Ranges of functors and elementary classes via topos theory

Peter Arndt
University of Düsseldorf, Germany

TACL, 19.6.2019

$\mathbb{S}, \mathbb{T}$ first order theories, $F\colon \mathcal{A} := Mod(\mathbb{T}) \to Mod(\mathbb{S}) =: \mathcal{B}$ a functor between their categories of models.

$\mathbb{S}, \mathbb{T}$ first order theories, $F\colon \mathcal{A} := Mod(\mathbb{T}) \to Mod(\mathbb{S}) =: \mathcal{B}$ a functor between their categories of models.

Questions:

(i) Is every $B \in \mathcal{B}$ isomorphic to $F(A)$ for some $A \in \mathcal{A}$?

$\mathbb{S}, \mathbb{T}$ first order theories, $F\colon \mathcal{A} := Mod(\mathbb{T}) \to Mod(\mathbb{S}) =: \mathcal{B}$ a functor between their categories of models.

Questions:

(i) Is every $B \in \mathcal{B}$ isomorphic to $F(A)$ for some $A \in \mathcal{A}$?

(ii) Is every $B \in \mathcal{B}$ elementarily equivalent to $F(A)$ for some $A \in \mathcal{A}$?

$\mathbb{S}, \mathbb{T}$ first order theories, $F\colon \mathcal{A} := Mod(\mathbb{T}) \to Mod(\mathbb{S}) =: \mathcal{B}$ a functor between their categories of models.

Questions:

(i) Is every $B \in \mathcal{B}$ isomorphic to $F(A)$ for some $A \in \mathcal{A}$?

(ii) Is every $B \in \mathcal{B}$ elementarily equivalent to $F(A)$ for some $A \in \mathcal{A}$?

(iii) What is the $\underbrace{\text{elementary class generated by the essential image of } F}_{:= Mod(\bigcap_{A \in \mathcal{A}} Th(F(A)))}$?

$\mathbb{S}, \mathbb{T}$ first order theories, $F\colon \mathcal{A} := Mod(\mathbb{T}) \to Mod(\mathbb{S}) =: \mathcal{B}$ a functor between their categories of models.

Questions:

(i) Is every $B \in \mathcal{B}$ isomorphic to $F(A)$ for some $A \in \mathcal{A}$?

(ii) Is every $B \in \mathcal{B}$ elementarily equivalent to $F(A)$ for some $A \in \mathcal{A}$?

(iii) What is the $\underbrace{\text{elementary class generated by the essential image of } F}_{:= Mod(\bigcap_{A \in \mathcal{A}} Th(F(A)))}$?

Rephrasing of (iii): Is there some first order statement true for every $F(A)$, but not for all Bs ?

Examples

Example 1: Dilworth's congruence lattice problem

$$\begin{array}{rcl} \textbf{Lat} & \to & \textbf{AlgDistLat} \\ L & \mapsto & Con(L) \end{array}$$

Question (Dilworth 1940s) **:** Is every AlgDistLat of the form $Con(L)$?

Examples

Example 1: Dilworth's congruence lattice problem

$$\begin{aligned} \mathbf{Lat} &\to \mathbf{DistSemLat} \\ L &\mapsto Con_c(L) \end{aligned}$$

Question: Is every DistSemLat of the form $Con_c(L)$?

Examples

Example 1: Dilworth's congruence lattice problem

$$\begin{array}{rcl} \textbf{Lat} & \to & \textbf{DistSemLat} \\ L & \mapsto & Con_c(L) \end{array}$$

Question: Is every DistSemLat of the form $Con_c(L)$?

Theorem (Huhn 1985)**:** Every DistSemLat of cardinality $\leq \aleph_1$ is of the form $Con_c(L)$.

Examples

Example 1: Dilworth's congruence lattice problem

$$\begin{array}{rcl} \textbf{Lat} & \to & \textbf{DistSemLat} \\ L & \mapsto & Con_c(L) \end{array}$$

Question: Is every DistSemLat of the form $Con_c(L)$?

