textvisiblespace} \vdash s}{\text{alice} \cdot (\text{found} \cdot \text{there}) \vdash s/np}\ /I^1$$

$$\dfrac{\dfrac{\text{that}}{(n\backslash n)/(s/np)} \qquad \text{alice} \cdot (\text{found} \cdot \text{there}) \vdash s/np}{\text{that} \cdot (\text{alice} \cdot (\text{found} \cdot \text{there})) \vdash n\backslash n}\ /E$$

$$\dfrac{\dfrac{\text{key}}{n} \qquad \text{that} \cdot (\text{alice} \cdot (\text{found} \cdot \text{there})) \vdash n\backslash n}{\text{key} \cdot (\text{that} \cdot (\text{alice} \cdot (\text{found} \cdot \text{there}))) \vdash n}\ \backslash E$$

# Associativity ×

$$\frac{\text{love}}{(np\backslash s)/np} \qquad \frac{\text{super\_mario}}{np} \qquad \frac{\text{kids}}{np} \qquad \frac{\text{but}}{(s\backslash s)/s} \qquad \frac{\text{parents}}{np} \qquad \frac{\text{hate}}{(np\backslash s)/np} \qquad \frac{\text{that}}{(n\backslash n)/(s/np)} \qquad \frac{\text{games}}{n}$$

$$\frac{(np\backslash s)/np \qquad [\,\llcorner \vdash np]^1}{\text{hate}\cdot\llcorner \vdash np\backslash s}\ /E$$

$$\frac{np \qquad \text{hate}\cdot\llcorner \vdash np\backslash s}{\text{parents}\cdot(\text{hate}\cdot\llcorner) \vdash s}\ \backslash E$$

$$\frac{(s\backslash s)/s \qquad \text{parents}\cdot(\text{hate}\cdot\llcorner) \vdash s}{\text{but}\cdot(\text{parents}\cdot(\text{hate}\cdot\llcorner)) \vdash s\backslash s}\ /E$$

$$\frac{(np\backslash s)/np \qquad np}{\text{love}\cdot\text{super\_mario} \vdash np\backslash s}\ /E$$

$$\frac{np \qquad \text{love}\cdot\text{super\_mario} \vdash np\backslash s}{\text{kids}\cdot(\text{love}\cdot\text{super\_mario}) \vdash s}\ \backslash E$$

$$\frac{\text{kids}\cdot(\text{love}\cdot\text{super\_mario}) \vdash s \qquad \text{but}\cdot(\text{parents}\cdot(\text{hate}\cdot\llcorner)) \vdash s\backslash s}{(\text{kids}\cdot(\text{love}\cdot\text{super\_mario}))\cdot(\text{but}\cdot(\text{parents}\cdot(\text{hate}\cdot\llcorner))) \vdash s}\ \backslash E$$

$$\frac{(\text{kids}\cdot(\text{love}\cdot\text{super\_mario}))\cdot(\text{but}\cdot(\text{parents}\cdot(\text{hate}\cdot\llcorner))) \vdash s}{(\text{kids}\cdot(\text{love}\cdot\text{super\_mario}))\cdot(\text{but}\cdot((\text{parents}\cdot\text{hate})\cdot\llcorner)) \vdash s}\ A$$

$$\frac{(\text{kids}\cdot(\text{love}\cdot\text{super\_mario}))\cdot(\text{but}\cdot((\text{parents}\cdot\text{hate})\cdot\llcorner)) \vdash s}{(\text{kids}\cdot(\text{love}\cdot\text{super\_mario}))\cdot((\text{but}\cdot(\text{parents}\cdot\text{hate}))\cdot\llcorner) \vdash s}\ A$$

$$\frac{(\text{kids}\cdot(\text{love}\cdot\text{super\_mario}))\cdot((\text{but}\cdot(\text{parents}\cdot\text{hate}))\cdot\llcorner) \vdash s}{((\text{kids}\cdot(\text{love}\cdot\text{super\_mario}))\cdot(\text{but}\cdot(\text{parents}\cdot\text{hate})))\cdot\llcorner \vdash s}\ A$$

