The continuous weak (Bruhat) order and mix $\star$-autonomous quantale(oid)s

Maria João Gouveia[1] and Luigi Santocanale[2]

TACL@Nice, June 18, 2019

[1]Faculdade de Ciências da Universidade de Lisboa, Portugal
[2]LIS, Aix-Marseille Université, France

Plan

Permutations, words, paths

The quantaloid of discrete paths

Adding the continuum

The continuous Bruhat order

Idempotents, a dive into combinatorics

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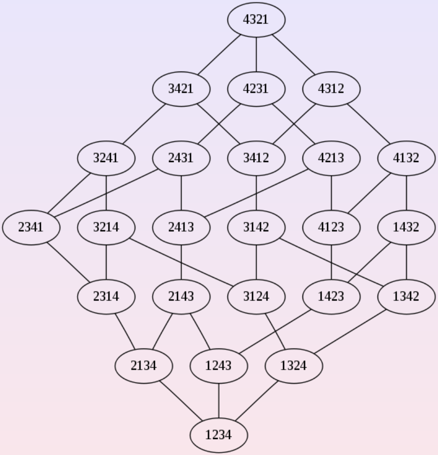
The weak Bruhat order, i.e. the permutohedron P(n)



Theorem (Santocanale & Wehrung, 2018)

The equational theory of the lattices P(n) *is non-trivial and decidable.*

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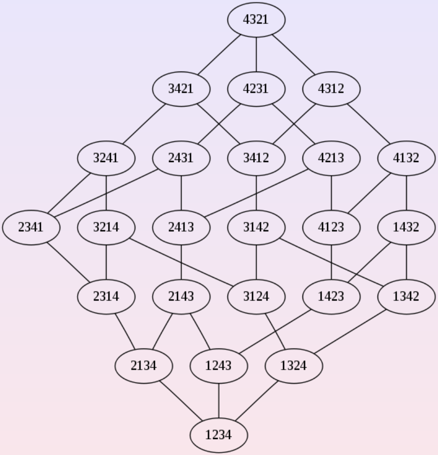

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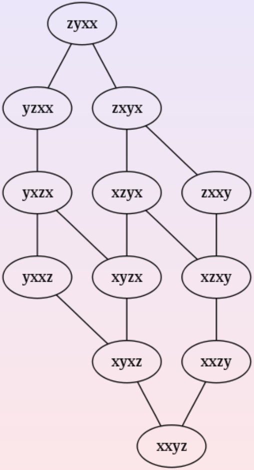
The multinomial lattice P(2, 1, 1)



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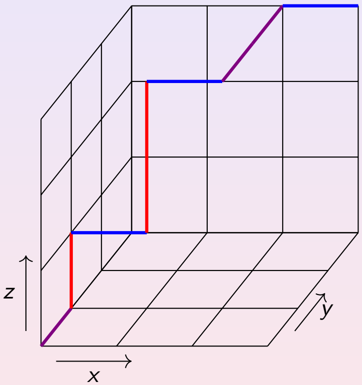

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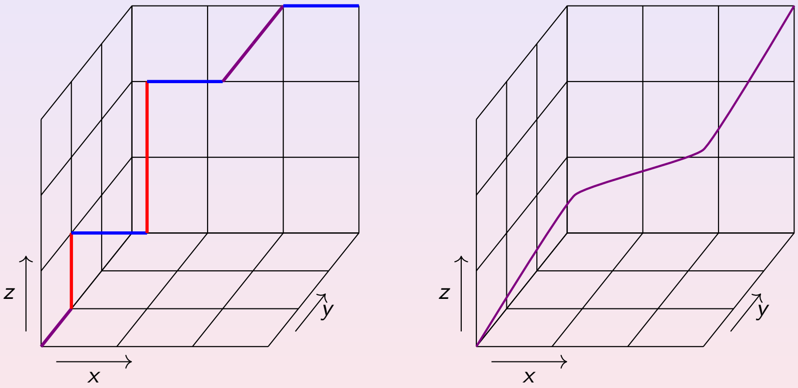
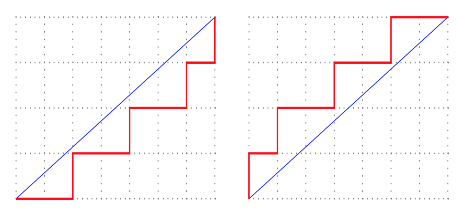


Motivations: discrete geometry and Christoffel words



Christoffel words are images of the diagonal via right/left adjoints:

Are there generalizations of these ideas in dimensions ≥ 3?

