On the structure of finite (commutative) idempotent involutive residuated lattices

Peter Jipsen, Olim Tuyt, Diego Valota

TACL 2019, June 17th

Overview

- Introduce CIdInRLs
- Examples
- Some properties
- Constructions
- Conclusion and remarks

Definition

Definition

A *(pointed) commutative residuated lattice* is a tuple
$\mathbf{L} = \langle L, \wedge, \vee, \cdot, \to, 1, 0 \rangle$ such that

- $\langle L, \wedge, \vee \rangle$ is a lattice;
- $\langle L, \cdot, 1 \rangle$ is a commutative monoid;
- $\cdot$ is residuated with residual $\to$, i.e. for $a, b, c \in L$,

$$a \cdot b \leq c \qquad \Longleftrightarrow \qquad a \leq b \to c.$$

We call $\mathbf{L}$

- *idempotent* if $a \cdot a = a$ for all $a \in L$;
- *involutive* if $--a = a$, where $-a := a \to 0$ for all $a \in L$.

Let CIdInRL denote the variety of commutative idempotent involutive residuated lattices.

Some properties

Let $\mathbf{L} \in \mathsf{CIdInRL}$.

- The residual is expressible, i.e. for $a, b \in L$,

$$a \to b = -(-b \cdot a).$$

- $\langle L, \cdot, 1 \rangle$ is a meet-semilattice with top element 1 and order $\sqsubseteq$ (*monoidal order*) defined as

$$a \sqsubseteq b \quad :\Longleftrightarrow \quad a \cdot b = a.$$

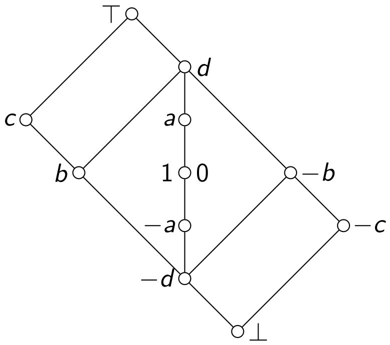
Hence, the orders $\leq$ and $\sqsubseteq$ together with the involution $-$ completely determine $\mathbf{L}$.

Examples I

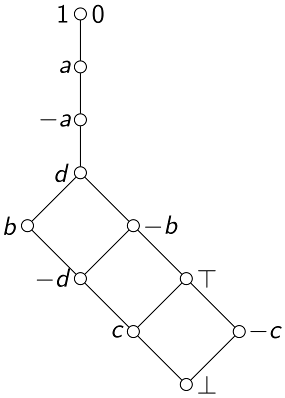
- Boolean algebras (where $\leq = \sqsubseteq$)
- Sugihara monoids (algebraic semantics for relevance logic RM^{t})

Examples I

- Boolean algebras (where $\leq = \sqsubseteq$)
- Sugihara monoids (algebraic semantics for relevance logic $\mathrm{RM^t}$)



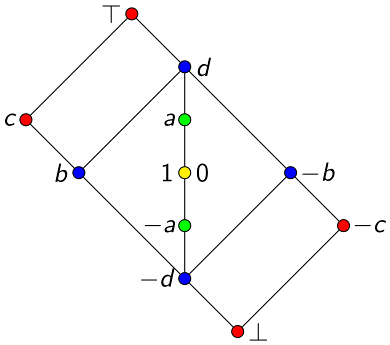
$\langle L, \leq \rangle$



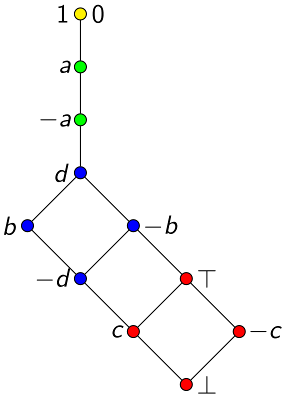
$\langle L, \sqsubseteq \rangle$

Examples I

- Boolean algebras (where $\leq = \sqsubseteq$)
- Sugihara monoids (algebraic semantics for relevance logic RM^{t})



$\langle L, \leq \rangle$



$\langle L, \sqsubseteq \rangle$

Some more properties

For each $a \in L$, define

$$\bot_a := a \cdot -a = a \wedge -a$$
$$\top_a := a \vee -a = -(a \cdot -a) = a \to a.$$

Moreover, for $a, b \in L$, let

$$[a, b]_{\sqsubseteq} := \{c \in L \mid a \sqsubseteq c \sqsubseteq b\} \text{ and } [a, b]_{\leq} := \{c \in L \mid a \leq c \leq b\}.$$