$$\frac{((\text{kids}\cdot(\text{love}\cdot\text{super\_mario}))\cdot(\text{but}\cdot(\text{parents}\cdot\text{hate})))\cdot\llcorner \vdash s}{(\text{kids}\cdot(\text{love}\cdot\text{super\_mario}))\cdot(\text{but}\cdot(\text{parents}\cdot\text{hate})) \vdash s/np}\ /I^1$$

$$\frac{(n\backslash n)/(s/np) \qquad (\text{kids}\cdot(\text{love}\cdot\text{super\_mario}))\cdot(\text{but}\cdot(\text{parents}\cdot\text{hate})) \vdash s/np}{\text{that}\cdot((\text{kids}\cdot(\text{love}\cdot\text{super\_mario}))\cdot(\text{but}\cdot(\text{parents}\cdot\text{hate}))) \vdash n\backslash n}\ /E$$

$$\frac{n \qquad \text{that}\cdot((\text{kids}\cdot(\text{love}\cdot\text{super\_mario}))\cdot(\text{but}\cdot(\text{parents}\cdot\text{hate}))) \vdash n\backslash n}{\text{games}\cdot(\text{that}\cdot((\text{kids}\cdot(\text{love}\cdot\text{super\_mario}))\cdot(\text{but}\cdot(\text{parents}\cdot\text{hate})))) \vdash n}\ \backslash E$$

# Licensing rules in a controlled form - 1/2

[Moortgat 96, Kurtonina & Moortgat 97], [Morrill 17]

$$\dfrac{\text{found}}{(np\backslash s)/np} \qquad \dfrac{[\,\llcorner\!\lrcorner \vdash \Box np]^2}{\langle \llcorner\!\lrcorner \rangle \vdash np}\ \Box E$$

$$\dfrac{(np\backslash s)/np \qquad \langle \llcorner\!\lrcorner \rangle \vdash np}{\text{found} \cdot \langle \llcorner\!\lrcorner \rangle \vdash np\backslash s}\ /E$$

$$\dfrac{\dfrac{\text{alice}}{np} \qquad \text{found} \cdot \langle \llcorner\!\lrcorner \rangle \vdash np\backslash s}{\text{alice} \cdot (\text{found} \cdot \langle \llcorner\!\lrcorner \rangle) \vdash s}\ \backslash E$$

$$\dfrac{\text{alice} \cdot (\text{found} \cdot \langle \llcorner\!\lrcorner \rangle) \vdash s}{(\text{alice} \cdot \text{found}) \cdot \langle \llcorner\!\lrcorner \rangle \vdash s}\ cA$$

$$\dfrac{[\,\llcorner\!\lrcorner \vdash \Diamond\Box np]^1 \qquad (\text{alice} \cdot \text{found}) \cdot \langle \llcorner\!\lrcorner \rangle \vdash s}{(\text{alice} \cdot \text{found}) \cdot \llcorner\!\lrcorner \vdash s}\ \Diamond E^2$$

$$\dfrac{(\text{alice} \cdot \text{found}) \cdot \llcorner\!\lrcorner \vdash s}{\text{alice} \cdot \text{found} \vdash s/\Diamond\Box np}\ /I^1$$

$$\dfrac{\dfrac{\text{that}}{(n\backslash n)/(s/\Diamond\Box np)} \qquad \text{alice} \cdot \text{found} \vdash s/\Diamond\Box np}{\text{that} \cdot (\text{alice} \cdot \text{found}) \vdash n\backslash n}\ /E$$

$$\dfrac{\dfrac{\text{key}}{n} \qquad \text{that} \cdot (\text{alice} \cdot \text{found}) \vdash n\backslash n}{\text{key} \cdot (\text{that} \cdot (\text{alice} \cdot \text{found})) \vdash n}\ \backslash E$$