Theorem (Huhn 1985)**:** Every DistSemLat of cardinality $\leq \aleph_1$ is of the form $Con_c(L)$.

Theorem (Wehrung/Tůma 2007/9)**:** There are DistSemLats of cardinality $\geq \aleph_2$ *not* of the form $Con_c(L)$.

Examples

Example 1: Dilworth's congruence lattice problem

$$\begin{array}{rcl} \mathbf{Lat} & \to & \mathbf{DistSemLat} \\ L & \mapsto & Con_c(L) \end{array}$$

Question: Is every DistSemLat of the form $Con_c(L)$?

Theorem (Huhn 1985)**:** Every DistSemLat of cardinality $\leq \aleph_1$ is of the form $Con_c(L)$.

Theorem (Wehrung/Tůma 2007/9)**:** There are DistSemLats of cardinality $\geq \aleph_2$ *not* of the form $Con_c(L)$.

Corollary: $\nexists$ first order sentence holding for all $Con_c(L)$, but not for general DistSemLats.

Examples

My motivating examples:
– Representation problem for special groups (quadratic form theory)
– Which graded $\mathbb{Z}/2$-algebras arise as Milnor K-theory of a field? (possible applications to inverse Galois problem)

What are *your* examples?

Some fragments of $L_{\infty\kappa}$

Definition: Let κ be a cardinal, Σ a first order signature.

(i) A *κ-geometric formula* is a formula built from atomic formulas, $\top, \bot$, using $\bigvee_{j \in J}$ (J a set), $\bigwedge_{i \in I}$ ($|I| < \kappa$) and $\exists \{x_i\}_{i \in I}$ ($|I| < \kappa$).

(ii) A *κ-geometric theory* is a theory which can be axiomatized by formulas of the form $\forall \{x_i\}\ \phi \to \psi$, where ϕ, ψ are κ-geometric formulas. (*κ-geometric sequents*)

(iii) For a class of Σ-structures $\mathcal{C}$ denote by $Th_{\kappa\text{-geom}}(\mathcal{C})$ the *κ-geometric theory of $\mathcal{C}$*, i.e. the set of all κ-geometric sequents that are valid in every member of $\mathcal{C}$.

(iv) Denote by $Th_{\neg\text{-}\kappa\text{-geom}}(\mathcal{C})$ the set of negations of κ-geometric formulas (i.e. sequents of the form $\forall \bar{x}\ \phi \to \bot$), that are valid in every member of $\mathcal{C}$.

Accessible categories

Definition: Let κ be a regular cardinal.

(i) A diagram is κ-*directed* if...

$$A_i$$

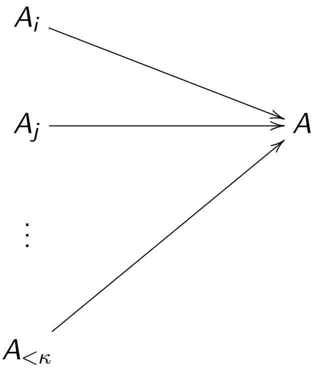
$$A_j$$

$$\vdots$$

$$A_{<\kappa}$$

Accessible categories

Definition: Let κ be a regular cardinal.
(i) A diagram is κ-*directed* if...



Accessible categories

Definition: Let κ be a regular cardinal.

(i) A diagram is κ-*directed* if...

(ii) An object A is κ-*presentable* if for all κ-directed diagrams D

$$Hom(A, colim\ D) \cong colim_{d \in Ob\, D}\ Hom(A, d)$$

Accessible categories

Definition: Let κ be a regular cardinal.

(i) A diagram is κ*-directed* if...

