Epimorphisms in Varieties of Heyting Algebras[1]

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[1] Opinions expressed and conclusions arrived at are those of the second author and are not necessarily to be attributed to the CoE-MaSS.

Heyting algebras

A *Heyting algebra* $\mathbf{A} = \langle A; \wedge, \vee, \rightarrow, 1, 0\rangle$ is a distributive lattice with bounds 1 and 0 which satisfies

$$x \wedge y \leq z \text{ iff } x \leq y \rightarrow z.$$

Heyting algebras are fully determined by their lattice reducts, because

$$y \rightarrow z = \bigvee\{x : x \wedge y \leq z\}.$$

Thus any *finite* distributive lattice is a Heyting algebra.

Varieties of Heyting algebras algebraize axiomatic extensions of intuitionistic logic (i.e., intermediate logics).

Esakia Duality

An *Esakia space* $\mathbf{X} = \langle X; \tau, \leq \rangle$, is a compact Hausdorff space with topology τ on X and a partial order $\leq$ such that

1. ${\uparrow}x$ is closed for all $x \in X$, and
2. ${\downarrow}\mathcal{U}$ is clopen for every clopen $\mathcal{U} \subseteq X$,

where ${\uparrow}x := \{z \in X : z \geq x\}$ and
${\downarrow}\mathcal{U} := \{z \in X : z \leq y \text{ for some } y \in \mathcal{U}\}$, and ${\downarrow}x$ and ${\uparrow}\mathcal{U}$ are defined similarly.

An *Esakia morphism* $f : \mathbf{X} \to \mathbf{Y}$ between Esakia spaces is a continuous order-preserving map such that for every $x \in X$,

$$\text{if } f(x) \leq y \in Y, \text{ then } y = f(z) \text{ for some } z \geq x.$$

Duality

The category of Heyting algebras (with algebraic homomorphisms) is dually equivalent to the category of Esakia spaces (with Esakia morphisms), as witnessed by the covariant functors $(-)_*$ and $(-)^*$ which we now define.

$(-)_*$:

Let $\mathrm{Pr}\,\mathbf{A}$ denote the set of (non-empty, proper) prime filters of a Heyting algebra $\mathbf{A}$. Define, for every $a \in A$,

$$\gamma^{\mathbf{A}}(a) := \{F \in \mathrm{Pr}\,\mathbf{A} : a \in F\}.$$

The structure $\mathbf{A}_* := \langle \mathrm{Pr}\,\mathbf{A}; \tau, \subseteq \rangle$ is an Esakia space, where the topology τ has subbasis $\{\gamma^{\mathbf{A}}(a) : a \in A\} \cup \{\gamma^{\mathbf{A}}(a)^c : a \in A\}$. Furthermore, for every homomorphism $f : \mathbf{A} \to \mathbf{B}$, the map $f_* : \mathbf{B}_* \to \mathbf{A}_*$ is defined by $F \mapsto f^{-1}[F]$.

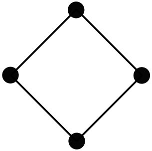
For a variety K of Heyting algebras, let $\mathsf{K}_* := \{\mathbf{A}_* : \mathbf{A} \in \mathsf{K}\}$.

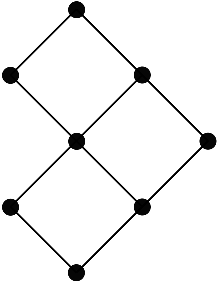
$(-)^*$:

Given an Esakia space $\mathbf{X}$, we let $\mathrm{Cu}\,\mathbf{X}$ denote the set of clopen up-sets of $\mathbf{X}$. Then the structure $\mathbf{X}^* := \langle \mathrm{Cu}\,\mathbf{X}; \cap, \cup, \rightarrow, \emptyset, X \rangle$ is a Heyting algebra, where $\mathcal{U} \rightarrow \mathcal{V} := X \setminus {\downarrow}(\mathcal{U} \setminus \mathcal{V})$. For every Esakia morphism $f : \mathbf{X} \rightarrow \mathbf{Y}$, we similarly define $f^* : \mathbf{Y}^* \rightarrow \mathbf{X}^*$ to be $\mathcal{U} \mapsto f^{-1}[\mathcal{U}]$.