$$\lambda x.((\textsc{key}\ x) \wedge ((\textsc{found}\ x)\ \textsc{alice}))$$

# Licensing rules in a controlled form - 2/2

$$\frac{\textrm{found}}{(np\backslash s)/np} \qquad \frac{[\textvisiblespace \vdash \Box np]^2}{\langle \textvisiblespace \rangle \vdash np}\,\Box E$$

$$\frac{(np\backslash s)/np \qquad \langle \textvisiblespace \rangle \vdash np}{\textrm{found} \cdot \langle \textvisiblespace \rangle \vdash np\backslash s}\,/E \qquad \frac{\textrm{there}}{(np\backslash s)\backslash(np\backslash s)}$$

$$\frac{\textrm{found} \cdot \langle \textvisiblespace \rangle \vdash np\backslash s \qquad (np\backslash s)\backslash(np\backslash s)}{(\textrm{found} \cdot \langle \textvisiblespace \rangle) \cdot \textrm{there} \vdash np\backslash s}\,\backslash E$$

$$\frac{\dfrac{\textrm{alice}}{np} \qquad (\textrm{found} \cdot \langle \textvisiblespace \rangle) \cdot \textrm{there} \vdash np\backslash s}{\textrm{alice} \cdot ((\textrm{found} \cdot \langle \textvisiblespace \rangle) \cdot \textrm{there}) \vdash s}\,\backslash E$$

$$\frac{\textrm{alice} \cdot ((\textrm{found} \cdot \langle \textvisiblespace \rangle) \cdot \textrm{there}) \vdash s}{\textrm{alice} \cdot ((\textrm{found} \cdot \textrm{there}) \cdot \langle \textvisiblespace \rangle) \vdash s}\,cMC$$

$$\frac{\textrm{alice} \cdot ((\textrm{found} \cdot \textrm{there}) \cdot \langle \textvisiblespace \rangle) \vdash s}{(\textrm{alice} \cdot (\textrm{found} \cdot \textrm{there})) \cdot \langle \textvisiblespace \rangle \vdash s}\,cA$$

$$\frac{[\textvisiblespace \vdash \Diamond\Box np]^1 \qquad (\textrm{alice} \cdot (\textrm{found} \cdot \textrm{there})) \cdot \langle \textvisiblespace \rangle \vdash s}{(\textrm{alice} \cdot (\textrm{found} \cdot \textrm{there})) \cdot \textvisiblespace \vdash s}\,\Diamond E^2$$

$$\frac{(\textrm{alice} \cdot (\textrm{found} \cdot \textrm{there})) \cdot \textvisiblespace \vdash s}{\textrm{alice} \cdot (\textrm{found} \cdot \textrm{there}) \vdash s/\Diamond\Box np}\,/I^1$$

$$\frac{\dfrac{\textrm{that}}{(n\backslash n)/(s/\Diamond\Box np)} \qquad \textrm{alice} \cdot (\textrm{found} \cdot \textrm{there}) \vdash s/\Diamond\Box np}{\textrm{that} \cdot (\textrm{alice} \cdot (\textrm{found} \cdot \textrm{there})) \vdash n\backslash n}\,/E$$

$$\frac{\dfrac{\textrm{key}}{n} \qquad \textrm{that} \cdot (\textrm{alice} \cdot (\textrm{found} \cdot \textrm{there})) \vdash n\backslash n}{\textrm{key} \cdot (\textrm{that} \cdot (\textrm{alice} \cdot (\textrm{found} \cdot \textrm{there}))) \vdash n}\,\backslash E$$

$$\lambda x.((\textsc{key}\ x) \wedge ((\textsc{there}\ (\textsc{found}\ x))\ \textsc{alice}))$$

