Some theorems concerning Grzegorczyk contact lattices

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Outline

Grzegorczyk contact lattices

Grzegorczyk points and their properties

Existence of GCLs

Set theoretical representation theorems for GCLs

The characterization of finite GCLs

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The characterization of finite GCLs

Grzegorczyk lattices – definition

A pair $\mathfrak{L} = \langle R, \leqslant \rangle$ is a Grzegorczyk lattice iff it is a lattice with zero element and satisfies the following strong polarization condition:

$$x \nleqslant y \to \exists_{z \in R}(z \leqslant x \wedge z \perp y \wedge \forall_{u \in R}(u \leqslant x \wedge u \perp y) \to u \leqslant z) \quad \text{(P)}$$

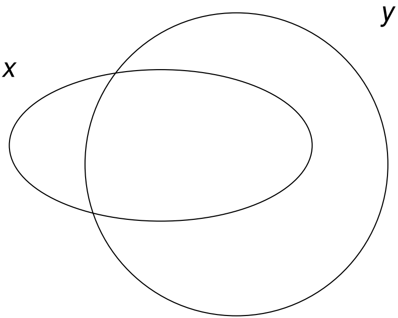
where $x \perp y \;:\longleftrightarrow\; x \sqcap y = 0$ (with $\sqcap$ being the standard meet operation).

Grzegorczyk lattices – definition

All Grzegorczyk lattices have the relative complement operation in $R \times R$:

$$x - y := \max\{z \in R \mid z \leqslant x \wedge z \perp y\}, \qquad (\mathtt{df}\,-)$$

which is well-defined thanks to (P).



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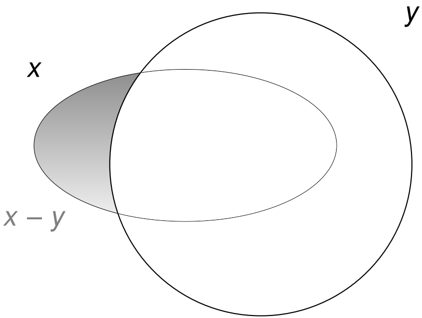


Grzegorczyk lattices – definition

- From model theoretical point of view, the class of Grzegorczyk lattices coincides with the class of generalized Boolean algebras.
- A family of finite subsets of $\mathbb{N}$ is an example of a Grzegorczyk lattice.

Pre-contact lattices

A pre-contact lattice is a triple $\mathfrak{C} = \langle R, \leqslant, \mathbf{C} \rangle$, where $\langle R, \leqslant \rangle$ is a Grzegorczyk lattice and $\mathbf{C} \subseteq R \times R$ (called pre-contact) satisfies:

$$0 \not\mathbf{C}\, x \tag{C0}$$

$$x \neq 0 \rightarrow x \mathbf{C} x \tag{C1}$$

$$x \mathbf{C} y \rightarrow y \mathbf{C} x \tag{C2}$$

$$x \mathbf{C} y \wedge y \leqslant z \rightarrow x \mathbf{C} z\,. \tag{C3}$$

Pre-contact lattices – motivations

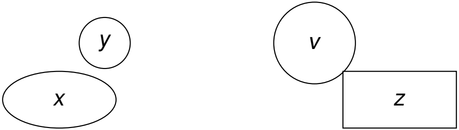


Figure: In both pairs regions are external to each other. Regions x and y are separated, but regions v and z are not—they are externally tangent to each other. The relation $\perp$ does not differentiate between these two situations.

Pre-contact lattices

In a standard way we define two further auxiliary relations, overlap and non-tangential part:

$$x \bigcirc y \;:\longleftrightarrow\; x \sqcap y \neq 0 \qquad (\mathtt{df}\bigcirc)$$

$$x \ll y \;:\longleftrightarrow\; \forall_{z \in R}(z \perp y \rightarrow z \not\mathbf{C}\, x)\,. \qquad (\mathtt{df} \ll)$$

Figure: Geometrical interpretation of non-tangential inclusion: x is non-tangentially included in y, while v touches the complement of z.