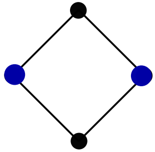
Note The topology of finite Esakia spaces is discrete (and can therefore be ignored).

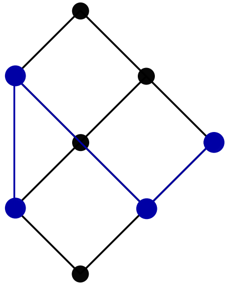
Example

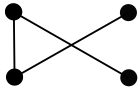




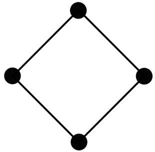
Example





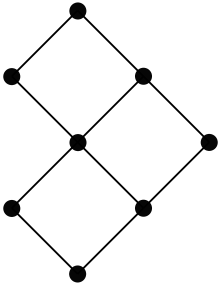


Example

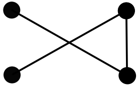


$\mathbf{D}_2{}^*$

$\mathbf{D}_2$



$\mathbf{X}_2{}^*$



$\mathbf{X}_2$

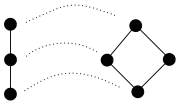
Epimorphisms

Let K be a variety of algebras and $\mathbf{A}, \mathbf{B} \in \mathsf{K}$. A homomorphism $f : \mathbf{A} \to \mathbf{B}$ is an *epimorphism* if, whenever $\mathbf{C} \in \mathsf{K}$ and $g, h : \mathbf{B} \to \mathbf{C}$ are homomorphisms,

$$\text{if } g \circ f = h \circ f, \text{ then } g = h.$$

All surjective homomorphisms are epimorphisms, but, the converse need not be true.

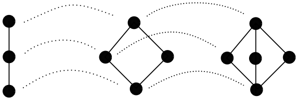
Example The embedding of the 3-element chain into the 4-element diamond is a (non-surjective) epimorphism in the variety of distributive lattices. This reflects the fact that lattice complements are *implicitly* defined (i.e., unique or non-existent) for distributive lattices, even though there is no unary term that defines them *explicitly*.



We say K has the *epimorphism surjectivity (ES) property* if all its epimorphisms are surjective.

The ES property is not in general inherited by subvarieties, because a (non-surjective) homomorphism in a variety may become an epimorphism in a subvariety.

Example The variety of all lattices has the ES property but the variety of distributive lattices does not. The embedding above is not an epimorphism in the variety of lattices, because the above diagram extends in two distinct ways to $\mathbf{M}_3$.



Equivalent conditions

Let K be a variety of Heyting algebras that algebraizes an intermediate logic $\vdash$. The following are equivalent:

- K has surjective epimorphisms.
- $\vdash$ satisfies the *infinite Beth property*, i.e., all *implicit* definitions of propositional functions in $\vdash$ can be made *explicit*.

We shall investigate the ES property for varieties of Heyting algebras, and by implication, the infinite Beth property for intermediate logics.

A subalgebra $\mathbf{A} \leq \mathbf{B} \in \mathsf{K}$ is *epic* if the inclusion map $\mathbf{A} \hookrightarrow \mathbf{B}$ is an epimorphism, i.e., homomorphisms from $\mathbf{B}$ to members of K are determined by their restrictions to A.

A *correct partition* R on an Esakia space $\mathbf{X}$ is a equivalence relation on X such that for every $x, y, z \in X$

1. if $\langle x, y \rangle \in R$ and $x \leq z$, then $\langle z, w \rangle \in R$ for some $w \geq y$, and
2. if $\langle x, y \rangle \notin R$, then there is a clopen $\mathcal{U}$ which is a union of equivalence classes of R, such that $x \in \mathcal{U}$ and $y \notin \mathcal{U}$.

Let K be a variety of Heyting algebras. The following are equivalent:

- K lacks the ES property.
- There is a member of K with a *proper* epic subalgebra.
- There is an Esakia space $\mathbf{X} \in \mathsf{K}_*$ with a non-identity correct partition R such that for every $\mathbf{Y} \in \mathsf{K}_*$ and every pair of Esakia morphisms $f, g \colon \mathbf{Y} \to \mathbf{X}$, if $\langle f(y), g(y) \rangle \in R$ for every $y \in Y$, then $f = g$.