# Blocking rules in a controlled form

$$\frac{\text{love}}{(np\backslash s)/np} \qquad \frac{[\, \sqcup \vdash \square np]^4}{\langle \sqcup \rangle \vdash np}\ \square E$$

$$\frac{(np\backslash s)/np \qquad \langle \sqcup \rangle \vdash np}{\text{love} \cdot \langle \sqcup \rangle \vdash np\backslash s}\ /E$$

$$\frac{\dfrac{\text{kids}}{np} \qquad \text{love} \cdot \langle \sqcup \rangle \vdash np\backslash s}{\text{kids} \cdot (\text{love} \cdot \langle \sqcup \rangle) \vdash s}\ \backslash E$$

$$\frac{\text{kids} \cdot (\text{love} \cdot \langle \sqcup \rangle) \vdash s}{(\text{kids} \cdot \text{love}) \cdot \langle \sqcup \rangle \vdash s}\ cA$$

$$\frac{[\, \sqcup \vdash \Diamond\square np]^3 \qquad (\text{kids} \cdot \text{love}) \cdot \langle \sqcup \rangle \vdash s}{(\text{kids} \cdot \text{love}) \cdot \sqcup \vdash s}\ \Diamond E^4$$

$$\frac{(\text{kids} \cdot \text{love}) \cdot \sqcup \vdash s}{\text{kids} \cdot \text{love} \vdash s/\Diamond\square np}\ /I^3$$

$$\frac{\dfrac{\text{but}}{((s/\Diamond\square np)\backslash\square(s/np))/(s/\Diamond\square np)} \qquad \vdots}{\text{but} \cdot (\text{parents} \cdot \text{hate}) \vdash (s/\Diamond\square np)\backslash\square(s/np)}$$

$$\frac{\text{kids} \cdot \text{love} \vdash s/\Diamond\square np \qquad \text{but} \cdot (\text{parents} \cdot \text{hate}) \vdash (s/\Diamond\square np)\backslash\square(s/np)}{(\text{kids} \cdot \text{love}) \cdot (\text{but} \cdot (\text{parents} \cdot \text{hate})) \vdash \square(s/np)}$$

$$\frac{(\text{kids} \cdot \text{love}) \cdot (\text{but} \cdot (\text{parents} \cdot \text{hate})) \vdash \square(s/np)}{\langle(\text{kids} \cdot \text{love}) \cdot (\text{but} \cdot (\text{parents} \cdot \text{hate}))\rangle \vdash s/np}\ \square E$$

$$\frac{\langle(\text{kids} \cdot \text{love}) \cdot (\text{but} \cdot (\text{parents} \cdot \text{hate}))\rangle \vdash s/np}{\langle(\text{kids} \cdot \text{love}) \cdot (\text{but} \cdot (\text{parents} \cdot \text{hate}))\rangle \cdot \text{super\_mario} \vdash s}\ /E$$

# Starting point: display calculi

- Natural generalization of Gentzen's sequent calculi;
- sequents $X \vdash Y$, where $X$ and $Y$ are **structures**:
  - formulas are **atomic structures**
  - built-up: **structural connectives** (generalizing meta-linguistic comma in sequents $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$)
  - generation **trees** (generalizing sets, multisets, sequences)
- **Display property**:

$$\dfrac{\dfrac{\dfrac{Y \vdash X \check{\backslash} Z}{X \hat{\otimes} Y \vdash Z}}{Y \hat{\otimes} X \vdash Z}}{X \vdash Y \check{\backslash} Z} \qquad \dfrac{X \vdash \check{\neg} Y}{Y \vdash \check{\sim} X}$$

display rules semantically justified by **adjunction/residuation**

- **Canonical proof of cut elimination (via metatheorem)**

# Proper display calculi

[Wansing 98]: proper, [Belnap 82, 89]: display logic, [Mints 72, Dunn 73, 75]: structural connectives

### Definition

A **proper DC** verifies each of the following conditions:

1. structures can disappear, formulas are **forever**;
2. **tree-traceable** formula-occurrences, via suitably defined *congruence* relation (same shape, position, non-proliferation);
3. **principal = displayed**
4. rules are closed under **uniform substitution** of congruent parameters (Properness!);
5. **reduction strategy** exists when cut formulas are principal.