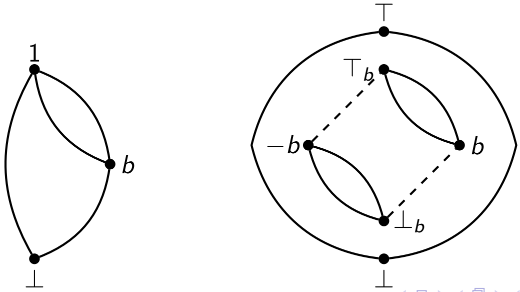
Lemma

- *For each $a \in L$, $\langle [\bot_a, \top_a]_{\sqsubseteq}, \cdot, \vee, -, \bot_a, \top_a \rangle$ is a Boolean algebra;*
- *The intervals $[\bot_a, \top_a]_{\sqsubseteq}$ partition L;*
- *$\langle \{\bot_a \mid a \in A\}, \cdot, \vee, 0 \rangle$ is a distributive lattice with top element 0.*

Hence, the monoidal semilattice consists of Boolean algebras in the 'skeleton' of a distributive lattice.

Goal: Give a structural characterization of all finite members of CIdInRL.

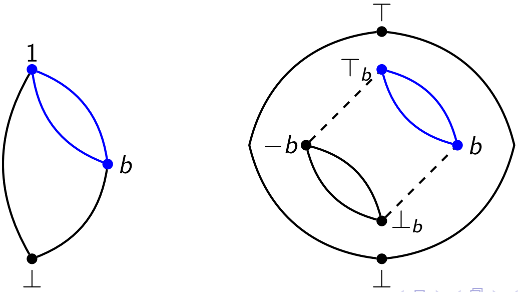
Idea: "Consider a principal $\sqsubseteq$-upset generated by $b \leq 1$. Construct a new member of our variety by adding a copy of this upset on top in the monoidal order and in the middle of the lattice order."



Goal: Give a structural characterization of all finite members of CIdInRL.

Idea: "Consider a principal $\sqsubseteq$-upset generated by $b \leq 1$. Construct a new member of our variety by adding a copy of this upset on top in the monoidal order and in the middle of the lattice order."

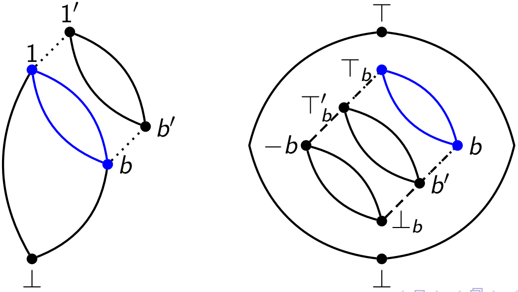
Crucial observation: for all $b \in L$, $[b, 1]_{\sqsubseteq} = [b, \top_b]_{\leq}$.



Goal: Give a structural characterization of all finite members of CIdInRL.

Idea: "Consider a principal $\sqsubseteq$-upset generated by $b \leq 1$. Construct a new member of our variety by adding a copy of this upset on top in the monoidal order and in the middle of the lattice order."

Crucial observation: for all $b \in L$, $[b, 1]_{\sqsubseteq} = [b, \top_b]_{\leq}$.

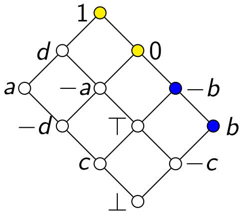


Constructions II

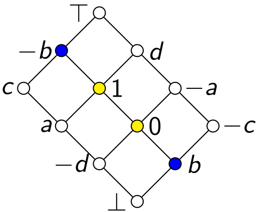
More specifically, we give two constructions that construct new members of the variety. The two constructions consider a $\mathbf{L} \in$ CIdInRL and an element $b \in L$ such that $b \leq 1$.

- If $b \leq 0$, construct a new member $\delta(\mathbf{L}, b)$ through a *doubling construction* (reminiscent of Day's doubling construction for lattices);
- If $b \not\leq 0$, construct a new member $\gamma(\mathbf{L}, b)$ through a *gluing construction*.

Doubling construction I ($b \leq 0$)



