Free Kleene algebras with domain

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Definitions

Definition

A **binary relation** on a set X is a subset f of $X \times X$.

There are various 'concrete' operations on binary relations (composition, union...)

Definition

An **algebra of binary relations** of the signature σ is:
an algebra $\mathfrak{A}$ of the signature σ for which...
...there is some set X such that...

- every element of $\mathfrak{A}$ is a binary relation on X
- the symbols of σ are interpreted as the *intended* operations

Definition

Let $\mathfrak{A}$ be an algebra of the signature σ.
A **representation** of $\mathfrak{A}$ is a isomorphism from $\mathfrak{A}$ to an algebra of binary relations

Operations

Composition

$R\,;S = \{(x,z) \in X^2 \mid \exists y \in X : (x,y) \in R \text{ and } (y,z) \in S\}$

Union

$R + S = \{(x,y) \in X^2 \mid (x,y) \in R \text{ or } (x,y) \in S\}$

Reflexive transitive closure

$$R^* := \{(x,y) \in X^2 \mid \exists n \in \mathbb{N} \;\; \exists x_0 \ldots x_n : \\ (x_0 = x) \wedge (x_n = y) \wedge (x_0, x_1) \in R \wedge \ldots \wedge (x_{n-1}, x_n) \in R\}$$

Zero $0 = \emptyset$

Identity $1 = \{(x,x) \in X^2\}$

—the Kleene algebra signature $\{;, +, *, 0, 1\}$

Domain

$\mathrm{D}(R) = \{(x,x) \in X^2 \mid \exists y \in X : (x,y) \in R\}$

The class $\mathsf{Rel}(;, +, ^*, 0, 1)$

Let $\mathsf{Rel}(;, +, ^*, 0, 1)$ denote the isomorphic closure of the class of all algebras of binary relations of the signature $\{;, +, ^*, 0, 1\}$.

- $\mathsf{Rel}(;, +, ^*, 0, 1)$ is not a first-order axiomatisable class (not closed under ultrapowers).
- The variety $\mathit{HSP}\,\mathsf{Rel}(;, +, ^*, 0, 1)$ has no finite equational axiomatisation (Redko 1964).
- **But** Kozen defined *Kleene algebras* using a finite number of quasiequations, and

$$\mathsf{Rel}(;, +, ^*, 0, 1) \subseteq \text{Kleene algebras} \subseteq \mathit{HSP}\,\mathsf{Rel}(;, +, ^*, 0, 1)$$

- In this variety, the free algebra over a finite set Σ is the set of regular languages over Σ.

Free algebra $\equiv$ regular languages

$$L_s = L_t \implies \mathrm{Rel}(;, +, *, 0, 1) \models s = t$$

In an algebra of relations (together with assigment to variables)

the term t holds holds on a pair (x, y)

$\iff$

there is a path from x to y labelled with an string from L_t

$$\mathrm{Rel}(;, +, *, 0, 1) \models s = t \implies L_s = L_t$$

Free algebras for the class $\mathsf{Rel}(;, +, {}^*, 0, 1, \mathsf{D})$

...finite labelled rooted trees...

Definition

Given a set Σ of labels, a **labelled rooted tree** is defined recursively as a set of pairs (a, T), where $a \in \Sigma$ and T is a labelled rooted tree.

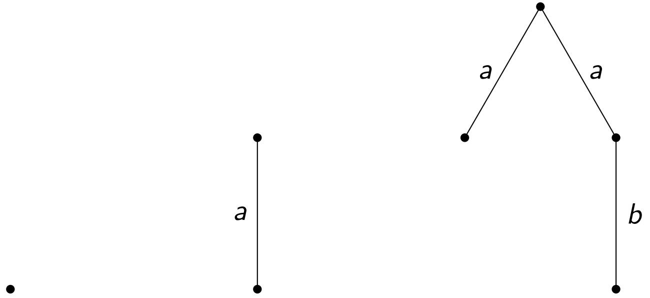


Figure: The labelled rooted trees encoded as $\emptyset$, $\{(a, \emptyset)\}$, and $\{(a, \emptyset), (a, \{(b, \emptyset)\})\}$, respectively (with roots at the top)

Free algebras for the class Rel(;, +, *, 0, 1, D)

Definition

A **pointed tree** is a tree with a distinguished vertex called the **point**.