Pre-contact lattices

A canonical interpretation of a pre-contact lattice is obtained by taking a Grzegorczyk lattice whose regions are regular open sets of some topological space, and defining:

$$x \mathbf{C} y \;:\longleftrightarrow\; \mathrm{Cl}\, x \cap \mathrm{Cl}\, y \neq \emptyset . \qquad (\mathtt{df}\,\mathbf{C})$$

In consequence:

$$x \ll y \longleftrightarrow \mathrm{Cl}\, x \subseteq y .$$

Pre-points in pre-contact lattices

A pre-point (or representative of a point) is any nonempty set Q or regions satisfying the following three conditions:

$$\forall_{x,y \in Q} \, (x = y \vee x \ll y \vee y \ll x) \tag{r1}$$

$$\forall_{x \in Q} \exists_{y \in Q} \, y \ll x \tag{r2}$$

$$\forall_{x,y \in R} (\forall_{u \in Q} \, (u \bigcirc x \wedge u \bigcirc y) \rightarrow x \mathbf{C} y) \, . \tag{r3}$$

Let $\mathbf{Q}$ be the set of all pre-points of a given pre-contact lattice.

Pre-points in pre-contact lattices

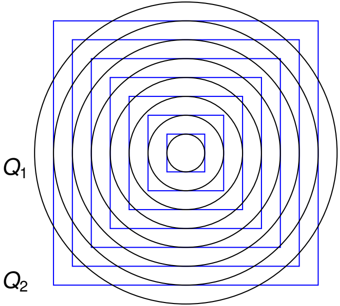


Figure: Q_1 and Q_2 represent the same point

Grzegorczyk contact lattices

Grzegorczyk axioms for pre-contact lattices postulate existence of pre-points:

$$x \bigcirc y \to \exists_{Q \in \mathbf{Q}} \exists_{z \in Q}\ z \leqslant x \sqcap y \qquad (\mathrm{G}_{\bigcirc})$$

$$x \mathbf{C}\, y \wedge x \perp y \to \exists_{Q \in \mathbf{Q}} \forall_{z \in Q} (z \bigcirc x \ \wedge\ z \bigcirc y) \qquad (\mathrm{G}_{\perp})$$

Intuitively, these can be geometrically interpreted as follows:

($\mathrm{G}_{\bigcirc}$) there is a pre-point in every non-zero region,

($\mathrm{G}_{\perp}$) there are pre-points at the loci of contact of regions.

Grzegorczyk contact lattice is any pre-contact lattice which satisfies the two axioms above.

Grzegorczyk contact lattices

Theorem

Axioms:

$$x \mathbf{C} (y \sqcup z) \to x \mathbf{C} y \vee x \mathbf{C} z \,, \tag{C4}$$

$$\forall_{z \in R} (z \mathbf{C} x \to z \mathbf{C} y) \to x \leqslant y \,. \tag{C5}$$

are true in any GCL (which justifies the name contact lattices).

Grzegorczyk contact lattices

$$x \mathbf{C} (y \sqcup z) \rightarrow x \mathbf{C} y \vee x \mathbf{C} z \qquad \text{(C4)}$$

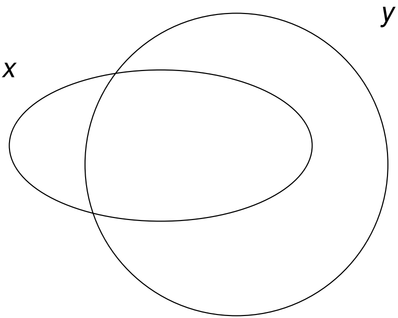
A proof of (C4).