$\langle L, \sqsubseteq \rangle$

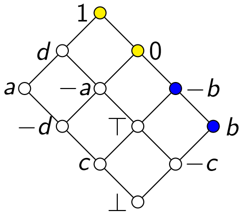


$\langle L, \leq \rangle$

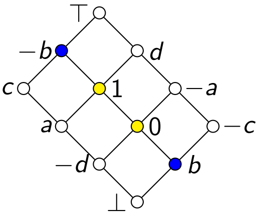
$\Downarrow$

$\langle \delta(\mathbf{L}, b), \sqsubseteq^{\delta} \rangle$

$\langle \delta(\mathbf{L}, b), \leq^{\delta} \rangle$

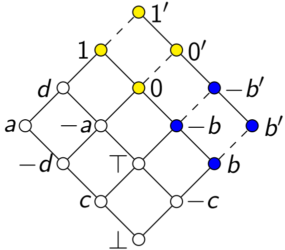


$\langle L, \sqsubseteq \rangle$



$\langle L, \leq \rangle$

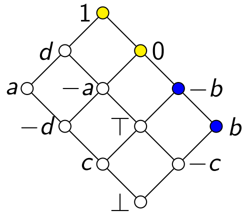
$\Downarrow$



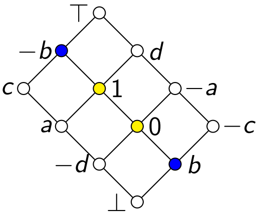
$\langle \delta(\mathbf{L}, b), \sqsubseteq^{\delta} \rangle$

$\langle \delta(\mathbf{L}, b), \leq^{\delta} \rangle$

Doubling construction I ($b \leq 0$)

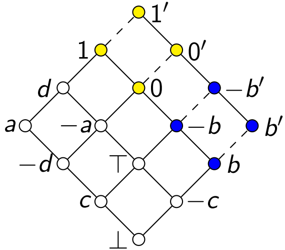


$\langle L, \sqsubseteq \rangle$

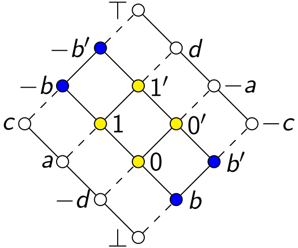


$\langle L, \leq \rangle$

$\Downarrow$

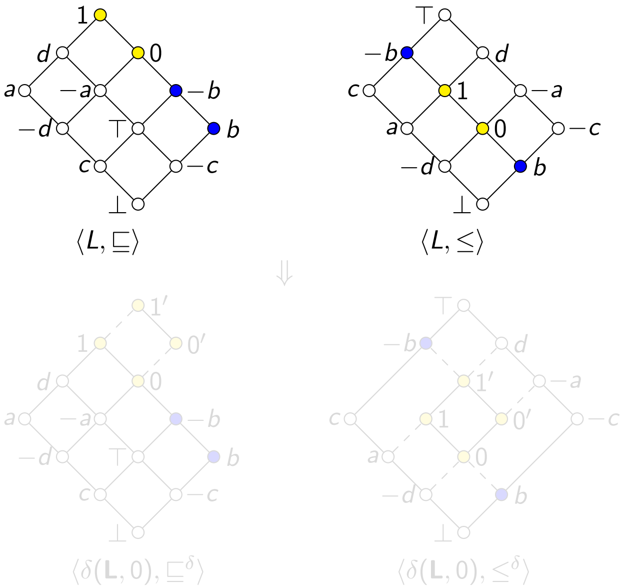


$\langle \delta(\mathbf{L}, b), \sqsubseteq^{\delta} \rangle$

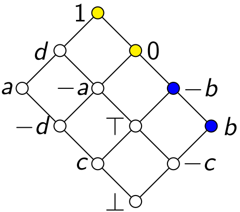


$\langle \delta(\mathbf{L}, b), \leq^{\delta} \rangle$

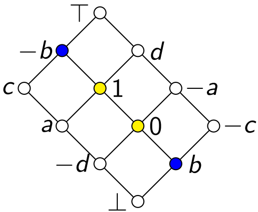
Doubling construction II ($b \leq 0$)



Doubling construction II ($b \leq 0$)

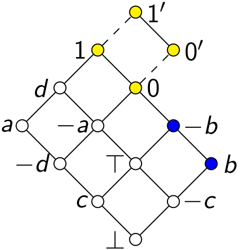


$\langle L, \sqsubseteq \rangle$

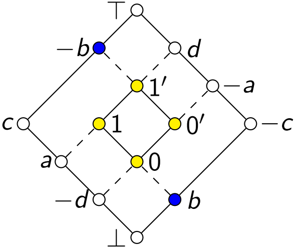


$\langle L, \leq \rangle$

$\Downarrow$

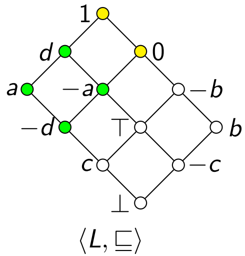


$\langle \delta(\mathbf{L}, 0), \sqsubseteq^{\delta} \rangle$



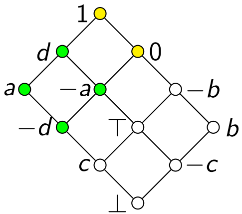
$\langle \delta(\mathbf{L}, 0), \leq^{\delta} \rangle$

Gluing construction ($b \not\leq 0$)

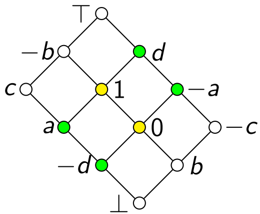


$\Downarrow$

Gluing construction ($b \not\leq 0$)