Known results

Thm (Maksimova) Only finitely many varieties of Heyting algebras K satisfy the following *stronger* variant of the ES property:

If $f : \mathbf{A} \to \mathbf{B}$ is a hom. in K and $b \in B \setminus f[A]$, then there are homs. $g, h : \mathbf{B} \to \mathbf{C} \in$ K such that $g \circ f = h \circ f$ and $g(b) \neq h(b)$.

Varieties with this stronger property include the respective classes of all Boolean algebras, Gödel algebras, and Heyting algebras.

Thm (Kreisel) *Every* variety K of Heyting algebras has the following *weak* variant of the ES property:

If $f : \mathbf{A} \to \mathbf{B}$ is a non-surjective hom. in K, where $\mathbf{B}$ is generated by $f[A]$ plus finitely many elements of B, then f is not an epimorphism.

Nevertheless, the (unqualified) ES property remains poorly understood.

Thm (Campercholi) Let K be an arithmetical variety whose FSI members form a universal class. Then K has the ES property iff its *FSI members* lack proper epic subalgebras.

Thm Every finitely generated variety K of Heyting algebras has surjective epimorphisms.

Proof Suppose, on the contrary, that K lacks the ES property. By Campercholi, there is a FSI $\mathbf{B} \in$ K with a proper epic subalgebra $\mathbf{A}$. Since K is fin. gen., Jónsson's Lemma guarantees that $\mathbf{B}$ is finite. But Kreisel's result then implies that $\mathbf{A}$ is not epic in $\mathbf{B}$, a contradiction.

One of the few general positive results is the following:

Thm (G. Bezhanishvili, T. Moraschini, J. G. Raftery) Varieties of Heyting algebras with finite *depth* have surjective epimorphisms.

- This implies that there is a continuum of such varieties with the ES property.
- The above authors also provided one example of a variety of Heyting algebras which *lacks* the ES property—thereby confirming a conjecture by Blok and Hoogland: the weak ES property is indeed strictly weaker than the ES property.

We will recall this example shortly and exhibit the failure of the ES property for many more varieties.

Recall that to find varieties of Heyting algebras without the ES property, we must avoid varieties that have finite depth. The following construction proves useful in this regard.

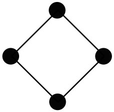
Infinite Sums

Let $\{\mathbf{Y}_n : n \in \omega\}$ be a family of Esakia spaces. Let $\sum \mathbf{Y}_n$ denote the Esakia space obtained by stacking the components above one another, increasing with n, and then adding a fresh top element $\top$ (as in a topological one-point compactification).

Dually, if $\{\mathbf{A}_n : n \in \omega\}$ is a family of Heyting algebras, we let $\sum \mathbf{A}_n$ denote the Heyting algebra obtained by stacking the components, decreasing with n, and identifying the bottom element of the component above with the top element of component below, and then adding a fresh bottom element.

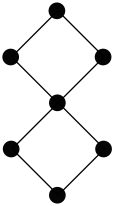
Then $\sum \mathbf{Y}_n^* \cong (\sum \mathbf{Y}_n)^*$.

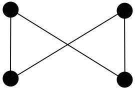
Infinite Sums Example



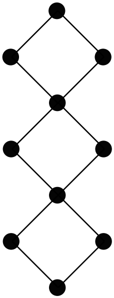
$\mathbf{D}_2^*$ $\mathbf{D}_2$

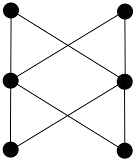
Infinite Sums Example



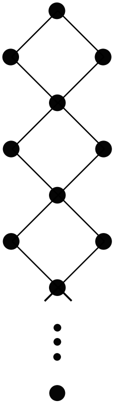


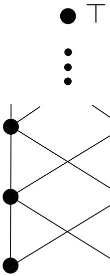
Infinite Sums Example





Infinite Sums Example



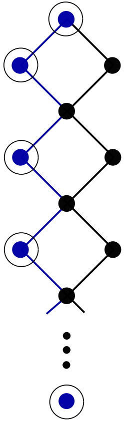


$\sum \mathbf{D}_2^* \cong (\sum \mathbf{D}_2)^* \cong (\mathbf{D}_2^\infty)^*$

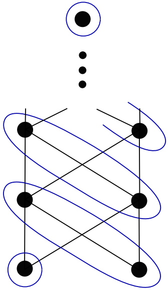
$\sum \mathbf{D}_2 =: \mathbf{D}_2^\infty$

Failure of the ES property

The variety discovered by Bezhanishvili, Moraschini and Raftery in which the ES property fails is $\mathbb{V}((\mathbf{D}_2^\infty)^*)$. In this variety $(\mathbf{D}_2^\infty)^*$ has an epic subalgebra consisting of the chain of its left-most elements.
In this algebra every element has a unique 'sibling' (an element order-incomparable with it) or no sibling. Siblinghood cannot be explicitly defined.



