Hyper-MacNeille Completions of Heyting Algebras

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Algebraic proof theory: Cuts and completions

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There is a close connection between the *admissibility* of the *cut-rule* in hypersequent calculi for substructural logics and closure under *hyper-MacNeille completions* of the corresponding algebraic semantics. (Ciabattoni, Galatos, & Terui 2017)

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Let E be a set of $\mathcal{P}_3$-equations.

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1. *The set E is (effectively) equivalent to a set of hypersequent rules R such that the cut-rule is redundant in the calculus $\mathsf{HLJ} + R$.*
2. *The variety of Heyting algebras axiomatized by E is closed under hyper-MacNeille completions.*

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$$L(Y) = \{w_0 \in W_0 : \forall w_1 \in Y \ w_0 N w_1\},$$
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3. The composition $LU\colon \wp(W_0) \to \wp(W_0)$ is a closure operation on $\langle \wp(W_0), \subseteq \rangle$ and so determines a complete lattice $\mathbb{P}^+$, with

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Example

Let $\mathbf{L}$ be a lattice. Then $\mathbb{P}_{\mathbf{L}} = (L, L, \leq)$ is a polarity and $\mathbb{P}^+_{\mathbf{L}}$ is the MacNeille completion of $\overline{\mathbf{L}}$ of $\mathbf{L}$.

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Proposition (Terui 2018, Galatos & Jipsen 2013)

If $\mathbb{F} = \langle W_0, W_1, N, \circ, \rightsquigarrow, \epsilon \rangle$ is a Heyting frame, then the induced lattice $\mathbb{F}^+$ is a complete Heyting algebra with

$$Z_1 \to Z_2 = \{w \in W_0 : \forall w' \in Z_1\ w' \circ w \in Z_2\}.$$

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If **A** is a Heyting algebra, then $\mathbb{G}_{\mathbf{A}} = \langle A, A, \leq, \wedge, \rightarrow, 1 \rangle$ is a Heyting frame, and $\mathbb{G}_{\mathbf{A}}^+ = \overline{\mathbf{A}}$, the MacNeille completion of **A**.

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If $\mathbf{A}$ is a Heyting algebra, then $\mathbb{H}_{\mathbf{A}} = \langle A^2, A^2, N, \circ, \rightsquigarrow, (0,1)\rangle$

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is a Heyting frame. Thus $\mathbb{H}^+_{\mathbf{A}} := \mathbf{A}^+$ is a complete Heyting algebra.

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Proposition (Ciabattoni, Galatos, & Terui 2017)

For each Heyting algebra $\mathbf{A}$ *there is an embedding of Heyting algebras* $\mathbf{A} \hookrightarrow \mathbf{A}^{+}$ *given by* $b \mapsto L(0, b) = \{(s, a) \in A^2 : s \vee (a \to b) = 1\}$.

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3. *If* $\mathbf{A}$ *is* externally distributive, *i.e., for all* $\{a\} \cup S \subseteq A$ *with* S *having a greatest lower bound in* $\mathbf{A}$,

$$\forall s \in S\ (a \vee s = 1) \implies a \vee \bigwedge S = 1,$$

then $\mathbf{A} \hookrightarrow \mathbf{A}^+$ *is a regular completion.*

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$$1 \approx x \vee y \implies 1 \approx x \text{ or } 1 \approx y.$$

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If $\mathbf{A}$ *is a De Morgan supplemented Heyting algebra, then the embedding* $\mathbf{A} \hookrightarrow \mathbf{A}^+$ *is both meet- and join-dense. Consequently,* $\mathbf{A}^+ \cong \overline{\mathbf{A}}$*.*

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(ii) MacNeille completions of Boolean product have been looked at before (Harding 1993; Crown, Harding, & Janowitz 1996).

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If $\mathbf{A}$ is a De Morgan supplemented Heyting algebra, then the embedding $\mathbf{A} \hookrightarrow \prod_{x \in \min(X)} \mathbf{A}/\theta_x$ gives a Boolean product representation.

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7. Note that $Q(\mathbf{A}) \in \mathcal{V}(\mathbf{A})$.

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Theorem

For all Heyting algebras $\mathbf{A}$ *there is an embedding* $Q(\mathbf{A}) \hookrightarrow \mathbf{A}^+$, *which is both meet- and join-dense. Consequently,* $\mathbf{A}^+ \cong \overline{Q(\mathbf{A})}$.

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2. *The class of De Morgan supplemented members of* $\mathcal{V}$ *is closed under MacNeille completions.*

Varieties closed under hyper-MacNeille completions

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Theorem (cf. Harding 1993)

Let $\mathbf{A}$ *be a Heyting algebra with dual Esakia space* X*. If there is* $n \in \omega$*, such that* $|\mathbf{A}/\theta_x| \leq n$ *for all* $x \in \min(X)$*, then the algebra* $Q(\mathbf{A})$ *is complete. In particular,* $\mathbf{A}^+ \cong Q(\mathbf{A})$*.*

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The variety axiomatized by the equation

$$x_2 \vee (x_2 \to (x_1 \vee \neg x_1)) \approx 1 \qquad (\mathbf{bd}_2)$$

is closed under hyper-MacNeille completions, but not axiomatizable by $\mathcal{P}_3$*-equations nor finitely generated.*

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Recall that a Heyting algebra is *externally distributive*, provided that

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Note that every variety of Heyting algebras $\mathcal{V} \supsetneq \mathcal{BA}$ contains an incomplete algebra which is not externally distributive.

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3. How to find Boolean product representations of $\mathbf{A}^+$ and $Q(\mathbf{A})$?
4. Can we find workable descriptions of MacNeille completions of Boolean products of (fsi) Heyting algebras?

Thank you very much for your time and attention