### Theorem (Canonical!)

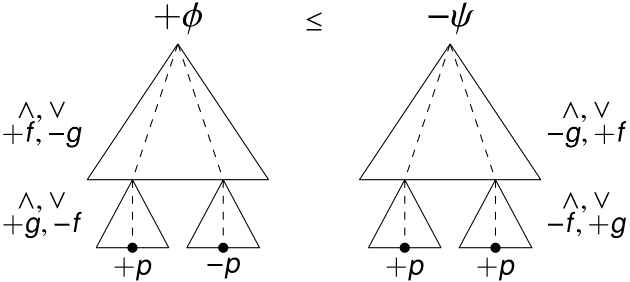
Cut elim. and subformula property hold for any **proper DC**.

# Which logics are properly displayable?

[Ciabattoni et al. 15, Greco et al. 16]

## Complete characterization:

1. the logics of any **basic** normal (D)LE;
2. axiomatic extensions of these with **analytic inductive inequalities**: $\rightsquigarrow$ unified correspondence




**Fact:** cut-elim., subfm. prop., sound-&-completeness, conservativity **guaranteed** by metatheorem + ALBA-technology.

# Examples

The definition of analytic inductive inequalities is uniform in each signature.

- Analytic inductive axioms

  $(A \to (B \vee C)) \to ((A \to B) \vee C)$

  $(\Diamond A \to \Box B) \to \Box(A \to B)$

- Sahlqvist but non-analytic axioms

  $A \to \Diamond\Box A$

  $(\Box A \to \Diamond B) \to (A \to B)$

The excluded middle is derivable using *Grishin's rule*:

$$
\begin{array}{c}
\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A \mathbin{\hat{\wedge}} \top \vdash A}}{A \mathbin{\hat{\wedge}} \top \vdash \bot \mathbin{\check{\vee}} A}}{\top \vdash A \mathbin{\check{\rightarrow}} (\bot \mathbin{\check{\vee}} A)}\;\mathit{Gri}}{\top \vdash (A \mathbin{\check{\rightarrow}} \bot) \mathbin{\check{\vee}} A} \\
\hline \\
\vdots \\
\top \vdash \neg A \vee A
\end{array}
$$

# For many... but not for all.

- The characterization theorem sets **hard boundaries** to the scope of proper display calculi.
- Interesting logics are **left out**:
  - First order logic
  - Non normal modal logics
  - Conditional logics
  - Dynamic epistemic logic
  - Inquisitive logic
  - Semi De Morgan logic
  - Bi-lattice logic
  - Rough algebras
  - ...

Can we **extend the scope** of proper display calculi?

Yes: proper display calculi $\rightsquigarrow$ proper **multi-type** calculi
(read: multi-sorted calculi)

# Multi-type ($\rightsquigarrow$ multi-sorted) proper display calculi

[Greco et al. 14...]

## Definition

A **proper mDC** verifies each of the following conditions:

1. structures can disappear, formulas are **forever**;
2. **tree-traceable** formula-occurrences, via suitably defined *congruence* relation (same shape, position, non-proliferation)
3. **principal = displayed**
4. rules are closed under **uniform substitution** of congruent parameters within each type (Properness!);
5. **reduction strategy** exists when cut formulas are principal.
6. **type-uniformity** of derivable sequents;
7. **strongly uniform cuts** in each/some type(s).

## Theorem (Canonical!)

Cut elim. and subformula property hold for any **proper mDC**.

# Language expansion: structural control operators 1/2

- Display rules (adjunction)

$$\text{adj}\ \frac{X \vdash \check{\blacksquare}\, Y}{\hat{\Diamond}\, X \vdash Y}$$

- Logical rules (arity and tonicity)

$$\Diamond_L\ \frac{\hat{\Diamond}\, A \vdash X}{\Diamond A \vdash X} \qquad \frac{X \vdash A}{\hat{\Diamond}\, X \vdash \Diamond A}\ \Diamond_R$$

$$\check{\blacksquare}_L\ \frac{A \vdash X}{\blacksquare A \vdash \check{\blacksquare}\, X} \qquad \frac{X \vdash \check{\blacksquare}\, A}{X \vdash \blacksquare A}\ \blacksquare_R$$