$(\mathbf{D}_2^\infty)^*$



$\mathbf{D}_2^\infty$

New Failures of the ES property

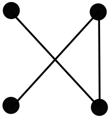
Let n be a positive integer. An Esakia space has *width* $\leq n$ if for every $x \in X$, the up-set $\uparrow x$ does not contain antichains of $n+1$ elements.

A Heyting algebra has *width* $\leq n$ if its Esakia dual does.

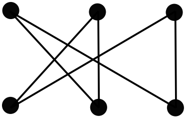
Thm (Baker) The class W_n of all Heyting algebras with width $\leq n$ is a variety. In particular, W_1 is the variety of Gödel algebras.

We shall show that W_n lacks the ES property for any $n \geq 2$.

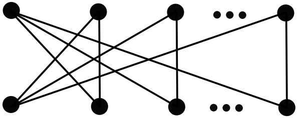
Consider the following Esakia space, for $n \geq 2$:



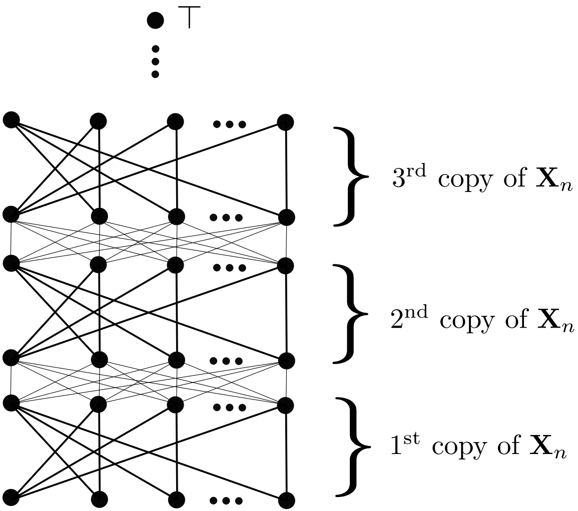
$\mathbf{X}_2$



$\mathbf{X}_3$

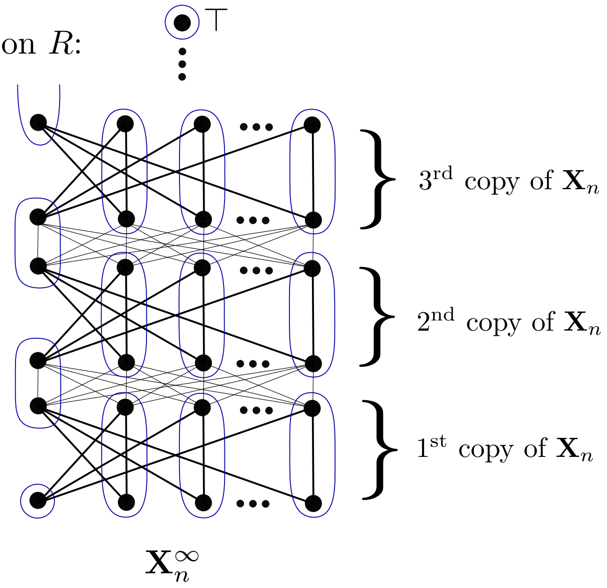


$\mathbf{X}_n$



$\mathbf{X}_n^{\infty}$

A correct parition R:



For every $\mathbf{Y} \in (\mathsf{W}_n)_*$ and every pair of Esakia morphisms $f, g : \mathbf{Y} \to \mathbf{X}_n^\infty$, if $\langle f(y), g(y) \rangle \in R$ for every $y \in Y$, then $f = g$.

Thm For every integer $n \geq 2$ and variety $\mathsf{K} \subseteq \mathsf{W}_n$, if $\mathbf{X}_n^\infty \in \mathsf{K}_*$, then K lacks the ES property.

We sketch the proof for $n = 2$, making use of the following technical Lemma.