- If $x \mathbf{C} (y \sqcup z)$, then by the Grzegorczyk axiom's there is a pre-point Q such that (a) $\forall_{u \in Q}\, u \bigcirc x$ and (b) $\forall_{u \in Q}\, u \bigcirc y \sqcup z$.
- Divide Q into: $Q_y := \{u \in Q \mid u \bigcirc y\}$ and $Q_z := \{u \in Q \mid u \bigcirc z\}$. Assume there is $q \in Q \setminus Q_y$ (i.e. $q \perp y$).
- Pick an arbitrary $u \in Q$. We have that $q \leqslant u$ or $u \leqslant q$.
- In the first case, $q \bigcirc z$ and the more so $u \bigcirc z$.
- In the second case, $u = q \sqcap u$ and so $u \perp y$, so $u \bigcirc z$.
- Therefore $Q \subseteq Q_z$ and from this, (a) and properties of pre-points we have that $x \mathbf{C} z$. □

Grzegorczyk contact lattices

$$\forall_{z \in R}\,(z \mathbf{C}\, x \rightarrow z \mathbf{C}\, y) \rightarrow x \leqslant y \qquad \text{(C5)}$$

A proof of (C5).

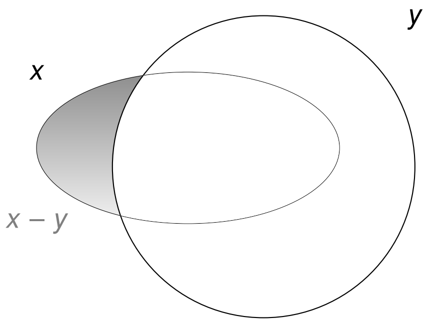


□

Grzegorczyk contact lattices

$$\forall_{z \in R} \,(z \,\mathbf{C}\, x \to z \,\mathbf{C}\, y) \to x \leqslant y \tag{C5}$$

A proof of (C5).

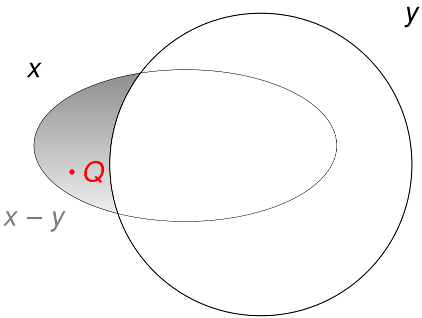


□

Grzegorczyk contact lattices

$$\forall_{z \in R} \, (z \; \mathbf{C} \; x \to z \; \mathbf{C} \; y) \to x \leqslant y \tag{C5}$$

A proof of (C5).



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Grzegorczyk contact lattices

$$\forall_{z \in R}\,(z \mathbf{C}\, x \rightarrow z \mathbf{C}\, y) \rightarrow x \leqslant y \qquad \text{(C5)}$$

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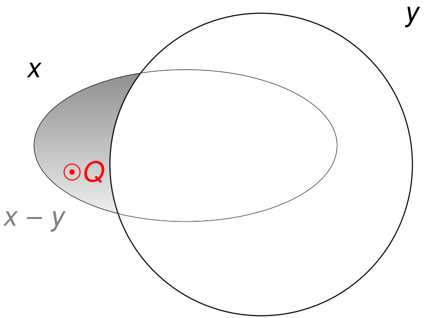


□

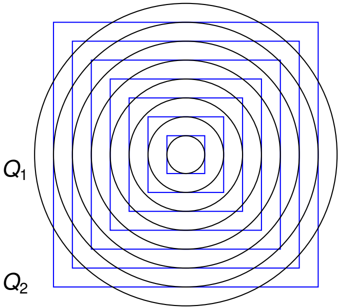
Outline

Grzegorczyk points and their properties

Points are (proper) filters generated by pre-points:

$$\mathfrak{p} \in \mathbf{Pt} \;:\longleftrightarrow\; \exists_{Q \in \mathbf{Q}}\, \mathfrak{p} = \left\{ x \in R \,\middle|\, \exists_{y \in Q}\, y \leqslant x \right\} . \qquad (\mathtt{df}\,\mathbf{Pt})$$

Points will be denoted by small Greek letters '$\mathfrak{p}$', '$\mathfrak{q}$', '$\mathfrak{r}$', '$\mathfrak{s}$'.



