Frege's Basic Law V via Partial Orders

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SOL

A standard second-order structure is a sequence:

$$\mathfrak{S} = \langle A, A^*, c^*, R^* \rangle,$$

wherein:

- $A^* = \langle A_n | n \in N \rangle$;
- $c^* = \{c_i | i \in N\} \subseteq A$;
- $R = \langle R_i^n | i, n \in N \rangle$ and $A_n \subseteq \wp(A^n)$, $R_i^n \in A_n$.

Roughly speaking, a second-order structure consists of a universe A of individuals, a second-order universe for n-ary relations, for $n \geq \emptyset$ and individual constants.

Remark. In the case that $A_n = \wp(A^n)$, i.e. A_n contains all n-ary relations, we call $\mathfrak{S}$ full.

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BLV

$$\text{BLV}: \quad \forall F \forall G[\epsilon Fx = \epsilon Gx \longleftrightarrow \forall x(Fx \leftrightarrow Gx)].$$

Basic law V axiomatizes the behavior of a type-lowering operator (ϵ), from the second-order entities to first-order individuals. ϵ is called *extension operator*. Indeed, BLV postulates that this operator is an injective fuction.

ϵ takes a second-order entitie F as argument and returns an object ϵF.

Remark. ϵFx as $\{x : Fx\}$.

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Standard model

Models for BLV have the following form:

$$\mathcal{M} = (M, S_1(M), S_2(M), \ldots, \pi),$$

wherein:

- $M \neq \emptyset$ serves for the interpretation of the first-order individuals;
- $S_n(M) \subseteq \wp(M^n)$ serves for the interpretation of second-order n-ary predicates;
- $\pi : S_1(M) \to M$ is an injection.

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Syntax

- Standard SOL with $A_n = \wp(A_n)$;
- A sort of first order variables, $x, y, z, \ldots$ and a sort of second-order variables, $F, G, H, \ldots$;
- Unary function symbol ϵ.

Semantics

Let $\vartheta(x)$ be a metavariable for any second-order variable with at most one free variable, $\mathcal{M}_1$ the first-order domain and $\mathcal{M}_2 = \wp(\mathcal{M}_1)$ the second-order domain.

- $\mathcal{E}(\vartheta(x)) \subseteq \mathcal{M}_1$;
- *Remark.* $\mathcal{E}(\vartheta)$ is the set that is specified by ϑ.

- $\mathcal{A}(\vartheta(x)) := \mathcal{M}_1 - \mathcal{E}(\vartheta(x))$, with $\mathcal{E} \cap \mathcal{A} = \emptyset$ and $\mathcal{E} \cup \mathcal{A} = \mathcal{M}_1$;
- ϵ is interpreted by the function $\pi : \mathcal{M}_2 \to \mathcal{M}_1$;
- $\mathfrak{A} \models \forall F^n(Fx)$ if $\mathfrak{A} \models F^n x$ for all $F^n \in \mathcal{M}_2$ and $\mathfrak{A} \models \exists X(Xx)$ if $\mathfrak{A} \models X^n x$ for some $X^n \in \mathcal{M}_2$.

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Hierarchy of interpretations

Definition (*Hierarchy of Interpretations*)

- $\mathcal{S}_{\emptyset}$: $\mathcal{M}_1 = \emptyset$, namely, $(\mathcal{E}) = \emptyset$;
- $\mathcal{S}_{n+1}$: $\mathfrak{A} \models \vartheta$, for any $x \in \mathcal{E}(\vartheta)$;
- $\mathcal{S}_{\sigma}$: $\bigcup\limits_{\lambda<\sigma} \mathcal{E}_{\lambda}$.

Remark. Only at the limit stage of this hierarchy, $\mathcal{E}(\vartheta)$ will be fixed, namely, $\mathcal{E}(\vartheta)$ is in $\mathcal{M}_2$.

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Poset

Definition (*Poset*)

Let $\mathcal{M} = \langle \mathcal{D}, \subseteq \rangle$ be a poset where $\mathcal{D} = \wp(\omega)$ and $\subseteq$ is a relation, reflexive, antisymmetric, and transitive over $\mathcal{D}$.

By poset properties is possibile to define a function ϕ over $\mathcal{M}$ such that:

Definition (*Monotonicity*)

Let ϕ an unary-function and $\mathcal{D}$ a domain, if $\forall x, y$ such that $x \leq y$ then $\phi(x) \leq \phi(y)$, where ϕ is ordered preserving, ϕ is called *monotone*.

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Monotonicity

Lemma

The Hierarchy of interpretation is a non decreasing sequence.