Lemma

For $0 < n \in \omega$, let $\mathbf{X}$ and $\mathbf{Y}$ be Esakia spaces of width $\leq n$, such that $\mathbf{Y}$ has a bottom element $\perp$. Let $f : \mathbf{Y} \to \mathbf{X}$ be an Esakia morphism and define

$$X' = \{x \in X : f(\perp) \leq x \text{ and } x \text{ is non-maximum}\}.$$

Suppose that whenever $f(\perp) < x \in X'$ there is an antichain of n elements in X' which contains x.

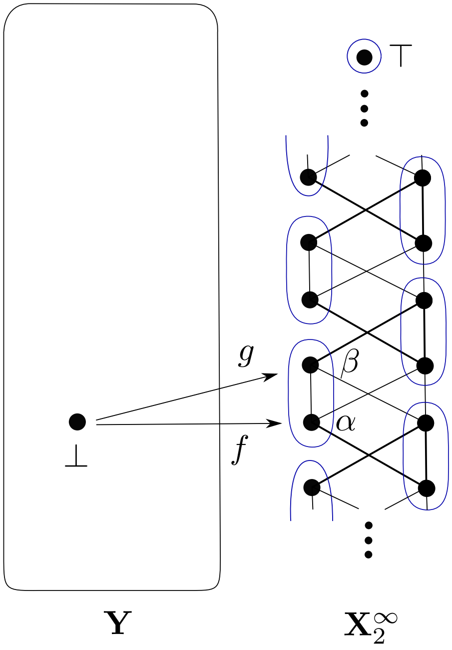
Then there is a subposet Z of $\mathbf{Y}$ such that the restriction

$$f : Z \to X'$$

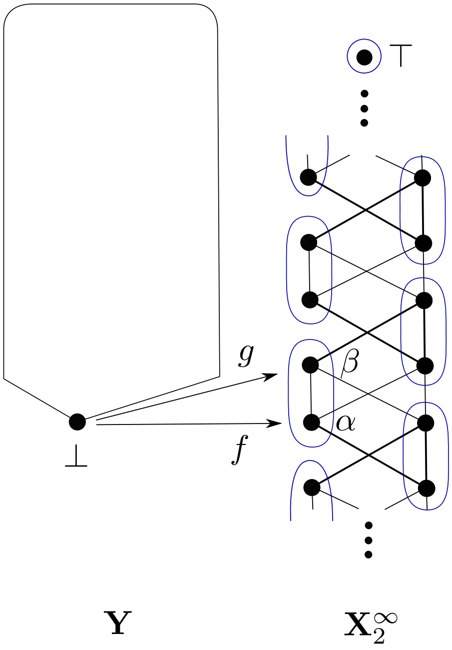
is a poset isomorphism.

Proof for $n = 2$

Let $\mathbf{Y} \in (\mathsf{W}_2)_*$ and $f, g : \mathbf{Y} \to \mathbf{X}_2^\infty$ *different* Esakia morphisms such that $\langle f(y), g(y) \rangle \in R$ for every $y \in Y$.
One can show that there exists $\perp \in Y$ such that $f(\perp) = \alpha$ and $g(\perp) = \beta$ in some component of $\mathbf{X}_2^\infty$.



We may suppose w.l.o.g. that $Y = {\uparrow}\bot$.

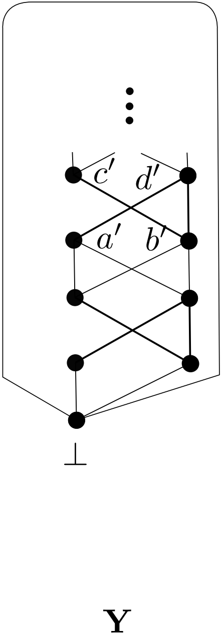


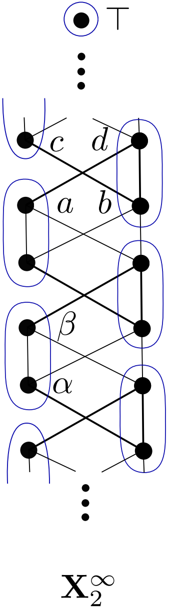
By the Lemma, there is a subposet $Z \subseteq Y$ such that the restriction

$$f : Z \to {\uparrow}\alpha \setminus \{\top\}$$

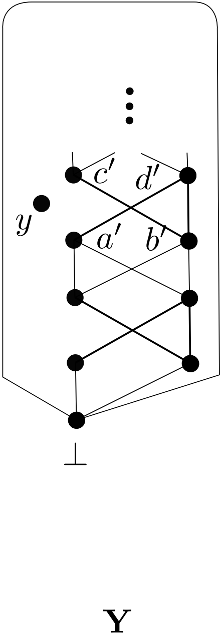
is a poset isomorphism.
We label every element of Z as the 'prime' of its copy under f in X.
We can show that for $x' \in Z$