Grzegorczyk points and their properties

Definition (of round filters and ends)

- A filter $\mathscr{F}$ of GCL is a round (contracting, concordant) filter iff for every $x \in \mathscr{F}$ there is $y \in \mathscr{F}$ such that $y \ll x$.
- $\mathscr{F}$ is an end iff it is a maximal round filter.

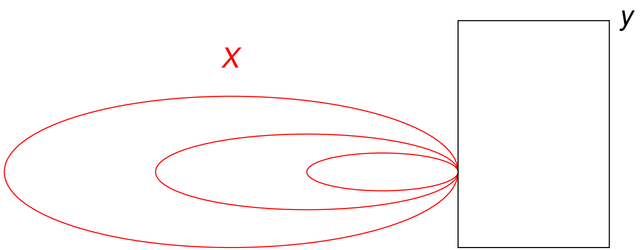
Grzegorczyk points and their properties

Theorem

Every Grzegorczyk point is an end (but not vice versa).

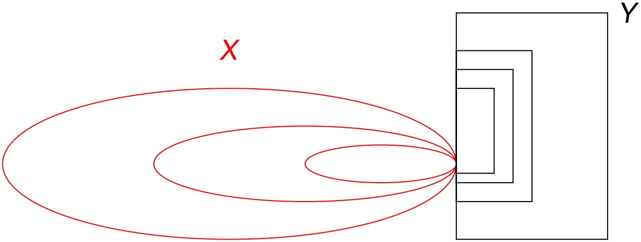
Grzegorczyk points and their properties

$$y \infty X \;:\longleftrightarrow\; \forall_{x\in X}\, y \,\mathbf{C}\, x \qquad (\mathtt{df}\,\infty)$$



Grzegorczyk points and their properties

$$X \infty Y \;:\longleftrightarrow\; \forall_{x \in X}\, x \infty Y \qquad (\mathtt{df}'\infty)$$



Grzegorczyk points and their properties

Lemma

If a round filter $\mathscr{F}$ satisfies the following condition:

$$x \infty \mathscr{F} \;\wedge\; x \ll y \to y \in \mathscr{F} \qquad (*)$$

then $\mathscr{F}$ is an end.

Proof.

- Suppose that a round filter $\mathscr{F}$ satisfies $(*)$ and let $\mathscr{F}'$ be any round filter such that $\mathscr{F} \subseteq \mathscr{F}'$.
- Notice that $\mathscr{F} \infty \mathscr{F}'$.
- Assume that $x \in \mathscr{F}'$. Then for some $x_0 \in \mathscr{F}'$ both $x_0 \ll x$ and $x_0 \infty \mathscr{F}$.
- Hence $x \in \mathscr{F}$, by $(*)$. □

Lemma

For any GCL:

$$x \ll y \longleftrightarrow \forall_{\mathfrak{p} \in \mathbf{Pt}} (y \in \mathfrak{p} \ \vee \ \exists_{z \in \mathfrak{p}} \, z \perp x) \, .$$

Proof.

- ($\to$) Let $x \ll y$. Assume for a contradiction that for some point $\mathfrak{p} \in \mathbf{Pt}$ we have (a) $y \notin \mathfrak{p}$ and $\forall_{z \in \mathfrak{p}} \, z \bigcirc x$.
- Hence (b) $\forall_{z \in \mathfrak{p}} \, z \nleqslant y$, and therefore (c) $\forall_{z \in \mathfrak{p}} \, z - y \not\mathbf{C} \, x$ (since $z - y \perp y$ and $x \ll y$).
- The point $\mathfrak{p}$ is generated by some $\mathbf{Q} \ni Q_{\mathfrak{p}} \subseteq \mathfrak{p}$.
- Thanks to (b) we have (d): $\forall_{u,v \in Q_{\mathfrak{p}}} \, u \bigcirc v - y$. Indeed, by (r1), either $v \leqslant u$ or $u \leqslant v$.
- In the first case: $v - y \leqslant v \leqslant u$.
- In the second case: $u - y \leqslant v - y$ and $u - y \leqslant u$; so $v - y \bigcirc u$.
- Since $Q_{\mathfrak{p}} \neq \emptyset$, we pick a member v_0 thereof. Thus, by (r3), (a) and (d), we have $v_0 - y \, \mathbf{C} \, x$, which contradicts (c). □

Grzegorczyk points and their properties

Lemma

For any GCL:

$$x \ll y \longleftrightarrow \forall_{\mathfrak{p} \in \mathbf{Pt}} (y \in \mathfrak{p} \ \vee \ \exists_{z \in \mathfrak{p}} \, z \perp x) \, .$$

Proof.