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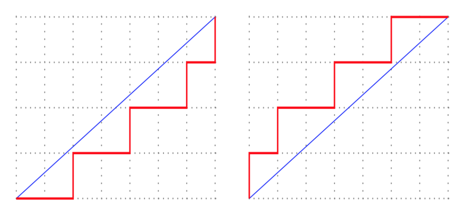

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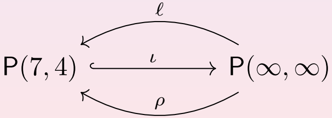

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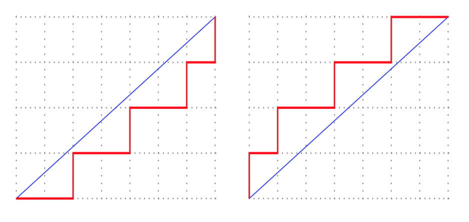

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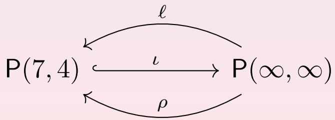

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The category P of discrete words/paths

- Objects : natural numbers $0, 1, \ldots, n, \ldots$
- Arrows:

$$\mathsf{P}(n, m) := \{ w \in \{ x, y \}^* \mid |w|_x = n, |w|_y = m \}$$

- Composition:

$xyxyyx \otimes yxxyxy$:

$$\begin{array}{ccccccc} \epsilon & y & xx & y & x & y & \epsilon \\ & | & & | & & | & \\ \epsilon & x & y & x & yy & x & \epsilon \end{array} \quad \rightsquigarrow \quad \epsilon | xxy | xyy | \epsilon \quad \rightsquigarrow \quad xxyxyy$$

It is a category

Let $[n] := \{ 1, \ldots, n \}, \mathbb{I}_n := \{ 0, 1 \ldots, n \}$ $(= [2]^{[n]})$. Standard bijections:

$$\mathrm{P}(n, m) \simeq Pos([n], \mathbb{I}_m) \simeq \mathrm{SLat}_{\bigvee}(\mathbb{I}_n, \mathbb{I}_m) \,.$$

$yxxxyzyyxy \in \mathrm{P}(5, 5)$:

$$f(1) = f(2) = f(3) = 1$$
$$f(4) = 2$$
$$f(5) = 4$$

Under the bijection, composition is function composition. Thus:

$$\mathrm{P} \simeq \mathrm{Kleisli}(\Delta, \mathbb{I}) \simeq \text{weakening relations over finite chains} \,.$$

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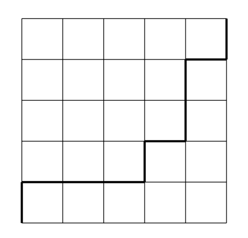

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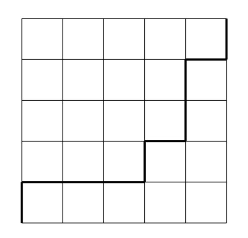

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Counting factorizations

$$\binom{n+m}{n}\binom{m+k}{k} = \sum_{i=0}^{m} \binom{n+m+k-i}{m-i}\binom{n}{i}\binom{k}{i}$$

In particular

$$\binom{2n}{n}^2 = \sum_{i=0}^{n} \binom{3n-i}{n-i}\binom{n}{i}^2 .$$

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Properties of P

- P is a quantaloid (sup-lattice enriched):

$$\mathsf{P}(n, m) \simeq \mathrm{SLat}_{\bigvee}(\mathbb{I}_n, \mathbb{I}_m)\,.$$

- The correspondence

$$f \mapsto f^{\wedge}\,, \qquad\qquad f^{\wedge}(x) := \bigwedge_{x<y} f(y)\,,$$

yields isos

$$\mathrm{SLat}_{\bigvee}(\mathbb{I}_n, \mathbb{I}_m) \simeq \mathrm{SLat}_{\bigwedge}(\mathbb{I}_n, \mathbb{I}_m) \simeq \mathrm{SLat}_{\bigvee}^{op}(\mathbb{I}_m, \mathbb{I}_n)\,.$$

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$\star$-autonomous structure

$$f^* := \text{left-adjoint-of}(f^\wedge) \quad (= \ (\text{right-adjoint-of}(f))^\vee \quad) .$$

On words: exchanges xs and ys.

Dual composition:

$$g \oplus f := (f^* \circ g^*)^* .$$

That is:

Proposition

P *is a $\star$-autonomous quantaloid (involutive residuated latticoid?). For each n,* P(n, n) *is $\star$-autonomous quantale, and an involutive residuated lattice.*

Clopens

Let $[d]_2 := \{ (i,j) \mid 1 \leq i < j \leq d \}$.