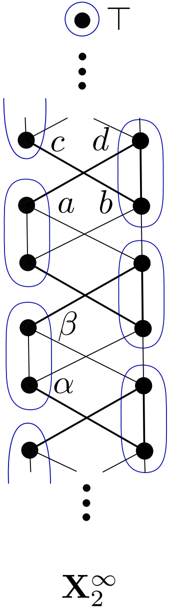
$$g(x') = \max(x/R).$$

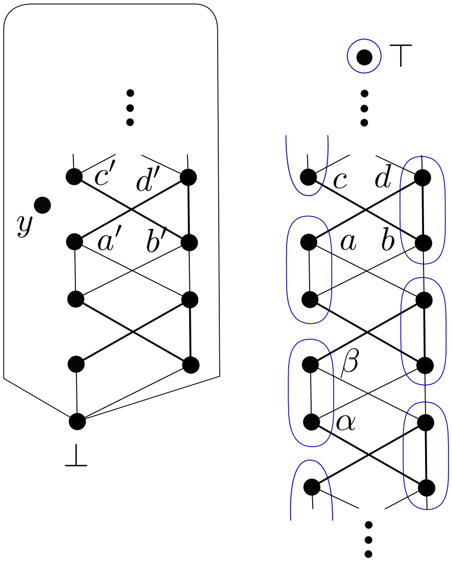




Since $g(\perp) = \beta \leq c$, there exists $y \in Y$ such that $g(y) = c$.
Since c is incomparable with $d = g(b')$, we have y incomparable with b'.
Then y is comparable with a', since $\mathbf{Y}$ has width ≤ 2.
But then, $g(y) = c$ is comparable with $g(a') = a$.
From this contradiction it follows that $f = g$.



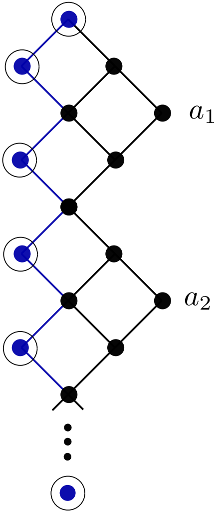




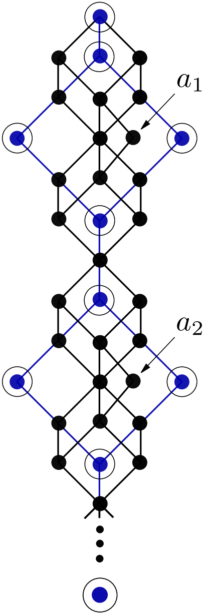
$\mathbf{Y}$ $\mathbf{X}_2^{\infty}$

Associated failure of the Beth property

$(\mathbf{X}_2^\infty)^*$:



$(\mathbf{X}_3^\infty)^*$:

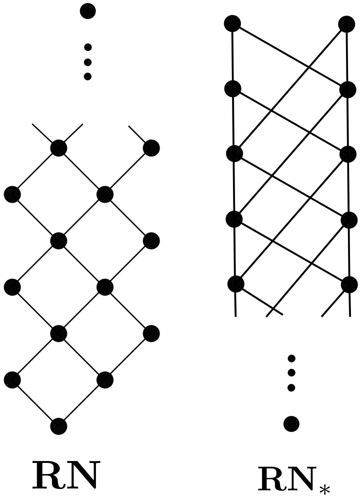


Each of the elements, labeled $a_1, a_2, \ldots$ above, can be considered a sibling of (each member of) a subset of the epic subalgebra.

Then, siblinghood is not explicitly definable.

Rieger-Nishimura lattice

Recall that the Rieger-Nishimura lattice **RN** is the one-generated free Heyting algebra depicted on the right.