- ($\leftarrow$) Suppose that $x \not\ll y$, i.e., there is $u_0 \in R$ such that (a) $u_0 \perp y$ and (b) $u_0 \mathbf{C} x$.
- Then, by (b), there is $\mathfrak{p}_0 \in \mathbf{Pt}$ such that (c): $\forall_{z \in \mathfrak{p}_0} (z \bigcirc u_0 \wedge z \bigcirc x)$.
- Thus $y \notin \mathfrak{p}$, by (a) and (c). □

Grzegorczyk points and their properties

Theorem

Every Grzegorczyk point is an end.

Proof.

It is easy to see that every point is a round filter. By the previous lemma every point satisfies $(*)$:

$$x \infty \mathscr{F} \;\wedge\; x \ll y \to y \in \mathscr{F}\,.$$

So every $\mathfrak{p} \in \mathbf{Pt}$ is an end. □

Grzegorczyk points and their properties

Theorem

Not every end is a Grzegorczyk point.

Proof.

Take $\mathcal{S} := \{(n, +\infty) \mid n \in \mathbb{N}\}$, a family of open infinite segments in $\mathbb{R}$. We consider the contracting filter $\mathscr{F}_{\mathcal{S}}$ generated by $\mathcal{S}$ and its contracting maximal extension $\mathscr{F}^*_{\mathcal{S}}$. Notice that $\mathscr{F}^*_{\mathcal{S}}$ does not satisfy:

$$x \mathbin{\mathbf{C}} y \longleftrightarrow \forall_{z \in \mathfrak{p}} (z \bigcirc x \wedge z \bigcirc y),$$

so it is not a member of **Pt**. To see that, we define two open subsets of $\mathbb{R}$:

$$U := \operatorname{Int} \operatorname{Cl} \bigcup_{n \in \mathbb{N}} (4n, +\infty) \quad \text{and} \quad V := \operatorname{Int} \operatorname{Cl} \bigcup_{n \in \mathbb{N}} (4n + 2, +\infty).$$

We have $V \propto \mathscr{F}^*_{\mathcal{S}} \propto U$, yet $\operatorname{Cl} V \cap \operatorname{Cl} U = \emptyset$, i.e., $V \mathbin{\not\mathbf{C}} U$. □

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Existence of GCLs

Set theoretical representation theorems for GCLs

The characterization of finite GCLs

Existence of GCLs

$$\forall_{x,y\in Q}\,(x = y \vee x \ll y \vee y \ll x) \tag{r1}$$

$$\forall_{x\in \mathbf{Q}}\exists_{y\in \mathbf{Q}}\, y \ll x \tag{r2}$$

$$\forall_{x,y\in R}(\forall_{u\in Q}\,(u \bigcirc x \wedge u \bigcirc y) \rightarrow x \,\mathbf{C}\, y) \tag{r3}$$

Lemma

For any pre-contact lattice in which $\mathbf{C} = \bigcirc$: *if* $a \in \mathrm{At}$, *then* $\{a\} \in \mathbf{Q}$.

Proof.

- For any atom a the singleton $\{a\}$ trivially satisfies (r1).
- (r2) is satisfied since $\mathbf{C} = \bigcirc$ entails $\ll\, = \,\leqslant$.
- For (r3): If $a \bigcirc x$ and $a \bigcirc y$, then $a \leqslant x$ and $a \leqslant y$. Hence $x \bigcirc y$, i.e., $x \,\mathbf{C}\, y$. □

Existence of GCLs

Fact

Every atomic Grzegorczyk lattice in which $\mathbf{C} = \bigcirc$ *is a* GCL.