Proof.

By transfinite induction on α:

- $\alpha = \emptyset$: $(\mathcal{E}) = \emptyset$;
- $\alpha = n + 1$: $(\mathcal{E}_{n+1})$ *extends the interpretation of* $(\mathcal{E}_n)$: *if* $(\mathcal{E}_n) \leq (\mathcal{E}_{n+1})$, *by monotonicity, then* $(\mathcal{E}_n) \leq \phi((\mathcal{E}_{n+1}))$.
- $\alpha = \sigma$ *with* σ *limit, I have* $\mathcal{E}_\sigma = \mathcal{E}_{\sigma+1}$; *by monotonicity,* $(\mathcal{E}_\sigma) = \phi(\mathcal{E}_{\sigma+1})$, *i.e.* $\mathcal{E}_\sigma = \bigcup_{\lambda<\sigma} \mathcal{E}_\lambda = (\mathcal{E}_{\sigma+1})$. *According to definition 1,* $\phi(\mathcal{E}_{\sigma+1}) = \phi(\mathcal{E}_\sigma)$.

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Least fixed point

Theorem

ϕ has least fixed point.

Proof.

Every monotone mapping $\nu : \mathcal{D} \to \mathcal{D}$ on an partially-ordered set has a unique least fixed point, i.e. for some $x \in \mathcal{D}$, $\nu(x) = x$. Since ϕ is a monotone function from $\wp(\omega) \to \wp(\omega)$ and that $\wp(\omega)$ is a chain-complete poset, i.e. every chain in $\mathcal{D}$ has least upper bound, ϕ has least fixed point. □

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Posets and BLV

- At the least fixed point level, there will not be new interpretation of $\vartheta(x)$, namely, his extension will be fixed in $\mathcal{M}_2$ and the application of the extension operator ϵ to it delivers an ordered first-order individual.
- Existence of a least element in $\mathcal{M}$;
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Theorem

$\mathcal{E}(x \neq x)$ is in the least fixed point of ϕ.

Proof.

The proof is given by contradiction. Let me assume that $\mathcal{E}(x \neq x)$ is not in the least fixed point of ϕ. Then, according to definition 1 and lemma 4, $\mathcal{E}(x \neq x)$ has no fixed extension, his extension increases. However, under $\mathcal{E}(x \neq x)$ no objects ever falls, so $\mathcal{E}(x \neq x)$ is alway empty. Thus, at the least fixed point level I have that $\phi(\mathcal{E}_{\sigma+1}(x \neq x)) = (\mathcal{E}_{\sigma}(x \neq x))$, namely, ϵ delivers from $\mathcal{M}_2$ the individual $\epsilon(x \neq x)$ to $\mathcal{M}_1$. □

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Least element

Claim

$\emptyset$ is the least element of $\mathcal{M}$, $\perp$.

Proof.

The object $(x \neq x)$ does not contain elements. □

Claim

$\{\perp\}$ is the simplest non empty poset. Moreover, $\{\perp\}$ is both discrete and flat.

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$\emptyset$ is the least element of $\mathcal{M}$, $\bot$.

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Upper bound?

Theorem

$\mathcal{E}(x = x)$ is not in the least fixed point of ϕ.

Proof.

The proof is given by contradiction. Let me assume that $\mathcal{E}(x = x)$ is in the least fixed point of ϕ. Thus, there must be a corresponding VR-term $\epsilon(x = x) \in M_I$. Since that it is true, then $\epsilon(x = x)$ is a new VR-term in M_I for which the concept $x = x$ has not yet been evaluated. But if $\mathcal{E}(x = x)$ was in the least fixed point of ϕ, $\mathcal{E}(x = x)$ should have specified all instances of $\mathcal{T}_\omega$. However, $\epsilon(x = x)$ is not in such set. Thus, $\mathcal{E}(x = x)$ is not in the least point of ϕ and it is in the non ordered portion of M_I. □

Upper bound?

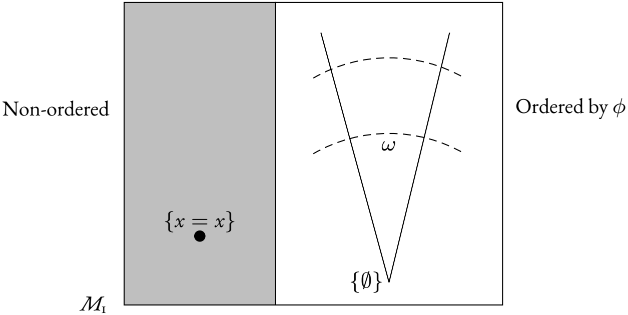
Theorem

$\mathcal{E}(x = x)$ is not in the least fixed point of ϕ.