Kuznetsov-Gerčiu Varieties

The Kuznetsov-Gerčiu variety is defined as
$\mathsf{KG} := \mathbb{V}\{\mathbf{A}_1 + \cdots + \mathbf{A}_n : 0 < n \in \omega \text{ and } \mathbf{A}_1, \ldots, \mathbf{A}_n \in \mathbb{H}(\mathbf{RN})\}$.

Thm A variety $\mathsf{K} \subseteq \mathsf{KG}$ has the ES property iff it excludes all sums of the form $\sum \mathbf{A}_n$ where each $\mathbf{A}_n$ is either $(\mathbf{X}_2)^*$ or $(\mathbf{D_2})^*$.

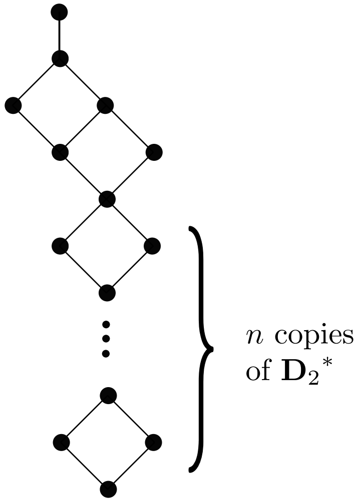
- This provides an alternative explanation of the fact that all varieties of Gödel algebras have the ES property.
- All subvarieties of KG that have the ES property are *locally finite*.
- Within KG, the ES property is *inherited* by subvarieties.
- The variety $\mathbb{V}(\mathbf{RN})$ lacks the ES property.

A Continuum of Varieties Lacking the ES Property

For every $n \in \omega$, consider the depicted algebra $\mathbf{B}_n$. Let $F := \{\mathbf{B}_n : n \in \omega\}$. Bezhanishvili, Bezhanishvili and de Jongh showed that, for every different pair $T, S \subseteq F$, we get $\mathbb{V}(T) \neq \mathbb{V}(S)$.
For every $T \subseteq F$, we show that $\mathbb{V}(T, (\mathbf{D}_2^{\infty})^*)$ is a locally finite subvariety of $\mathbb{V}(\mathbf{RN})$, and the map

$$\mathbb{V}(T) \mapsto \mathbb{V}(T, (\mathbf{D}_2^{\infty})^*)$$

is injective.



$\mathbf{B}_n$

Thm There is a continuum of locally finite subvarieties of $\mathbb{V}(\mathbf{RN})$ without the ES property.

thank you

Implicit definitions

Let K be a variety of Heyting algebras. The following are equivalent:

- K has surjective epimorphisms.
- Whenever an expression $\exists \vec{y} \Sigma(\vec{x}, \vec{y}, v)$-where Σ is a set of equations-defines v implicitly i.t.o $\vec{x}$ over K (in the sense that K satisfies

$$\&\big(\Sigma(\vec{x}, \vec{y}, v_1) \cup \Sigma(\vec{x}, \vec{z}, v_2)\big) \implies v_1 \approx v_2)$$

  and all elements of some $\mathbf{B} \in$ K are define implicitly i.t.o. elements of a subalgebra $\mathbf{A}$ of $\mathbf{B}$ (in the same sense, i.e.,

$$\forall b \in B \quad \exists \vec{a} \in A \quad \exists \vec{y} \in B \quad \Sigma(\vec{a}, \vec{y}, b)),$$

  then $\mathbf{A} = \mathbf{B}$.

Let $\vdash$ be an intermediate logic .

Consider two disjoint sets X and Z of variables, with $X \neq \emptyset$, and a set Γ of formulas over $X \cup Z$. We say that Z is *defined implicitly* in terms of X by means of Γ in $\vdash$ if

$$\Gamma \cup \sigma[\Gamma] \vdash z \leftrightarrow \sigma(z)$$

for every substitution σ such that $\sigma(x) = x$ for all $x \in X$. On the other hand, Z is said to be *defined explicitly* in terms of X by means of Γ in $\vdash$ when, for every $z \in Z$, there exists a formula φ_z over X such that

$$\Gamma \vdash z \leftrightarrow \varphi_z.$$

Then the infinite Beth property postulates the equivalence of implicit and explicit definability in $\vdash$ (for all X, Z, Γ as above).