Let $\vec{v} = (v_1, \ldots, v_d)$ with $v_i \in \mathbb{N}$, so $\vec{v} : [d] \to \mathrm{P}_0$.

We say that $\delta : [d]_2 \to \mathrm{P}_1$ (over P_0) is

- *closed* if
$$\delta_{i,j} \otimes \delta_{j,k} \leq \delta_{i,k}\,, \qquad \text{for each } i < j < k,$$
- *open* if
$$\delta_{i,k} \leq \delta_{i,j} \oplus \delta_{j,k}\,, \qquad \text{for each } i < j < k,$$
- *clopen* if it is both closed and open.

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The poset of clopens

- Clopens form a poset: $\delta \leq \delta'$ iff $\delta_{i,j} \leq \delta'_{i,j}$ $(1 \leq i < j \leq d)$
- The poset structure depends on the linear ordering of $[d]$.
- Closed (resp., open) tuples form a lattice.
- Clopens form a lattice as well, because of MIX:

$$g \otimes f \leq g \oplus f \,.$$

Proposition

Clopens bijectively correspond to maximal chains in the product lattice $\prod_{i=1,\ldots,n} \mathbb{I}_{v_i}$. Under this bijection, the lattice of clopens is the mutlinomial lattice $\mathrm{P}(v_1, \ldots, v_n)$.

Proposition

For every $\star$-autonomous quantale(oid) or involutive residuated lattice satisfying MIX Q (and each $d \geq 3$), the poset of clopens $\mathrm{Q}(d)$ is a lattice.

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A category P_+ of words/paths

- Objects: natural numbers $0, 1, \ldots, n, \ldots, \infty$.
- Arrows: $\mathsf{P}_+(n, m) = \mathrm{SLat}_{\bigvee}(\mathbb{I}_n, \mathbb{I}_m)$, where

$$\mathbb{I}_\infty := [0, 1]\,.$$

Join-continuous functions as continuous words

Lemma

Bijection/equality between the following kind of data:

- *maximal chains in* $[0,1]^2$,
- *images of continuous monotone functions* $\pi : [0,1] \to [0,1]^2$ *preserving endpoints,*
- *join-continuous (or meet-continuous) functions from* $[0,1]$ *to* $[0,1]$.

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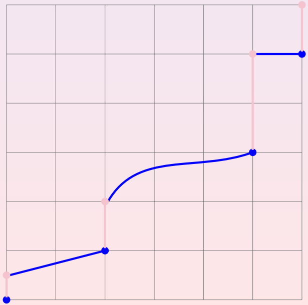

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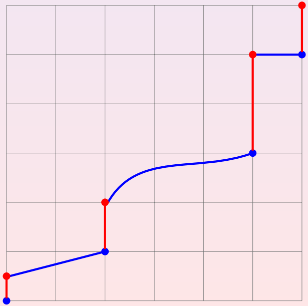

Generalized results

Proposition

P_+ *is a $\star$-autonomous quantaloid (satisfying mix: $\otimes \leq \oplus$).*

Let $\vec{v} = (v_1, \ldots, v_d)$ with $v_i \in \mathbb{N} \cup \{ \infty \}$, so $v : [d] \to (\mathsf{P}_+)_0$.

Proposition

Clopens over $\vec{v}$ bijectively correspond to maximal chains in the product lattice $\prod_{i=1,\ldots,n} \mathbb{I}_{v_i}$. Therefore, these maximal chains can be ordered so they form a lattice.

Remark. Bijection/equality between the following kind of data:

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The continuous Bruhat order of dimension d

- The lattice structure of $\mathsf{P}_+(\vec{\omega})$, $\vec{\omega} := (\underbrace{\infty, \ldots, \infty}_{d-\text{times}})$,
- For every $\vec{v} \in \mathbb{N}^d$ and every collection of lattice embeddings $\iota = \{\, \mathbb{I}_{v_i} \to \mathbb{I}_\infty \mid i = 1, \ldots, d \,\}$, there is a lattice embedding

$$P(\vec{v}, \iota) : \mathsf{P}(\vec{v}) \longrightarrow \mathsf{P}_+(\vec{\infty})$$

- $\mathsf{P}_+(\vec{\infty})$ is the Dedekind-MacNeille completion of the colomit of these embeddings.