# Language expansion: structural control operators 2/2

- Display rules (adjunction)

$$\text{adj}\ \frac{X \vdash \check{\blacksquare}\, \Gamma}{\hat{\Diamond}\, X \vdash \Gamma}$$

- Logical rules (arity and tonicity)

$$\Diamond_L\ \frac{\hat{\Diamond}\, \alpha \vdash X}{\Diamond\alpha \vdash X} \qquad \frac{\Gamma \vdash \alpha}{\hat{\Diamond}\, \Gamma \vdash \Diamond\alpha}\ \Diamond_R$$

$$\check{\blacksquare}_L\ \frac{A \vdash X}{\blacksquare A \vdash \check{\blacksquare}\, X} \qquad \frac{\Gamma \vdash \check{\blacksquare}\, A}{\Gamma \vdash \blacksquare A}\ \blacksquare_R$$

# Axiomatic extensions via analytic structural rules - 1/2

- Structural rules

$$A\ \frac{X \hat{\otimes} (Y \hat{\otimes} Z) \vdash W}{(X \hat{\otimes} Y) \hat{\otimes} Z \vdash W} \qquad MC\ \frac{(X \hat{\otimes} Z) \hat{\otimes} Y \vdash W}{(X \hat{\otimes} Y) \hat{\otimes} Z \vdash W}$$

- Controlled structural rules

$$cA\ \frac{X \hat{\otimes} (Y \hat{\otimes} \hat{\Diamond} Z) \vdash W}{(X \hat{\otimes} Y) \hat{\otimes} \hat{\Diamond} Z \vdash W} \qquad cMC\ \frac{(X \hat{\otimes} \hat{\Diamond} Z) \hat{\otimes} Y \vdash W}{(X \hat{\otimes} Y) \hat{\otimes} \hat{\Diamond} Z \vdash W}$$

# Axiomatic extensions via analytic structural rules - 2/2

- Structural rules

$$A\ \frac{X \hat{\otimes} (Y \hat{\otimes} Z) \vdash W}{(X \hat{\otimes} Y) \hat{\otimes} Z \vdash W} \qquad MC\ \frac{(X \hat{\otimes} Z) \hat{\otimes} Y \vdash W}{(X \hat{\otimes} Y) \hat{\otimes} Z \vdash W}$$

- Controlled structural rules

$$cA\ \frac{X \hat{\otimes} (Y \hat{\otimes} \hat{\Diamond} \Gamma) \vdash W}{(X \hat{\otimes} Y) \hat{\otimes} \hat{\Diamond} \Gamma \vdash W} \qquad cMC\ \frac{(X \hat{\otimes} \hat{\Diamond} \Gamma) \hat{\otimes} Y \vdash W}{(X \hat{\otimes} Y) \hat{\otimes} \hat{\Diamond} \Gamma \vdash W}$$

# Licensing rules: the case of Linear Logic

[Belnap 92]: **not** a **proper** display calculus:

$$\frac{A \vdash X}{!A \vdash X} \qquad \frac{Y \vdash A}{Y \vdash !A}$$

$$\frac{A \vdash Z}{?A \vdash Z} \qquad \frac{X \vdash A}{X \vdash ?A}$$

$Y$ and $Z$ not arbitrary but *exponentially restricted*.

$!!A \dashv\vdash !A$

$!A \vdash A$

$A \vdash B$ implies $!A \vdash !B$

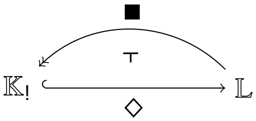
$!\top \dashv\vdash 1$

$!(A \& B) \dashv\vdash !A \otimes !B$ analytic?