Proof.

- First, if $x \bigcirc y$, then $x \sqcap y \in R^+$ and there is $a \in \mathrm{At}$ such that $a \leqslant x \sqcap y$. But $\{a\} \in \mathbf{Q}$.
- Second, since $\mathbf{C} = \bigcirc$, the condition '$x \mathbf{C} y \wedge x \perp y$' is false for all $x, y \in R$. Hence $(\mathrm{G}_{\perp})$ also holds. □

Outline

Representation theorems for GCLs

Definition

- A representation of a GCL $\mathfrak{G}$ is an isomorphism ι from $\mathfrak{G}$ into a GCL whose domain is contained in $\mathcal{P}(\mathbf{Pt})$ (the power set of the set of Grzegorczyk points of $\mathfrak{G}$).
- A representation ι is reduced if the image $\iota[R]$ separates points of **Pt**: for any $\mathfrak{p} \neq \mathfrak{q} \in \mathbf{Pt}$ there is a region x such that $\mathfrak{p} \in \iota(x)$ but $\mathfrak{q} \notin \iota(x)$.
- A representation ι is perfect if for all $x \in R$ and $\mathfrak{p} \in \mathbf{Pt}$:

$$x \in \mathfrak{p} \longleftrightarrow \mathfrak{p} \in \iota(x)\,.$$

Representation theorems for GCLs

Definition

For any region x we define:

$$\mathbf{Irl}(x) := \{\mathfrak{p} \in \mathbf{Pt} \mid x \in \mathfrak{p}\}$$

the set of internal points of x.

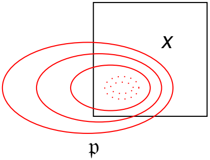


Figure: An internal point of the region x

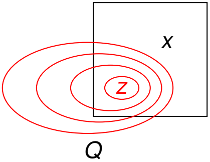
Representation theorems for GCLs

Fact

Every non-zero region has a point.

Proof.

Reflexivity of $\bigcirc$ gives $x \bigcirc x$, so by $(G_{\bigcirc})$ there is $Q \in \mathbf{Q}$ and $z \in Q$ such that $z \leqslant x$:



□

Representation theorems for GCLs

Lemma

The operation $\mathbf{Irl}\colon R \to \mathcal{P}(\mathbf{Pt})$ *has the following properties:*

$$\mathbf{Irl}(x) = \emptyset \longleftrightarrow x = 0$$

$$x \bigcirc y \longleftrightarrow \mathbf{Irl}(x) \cap \mathbf{Irl}(y) \neq \emptyset$$

$$\mathbf{Irl}(x \sqcap y) = \mathbf{Irl}(x) \cap \mathbf{Irl}(y)$$

$$x \leqslant y \longleftrightarrow \mathbf{Irl}(x) \subseteq \mathbf{Irl}(y)$$

$$x = y \longleftrightarrow \mathbf{Irl}(x) = \mathbf{Irl}(y)$$

Representation theorems for GCLs

Definition

For any region x we define:

$$\mathbf{Adh}(x) := \{\mathfrak{p} \in \mathbf{Pt} \mid x \infty \mathfrak{p}\}$$

the set of adherent points of x.

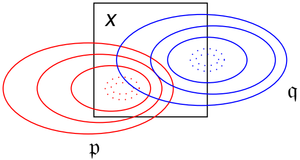


Figure: Both points $\mathfrak{p}$ and $\mathfrak{q}$ are adherent to the region x, but $\mathfrak{q}$ is not internal point of x

Representation theorems for GCLs

Basic properties of **Adh**: $R \to \mathcal{P}(\mathbf{Pt})$ operation:

$$\mathbf{Irl}(x) \subseteq \mathbf{Adh}(x)$$

$$\mathfrak{p} \in \mathbf{Adh}\, x \;:\longleftrightarrow\; \forall_{y \in \mathfrak{p}}\, y \bigcirc x \longleftrightarrow x \infty \mathfrak{p}$$

$$\mathbf{Adh}(x \sqcup y) = \mathbf{Adh}(x) \cup \mathbf{Adh}(y)$$

$$x = y \longleftrightarrow \mathbf{Adh}(x) = \mathbf{Adh}(y)$$

$$x \,\mathbf{C}\, y \longleftrightarrow \mathbf{Adh}(x) \cap \mathbf{Adh}(y) \neq \emptyset$$