Generation and discrete approximations

- Canonical cocone ι_v, with $\iota_{v_i}(k) = \frac{k}{v_i}$.
- $\mathsf{P}_+(\vec{\infty})$ is a $\bigvee\bigwedge$-completion of the colomit of the $\mathsf{P}(\vec{v})$.
- The diagonal lives in $\mathsf{P}_+(\vec{\infty})$, it is a join of elements of thos colimit.
- Open problem: characterize those elements from $\mathsf{P}_+(\vec{\infty})$ that are a join of elements of this colimit.

Open problems

- determine the largest set of chains extending P into a $\star$-autonomous quantaloid ...
- equational theories of $\mathsf{P}(\vec{n})$, $n = 0, 1, \ldots, n, \ldots \infty$ as a residuated lattices,
- determine congruences of $\mathsf{P}(\vec{\omega})$ a residuated lattice,
- determine idempotents (actually a closed problem, see next slides),
- determine their Karoubi completion,
- ...

Thank you (bis) !!!

Plan

Idempotents as emmentalers[3]

Definition

Le A be a complete join-semilattice. An emmentaler on A is a collection $\{ [y_i, x_i] \mid i \in I \}$ of pairwise disjoint intervals of A such that

- $\{ y_i \mid i \in I \}$ closed under meets,
- $\{ x_i \mid i \in I \}$ closed under joins.

Lemma

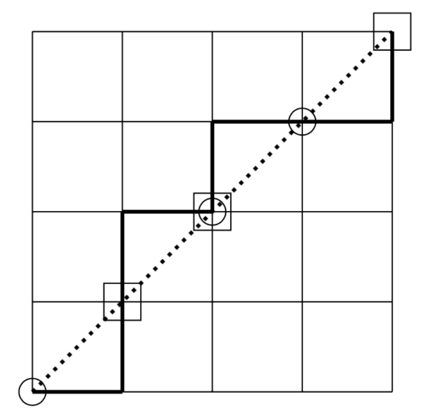
Let A be a complete join-semilattice, let $f \in \mathrm{SLat}_{\bigvee}(A, A)$ be idempotent, and let $f \dashv g$. Then $\{ [f(x), g(f(x))] \mid x \in A \}$ is an emmentaler of A. This sets up a bijective correspondence between idempotents and emmentalers.

[3] Thanks to Daniela Muresan

An emmentaler on $\mathbb{I}_n$

. . . is a sequence

$$0 = y_0 \leq x_0 < y_1 \leq x_1 < \ldots y_k \leq x_k = n$$



Every NE-turn is above $y = x + \frac{1}{2}$, every EN-turn is below this line.

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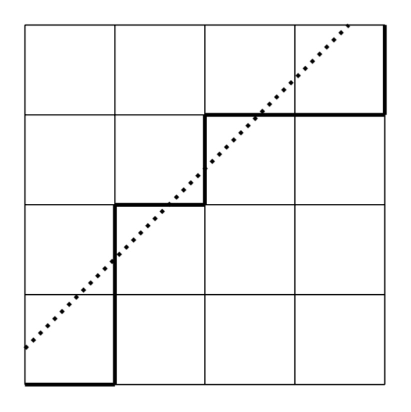

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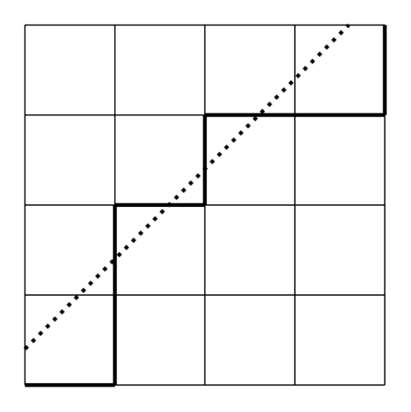

Every NE-turn is above $y = x + \frac{1}{2}$, every EN-turn is below this line.

Counting idempotents

Let f_n be the sequence of Fibonacci numbers.

Proposition

The number of idempotents in $\mathrm{SLat}_{\bigvee}(\mathbb{I}_n, \mathbb{I}_n)$ *equals* f_{2n+1}.

Remark:

$$Pos([n],[n]) = \text{strict maps in } \mathrm{SLat}_{\bigvee}(\mathbb{I}_n, \mathbb{I}_n)$$

$Pos([n],[n])$ is a submonoid of $\mathrm{SLat}_{\bigvee}(\mathbb{I}_n, \mathbb{I}_n)$.

Proposition (Howie 1971)

The number of idempotents in $Pos([n],[n])$ *equals* f_{2n}.

Proposition (Laradji and Umar 2006)

The number of idempotents in $f \in Pos([n],[n])$ *such that* $f(n) = n$ *equals* f_{2n-1}.