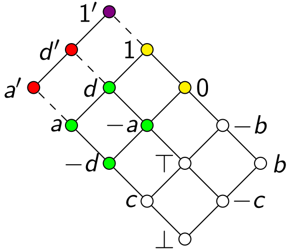


$\langle L, \sqsubseteq \rangle$



$\langle L, \leq \rangle$

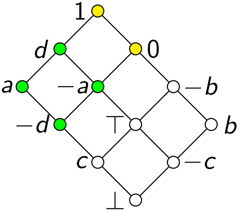
$\Downarrow$



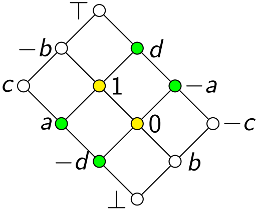
$\langle \gamma(\mathbf{L}, a), \sqsubseteq^{\gamma} \rangle$

$\langle \gamma(\mathbf{L}, a), \leq^{\gamma} \rangle$

Gluing construction ($b \not\leq 0$)

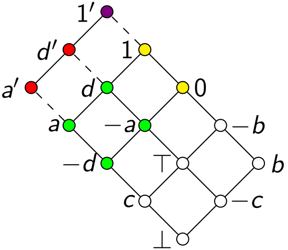


$\langle L, \sqsubseteq \rangle$

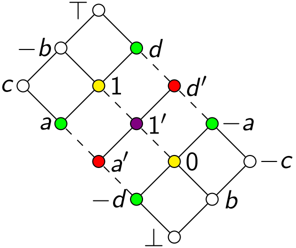


$\langle L, \leq \rangle$

$\Downarrow$



$\langle \gamma(\mathbf{L}, a), \sqsubseteq^{\gamma} \rangle$



$\langle \gamma(\mathbf{L}, a), \leq^{\gamma} \rangle$

Conclusions

- We have seen:
  - each CIdInRL can be partitioned into Boolean algebras
  - Two constructions to construct new CIdInRLs: doubling construction and gluing construction
- We have not seen: the desired structural characterization

Conjecture: The doubling and gluing constructions suffice to characterize all finite CIdInRLs starting from the trivial algebra.

- Many other questions to answer:
  - Is $\langle L, \cdot, 1 \rangle$ distributive as a meet-semilattice?
  - What about non-commutative IdInRLs?
  - ...

Conclusions

- We have seen:
  - each CIdInRL can be partitioned into Boolean algebras
  - Two constructions to construct new CIdInRLs: doubling construction and gluing construction
- We have not seen: the desired structural characterization

Conjecture: The doubling and gluing constructions suffice to characterize all finite CIdInRLs starting from the trivial algebra.

- Many other questions to answer:
  - Is $\langle L, \cdot, 1 \rangle$ distributive as a meet-semilattice?
  - What about non-commutative IdInRLs?
  - ...

Conclusions

- We have seen:
  - each CIdInRL can be partitioned into Boolean algebras
  - Two constructions to construct new CIdInRLs: doubling construction and gluing construction
- We have not seen: the desired structural characterization

Conjecture: The doubling and gluing constructions suffice to characterize all finite CIdInRLs starting from the trivial algebra.

- Many other questions to answer:
  - Is $\langle L, \cdot, 1 \rangle$ distributive as a meet-semilattice?
  - What about non-commutative IdInRLs?
  - ...

Conclusions

- We have seen:
  - each CIdInRL can be partitioned into Boolean algebras
  - Two constructions to construct new CIdInRLs: doubling construction and gluing construction
- We have not seen: the desired structural characterization

Conjecture: The doubling and gluing constructions suffice to characterize all finite CIdInRLs starting from the trivial algebra.

- Many other questions to answer:
  - Is $\langle L, \cdot, 1 \rangle$ distributive as a meet-semilattice?
  - What about non-commutative IdInRLs?
  - ...

References

[1] Alan Ross Anderson and Nuel D. Belnap, Jr. *Entailment. Volume I: The logic of relevance and necessity*. Princeton University Press, Princeton, N. J.-London, 1975.

[2] Alan Day. A simple solution to the word problem for lattices. *Canadian Mathematical Bulletin*, 13(2):253–254, 1970.

[3] J. Gil-Férez, P. Jipsen, and G. Metcalfe. Structure theorems for idempotent residuated lattices. *preprint*.

[4] W. McCune. Prover9 and Mace4. `http://www.cs.unm.edu/~mccune/prover9/`, 2005–2010.

[5] J. G. Raftery. Representable idempotent commutative residuated lattices. *Trans. Amer. Math. Soc.*, 359(9):4405–4427, 2007.

Thank you!