# Linear logic: algebraic analysis

$$!!a = !a \qquad !\top = 1$$
$$!a \leq a \qquad !(a \& b) = !a \otimes !b$$
$$a \leq b \text{ implies } !a \leq !b$$

$! : \mathbb{L} \to \mathbb{L}$ interior operator. Then $! = \Diamond\blacksquare$, where




**Fact:** Range(!) $::= \mathbb{K}_!$ has natural BA/HA-structure.

**Upshot:** natural semantics for the following **multi-type** language:

$$\text{Kernel} \ni \alpha ::= \blacksquare A \mid \mathtt{t} \mid \mathtt{f} \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \mid \alpha \to \alpha$$

$$\text{Linear} \ni A ::= p \mid \Diamond\alpha \mid 1 \mid \bot \mid A \otimes A \mid A \parr A \mid A \multimap A \mid$$
$$\top \mid 0 \mid A \,\&\, A \mid A \oplus A$$

# Reverse-engineering linear logic

**Problem:** the following axioms are **non-analytic**.

$$!\top \dashv\vdash 1 \quad \rightsquigarrow \quad \Diamond\blacksquare\top \dashv\vdash 1$$

$$!(A \mathbin{\&} B) \dashv\vdash !A \otimes !B \quad \rightsquigarrow \quad \Diamond\blacksquare(A \mathbin{\&} B) \dashv\vdash \Diamond\blacksquare A \otimes \Diamond\blacksquare B$$

**Solution:** $\blacksquare$ surjective and finitely meet-preserving $\Rightarrow$ axioms above semantically equivalent to the following analytic identities:

$$\Diamond \mathtt{t} = 1 \qquad \Diamond(\alpha \wedge \beta) = \Diamond\alpha \otimes \Diamond\beta$$

corresponding to the following analytic rules:

$$\text{co-nec}\ \frac{\hat{\Diamond}\ \hat{\mathtt{t}} \vdash X}{\hat{1} \vdash X} \qquad \frac{\hat{\Diamond}\,\Gamma\,\hat{\otimes}\,\hat{\Diamond}\,\Delta \vdash X}{\hat{\Diamond}\,(\Gamma\,\hat{\wedge}\,\Delta) \vdash X}\ \text{(co-)reg}$$

# Deriving $!(A \mathbin{\&} B) \vdash {!A} \otimes {!B}$

$$
\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A \mathbin{\&} B \vdash A}}{\blacksquare(A \mathbin{\&} B) \vdash \check{\blacksquare} A}}{\blacksquare(A \mathbin{\&} B) \vdash \blacksquare A}}{\hat{\Diamond}\, \blacksquare(A \mathbin{\&} B) \vdash \Diamond\blacksquare A}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{B \vdash B}{A \mathbin{\&} B \vdash B}}{\blacksquare(A \mathbin{\&} B) \vdash \check{\blacksquare} B}}{\blacksquare(A \mathbin{\&} B) \vdash \blacksquare B}}{\hat{\Diamond}\, \blacksquare(A \mathbin{\&} B) \vdash \Diamond\blacksquare B}
$$

$$\hat{\Diamond}\, \blacksquare(A \mathbin{\&} B)\, \hat{\otimes}\, \hat{\Diamond}\, \blacksquare(A \mathbin{\&} B) \vdash \Diamond\blacksquare A \otimes \Diamond\blacksquare B$$

reg

$$\hat{\Diamond}\, (\blacksquare(A \mathbin{\&} B)\, \hat{\wedge}\, \blacksquare(A \mathbin{\&} B)) \vdash \Diamond\blacksquare A \otimes \Diamond\blacksquare B$$

$$\blacksquare(A \mathbin{\&} B)\, \hat{\wedge}\, \blacksquare(A \mathbin{\&} B) \vdash \check{\blacksquare}\, (\Diamond\blacksquare A \otimes \Diamond\blacksquare B)$$

$C_K$

$$\blacksquare(A \mathbin{\&} B) \vdash \check{\blacksquare}\, (\Diamond\blacksquare A \otimes \Diamond\blacksquare B)$$

$$\hat{\Diamond}\, \blacksquare(A \mathbin{\&} B) \vdash \Diamond\blacksquare A \otimes \Diamond\blacksquare B$$

$$\Diamond\blacksquare(A \mathbin{\&} B) \vdash \Diamond\blacksquare A \otimes \Diamond\blacksquare B$$

$$=\!=\!=\!=\!=\!=\!=\!=\!=\!=$$

$$!(A \mathbin{\&} B) \vdash {!A} \otimes {!B}$$

# General strategy

- Define a multi-modal logic where linguistic composition is relativized to specific resource **management modes** via a language expansion.
- The extra expressivity is obtained in a controlled fashion via the addition of **interaction postulates**.
- It can be used to **licence** or to **block** the access to different regimes of resource management.