Representation theorems for GCLs

Definition

Let $\mathfrak{G}_1 = \langle R_1, \leqslant_1, \mathbf{C}_1 \rangle$ and $\mathfrak{G}_2 = \langle R_2, \leqslant_2, \mathbf{C}_2 \rangle$ be relational structures with binary relations. A strong homomorphism from $\mathfrak{R}_1$ into $\mathfrak{R}_2$ is a map $h\colon R_1 \to R_2$ such that for all $x, y \in R_1$:

$$x \leqslant_1 y \longleftrightarrow h(x) \leqslant_2 h(y),$$
$$x \mathbf{C}_1 y \longleftrightarrow h(x) \mathbf{C}_2 h(y).$$

Lemma

If $\mathfrak{G}_1$ is a GCL and e is an embedding from $\mathfrak{G}_1$ into $\mathfrak{G}_2$, then $\left\langle e[R_1], \leqslant_2|_{e[R_1]}, \mathbf{C}_2|_{e[R_1]} \right\rangle$ is also a GCL.

Representation theorems for GCLs

The operation **Irl** is one-to-one, so in the family $\mathbf{Irl}[R]$ we can introduce the following binary relation:

$$X \mathrel{\mathbf{C}} Y \;:\longleftrightarrow\; \mathbf{Adh} \circ \mathbf{Irl}^{-1}(X) \cap \mathbf{Adh} \circ \mathbf{Irl}^{-1}(Y) \neq \emptyset \,. \qquad (\mathrm{df}\,\mathbf{C})$$

It means that for any $x, y \in R$ we have:

$$\begin{aligned} \mathbf{Irl}(x) \mathrel{\mathbf{C}} \mathbf{Irl}(y) &\longleftrightarrow \mathbf{Adh}(x) \cap \mathbf{Adh}(y) \neq \emptyset \\ &\longleftrightarrow x \mathrel{\mathbf{C}} y \,. \end{aligned}$$

Representation theorems for GCLs

Thus, for a Grzegorczyk contact lattice $\mathfrak{G}$, we can put $\mathfrak{G}_1 := \langle \mathbf{Irl}[R], \subseteq, \mathbf{C} \rangle$, about which holds the following:

Theorem

1. *The operation* **Irl** *is an isomorphism of* $\mathfrak{G}$ *onto* $\mathfrak{G}_1$*.*
2. $\mathfrak{G}_1$ *is a G-structure.*
3. *The operation* **Irl** *is a reduced and perfect representation of* $\mathfrak{G}$*.*
4. *If* $\mathfrak{G}$ *has the unity* 1*, then* $\mathfrak{G}_1$ *has the unity* **Pt** *and* $\mathbf{Irl}(1) = \mathbf{Pt}$*.*
5. $\mathfrak{G}$ *is complete iff* $\mathfrak{G}_1$ *is complete.*

Outline

The characterization of finite GCLs

Theorem

A GCL $\langle R, \leqslant, \mathbf{C} \rangle$ is finite iff $\langle R, \leqslant \rangle$ is a finite Grzegorczyk lattice and $\mathbf{C} = \bigcirc$.

Theorem

A GCL $\langle R, \leqslant, \mathbf{C} \rangle$ is finite iff it is complete and the set of its Grzegorczyk points coincides with the set of ultrafilters of $\langle R, \leqslant \rangle$.

The characterization of finite GCLs

We have already proved the following:

Lemma

For any pre-contact lattice in which $\mathbf{C} = \bigcirc$: *if* $a \in \mathrm{At}$, *then* $\{a\} \in \mathbf{Q}$.

And we can prove this:

Lemma

If a *is an atom of a GCL, then* $a \ll a$, *and* $\{a\} \in \mathbf{Q}$.