Definition

The preorder $\preceq$ on (possibly pointed) labelled rooted trees is defined recursively as follows. For trees T_1 and T_2 with roots r_1 and r_2 respectively, $T_1 \preceq T_2$ if and only if

1. r_2 is not the point vertex,
2. for each child v_2 of r_2, there is a child v_1 of r_1 such that
   - the labels of the edges $r_1 v_1$ and $r_2 v_2$ are equal,
   - $T_{v_1} \preceq T_{v_2}$, where T_{v_1} and T_{v_2} are the v_1-rooted and v_2-rooted subtrees respectively.

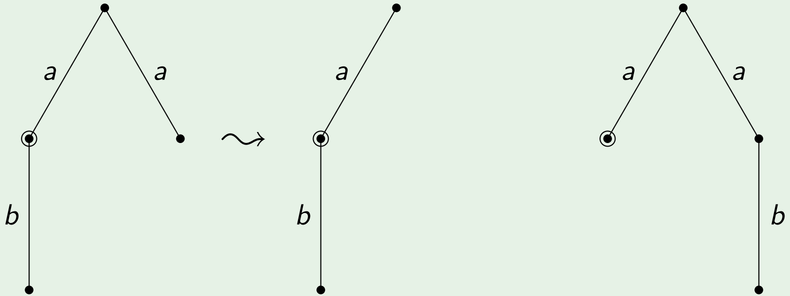
$$T_1 \preceq T_2 \iff \text{there exists a homomorphim } \theta\colon T_2 \to T_1.$$

Definition

Let T be a labelled tree with root r. The **reduced form** of T is the tree formed recursively as follows.

1. For each child v of r, replace the v-rooted subtree with its reduced form.
2. Remove all but the $\preceq$-minimal child subtrees of the tree obtained after the first step.

Example



Free algebras for the class $\mathsf{Rel}(;, +, {}^*, 0, 1, \mathsf{D})$

Definition

Let Σ be a set and let T and S be reduced pointed Σ-labelled rooted trees.

- The **pointed tree concatenation** $T\,;S$ of T and S is the tree formed by
  1. identifying the *point* of T and the *root* of S (the root is now the root of T and the point is the point of S),
  2. reducing the resulting tree to its reduced form.
- The **domain** $\mathsf{D}(T)$ of T is the tree formed by
  1. *reassigning* the point of T to the current root of T,
  2. reducing the resulting tree to its reduced form.

For $+$, *, and 0, lift to sets of reduced pointed Σ-labelled rooted trees, but only retain $\preceq$-maximal elements.

Free algebra is in Rel$(;, +, *, 0, 1)$

Two steps

1. Represent the free algebra for the signature $(;, 1)$ (the free monoid).
2. Include $+$, *, and 0 by lifting to the powerset.

$$w^\theta \subseteq \Sigma^* \times \Sigma^*$$

$$w^\theta = \{(w', w'w) \mid w' \in \Sigma\}$$

$$L^\theta = \bigcup_{w \in L} w^\theta$$

Free algebra is in $\mathrm{Rel}(;, +, *, 0, 1, \mathrm{D})$

$$T^\theta = \{(S \,;\, \mathrm{D}(T), S \,;\, T) \mid S \text{ a reduced tree}\}?$$

$$T^\theta = \{(S, S \,;_\phi\, T) \mid S \text{ a labelled rooted tree}\}$$

Closure properties

The regular sets of trees are closed under the following 'intersection' operation.

$$L_1 \cdot L_2 \coloneqq \text{maximal}(\downarrow L_1 \cap \downarrow L_2)$$

Problem

Are the regular sets of reduced trees closed under the following residuation operations?

$$L_1 \setminus L_2 \coloneqq \text{maximal}\{T \in \mathcal{R}_\Sigma \mid \forall S \in \downarrow L_1,\ S \,;\, T \in \downarrow L_2\}$$
$$L_1 \,/\, L_2 \coloneqq \text{maximal}\{T \in \mathcal{R}_\Sigma \mid \forall S \in \downarrow L_2,\ T \,;\, S \in \downarrow L_1\}$$

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