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- Define a multi-modal logic where linguistic composition is relativized to specific resource **management modes** via a language expansion.
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- It can be used to **licence** or to **block** the access to different regimes of resource management.

Ingredients:

- the sort of **general** elements that inhabit the more restrictive regime;
- the sorts of **special** elements that witness the licence of a more liberal regime;
- the sort(s) of **blocking** elements that provide the room to block structural transformations.

# Heterogeneous structural control algebras

For each $i \in I$, $\mathbb{H} := (\mathbb{G}, \mathbb{L}_i, \mathbb{R}_i, \mathbb{B})$ is a structure such that

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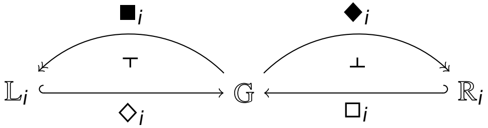

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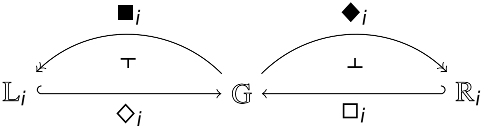


where the composition

| | | |
|---|---|---|
| $\Diamond_i \blacksquare_i$ | defines | an interior operator on $\mathbb{G}$ |
| $\Box_i \blacklozenge_i$ | defines | a closure operator on $\mathbb{G}$ |
| $\blacksquare_i \Diamond_i$ | defines | identity on $\mathbb{L}_i$ |
| $\blacklozenge_i \Box_i$ | defines | identity on $\mathbb{R}_i$ |

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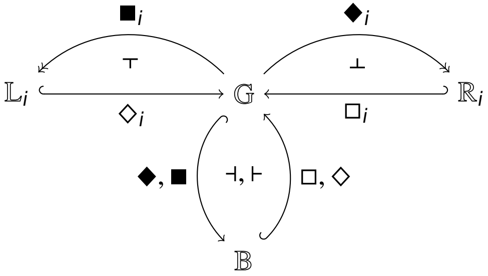

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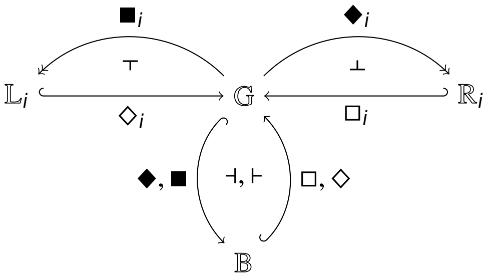


moreover,

- for each $f \in \mathcal{F}$ (resp. $g \in \mathcal{G}$) with domain $\mathbb{G}^n$, there exists a map $\mathcal{F}_B \ni f_B : \mathbb{B} \times \mathbb{G}^{n-1} \to \mathbb{G}$ (resp. $g \in \mathcal{G}$),
- $\mathcal{F}_B \cup \mathcal{G}_B$ is closed under adjoints/residuals.

# Beyond analiticity: towards a general theory

- Several examples of logics which are single-type not analytic but multi-type analytic.
- Patterns are emerging. Main guideline: discovering and exploiting hidden **adjunctions** / **representation theorems**.
- Can we make this practice into a uniform theory?

- What can we infer from interaction postulates?
- E.g. $\mathbb{L}_i$ and $\mathbb{R}_i$ can be systematically endowed with a compatible signature.
- What about the properties of the defined operations?
- What about the relation between $\mathbb{L}_i$ and $\mathbb{R}_i$?

**Thank you ◇■**