Proof.

It follows from Grzegorczyk's axioms that every region has non-tangential part. So there is x such that $x \ll a$. Thus $x \leqslant a$ and $x = a$. □

The characterization of finite GCLs

Corollary

In any GCL:

1. *For all* $a \in \mathrm{At}$ *and* $x \in R$:
   a) $a \not\mathbf{C}\, x$ *iff* $a \perp x$
   b) $a \,\mathbf{C}\, x$ *iff* $a \circ x$ *iff* $a \leqslant x$ *iff* $a \ll x$
   c) *if* $x \neq a$, *then* $a \not\mathbf{C}\, x - a$.
2. *For all atoms* $a \neq b$: $a \not\mathbf{C}\, b$.

Proof.

For every atom $a \ll a$, so if $x \in R$ and $x \perp a$, the definition of $\ll$ entails that $x \not\mathbf{C}\, a$. For the other implication: if $x \circ a$, then $x \,\mathbf{C}\, a$. □

The characterization of finite GCLs

Theorem

A GCL $\langle R, \leqslant, \mathbf{C}\rangle$ is finite iff $\langle R, \leqslant\rangle$ is a finite Grzegorczyk lattice and $\mathbf{C} = \bigcirc$.

Proof.

($\rightarrow$) If GCL is finite, then for any $x \in R$, x is the supremum of some set $\{a_1, \ldots, a_n\}$ of atoms. Thus if $x \mathbf{C} y$, $a_1 \sqcup \ldots \sqcup a_n \mathbf{C} y$ and the condition (C4) entails that for some $i \leqslant n$, $a_i \mathbf{C} y$, i.e. $a_i \leqslant y$. Thus $x \bigcirc y$.

($\leftarrow$) By assumption $\langle R, \leqslant\rangle$ is atomic, and earlier we proved that every atomic Grzegorczyk lattice in which $\mathbf{C} = \bigcirc$ is a GCL. □

The characterization of finite GCLs

Theorem

For every complete Grzegorczyk contact lattice $\mathfrak{G}$ the following conditions are equivalent:

1. $\mathfrak{G}$ *is finite*
2. **Pt** *is finite*
3. **Ult** $\subseteq$ **Pt**
4. **Ult** $=$ **Pt**.

Proof.

(1 $\leftrightarrow$ 2) If $\mathfrak{G}$ is not finite, it must have an infinite anti-chain A. Every region $x \in A$ has some point $\mathfrak{p}_x$, and if $x \neq y$, then $\mathfrak{p}_x \neq \mathfrak{p}_y$.

(1 $\rightarrow$ 3) If $\mathfrak{G}$ is finite, then every ultrafilter $\mathscr{U}$ is generated by an atom, and so it must be a point.

(3 $\rightarrow$ 4) If $\mathfrak{p} \in \mathbf{Pt}$, then it is a filter, so there is an ultrafilter $\mathscr{U} \supseteq \mathfrak{p}$. But $\mathscr{U}$ is a point by an assumption, so $\mathfrak{p} = \mathscr{U}$. □

The characterization of finite GCLs

Theorem

For every complete Grzegorczyk contact lattice $\mathfrak{G}$ the following conditions are equivalent:

1. $\mathfrak{G}$ *is finite*
2. **Pt** *is finite*
3. **Ult** $\subseteq$ **Pt**
4. **Ult** $=$ **Pt**.

Proof.

(4 $\to$ 2) If **Ult** $=$ **Pt**, then every ultrafilter of $\mathfrak{G}$ is generated by a chain. So every ultrafilter is principal, and therefore $\mathfrak{G}$ is finite (since every infinite and complete GCL has a free ultrafilter). □

The characterization of finite GCLs

Theorem

A GCL $\langle R, \leqslant, \mathbf{C} \rangle$ is finite iff it is complete and the set of its Grzegorczyk points coincides with the set of ultrafilters of $\langle R, \leqslant \rangle$.

Corollary

If a GCL is finite, then the set of its Grzegorczyk points coincides with the set of ends.

Thank you