Proof.

The proof is given by contradiction. Let me assume that $\mathcal{E}(x = x)$ is in the least fixed point of ϕ. Thus, there must be a corresponding VR-term $\epsilon(x = x) \in M_1$. Since that it is true, then $\epsilon(x = x)$ is a new VR-term in M_1 for which the concept $x = x$ has not yet been evaluated. But if $\mathcal{E}(x = x)$ was in the least fixed point of ϕ, $\mathcal{E}(x = x)$ should have specified all instances of $\mathcal{T}_\omega$. However, $\epsilon(x = x)$ is not in such set. Thus, $\mathcal{E}(x = x)$ is not in the least point of ϕ and it is in the non ordered portion of M_1. □

Figure: The first-order domain



Well and non well-ordered

- There are instances that works not in agreement with ϕ and then other first-order individuals that works in a non well-ordered way.

- There are first-order individuals ordered by the function ϕ and then they work in an iterative way because they are well ordered by ϕ and well founded by $\{\emptyset\}$.

A characterisation of $\mathcal{M}$

Definition (Product Order)

Given two poset M and N, the product order is a partial ordering on the cartesian product $M \times N$.

Thus, given two pairs (m_1, n_1) and $(m_1, n_1) + 1$ in a $\omega \times \omega$ sequence, $(m_1, n_1) \subseteq (m_1, n_1) + 1 \Leftrightarrow m_1 \subseteq m_1 + 1 \wedge n_1 \subseteq n_1 + 1$.

Generally, given a set $\mathcal{M}$, a product order on the Cartesian Product $\prod_{\mathcal{M}} \{1, 0\}$ is the inclusion ordering of subsets of $\mathcal{M}$.

Definition (Pairing function)

Let $f(m, n)$ and $g(m, n)$ be some pairing function. I define: $f_0(m, n) = 2 \times f(m, n)$ and $g(m, n) = 4 \times f(m, n) + 1$.

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Definition (Product Order)

Given two poset M and N, the product order is a partial ordering on the cartesian product $M \times N$.

Thus, given two pairs (m_1, n_1) and $(m_1, n_1) + 1$ in a $\omega \times \omega$ sequence, $(m_1, n_1) \subseteq (m_1, n_1) + 1 \Leftrightarrow m_1 \subseteq m_1 + 1 \wedge n_1 \subseteq n_1 + 1$.
Generally, given a set $\mathcal{M}$, a product order on the Cartesian Product $\prod_{\mathcal{M}} \{1, 0\}$ is the inclusion ordering of subsets of $\mathcal{M}$.

Definition (Pairing function)

Let $f(m, n)$ and $g(m, n)$ be some pairing function. I define:
$f_0(m, n) = 2 \times f(m, n)$ and $g(m, n) = 4 \times f(m, n) + 1$.

Model

Corollary ($\mathfrak{A}$)

The former structure is a smallest model for the theory: the triple be $\langle \mathcal{M}, \omega, \pi \rangle$ be a model $\mathfrak{A}$ wherein, $\mathcal{M} = \langle \mathcal{D}, \subseteq \rangle$ is the above mentioned poset; ω is the cardinality of $\mathfrak{A}$ and π is an interpretation for the extension operator.

- $\mathcal{M}$ is well ordered by $\{x \neq x\}$ that denotes the least element $\bot$ of $\mathcal{M}$
- Symmetrically, $\{\omega \times \omega\}$ denotes the upper bound $\top$, $M \in \mathcal{M} \wedge \forall x \in \mathcal{M}[x \leq M]$ with $M = \top$.

Model

Corollary ($\mathfrak{A}$)

The former structure is a smallest model for the theory: the triple be $\langle \mathcal{M}, \omega, \pi \rangle$ be a model $\mathfrak{A}$ wherein, $\mathcal{M} = \langle \mathcal{D}, \subseteq \rangle$ is the above mentioned poset; ω is the cardinality of $\mathfrak{A}$ and π is an interpretation for the extension operator.

- $\mathcal{M}$ is well ordered by $\{x \neq x\}$ that denotes the least element $\perp$ of $\mathcal{M}$
- Symmetrically, $\{\omega \times \omega\}$ denotes the upper bound $\top$, $M \in \mathcal{M} \wedge \forall x \in \mathcal{M}[x \leq M]$ with $M = \top$.

Model

Corollary ($\mathfrak{A}$)

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Thank You!

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