(ii) An object A is κ*-presentable* if for all κ-directed diagrams D

$$Hom(A, colim\ D) \cong colim_{d \in Ob\, D}\ Hom(A, d)$$

(iii) A category is κ*-accessible* if all κ-directed colimits exist and there is a *set* of κ-presentable objects, s.t. every object is a κ-directed colimit of these.

Accessible categories

Definition: Let κ be a regular cardinal.

(i) A diagram is κ-*directed* if...

(ii) An object A is κ-*presentable* if for all κ-directed diagrams D

$$Hom(A, colim\ D) \cong colim_{d \in Ob\, D}\ Hom(A, d)$$

(iii) A category is κ-*accessible* if all κ-directed colimits exist and there is a *set* of κ-presentable objects, s.t. every object is a κ-directed colimit of these.

Examples:

– Any variety is $\aleph_0$-accessible. ("finitely accessible")

Accessible categories

Definition: Let κ be a regular cardinal.

(i) A diagram is κ-*directed* if...

(ii) An object A is κ-*presentable* if for all κ-directed diagrams D

$$Hom(A, colim\ D) \cong colim_{d \in Ob\,D}\ Hom(A, d)$$

(iii) A category is κ-*accessible* if all κ-directed colimits exist and there is a *set* of κ-presentable objects, s.t. every object is a κ-directed colimit of these.

Examples:
– Any variety is $\aleph_0$-accessible. ("finitely accessible")
– **Fields** is $\aleph_0$-accessible

Accessible categories

Definition: Let κ be a regular cardinal.

(i) A diagram is κ-*directed* if...

(ii) An object A is κ-*presentable* if for all κ-directed diagrams D

$$Hom(A, colim\ D) \cong colim_{d \in Ob\, D}\ Hom(A, d)$$

(iii) A category is κ-*accessible* if all κ-directed colimits exist and there is a *set* of κ-presentable objects, s.t. every object is a κ-directed colimit of these.

Examples:

– Any variety is $\aleph_0$-accessible. ("finitely accessible")

– **Fields** is $\aleph_0$-accessible

– **Theorem:** $\mathbb{T}$ finitary first order theory over a countable signature. Then $Mod(\mathbb{T})$ is $\aleph_1$-accessible.

Accessible categories

Definition: Let κ be a regular cardinal.

(i) A diagram is κ-*directed* if...

(ii) An object A is κ-*presentable* if for all κ-directed diagrams D

$$Hom(A, colim\ D) \cong colim_{d \in Ob\,D}\ Hom(A, d)$$

(iii) A category is κ-*accessible* if all κ-directed colimits exist and there is a *set* of κ-presentable objects, s.t. every object is a κ-directed colimit of these.

Examples:

– Any variety is $\aleph_0$-accessible. ("finitely accessible")

– **Fields** is $\aleph_0$-accessible

– **Theorem:** $\mathbb{T}$ finitary first order theory over a countable signature. Then $Mod(\mathbb{T})$ is $\aleph_1$-accessible.

– **Theorem:** The κ-accessible categories are exactly the ones of the form $Mod(\mathbb{T})$ for a κ-geometric theory $\mathbb{T}$.

Theorem (A.): Let $\mathcal{A}$, $\mathcal{B}$ be κ-accessible categories, $\mathcal{A} = Mod(\mathbb{T})$, $\mathcal{B} = Mod(\mathbb{S})$ for κ-geometric theories. Denote by $\mathcal{A}_\kappa, \mathcal{B}_\kappa$ the subcategories of κ-presentable objects. Suppose we have

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ \uparrow & & \uparrow \\ \mathcal{A}_\kappa & \xrightarrow{F_\kappa} & \mathcal{B}_\kappa \end{array}$$

preserving κ-filtered colimits

...and restricting to

The result

Theorem (A.): Let $\mathcal{A}$, $\mathcal{B}$ be κ-accessible categories, $\mathcal{A} = Mod(\mathbb{T})$, $\mathcal{B} = Mod(\mathbb{S})$ for κ-geometric theories. Denote by $\mathcal{A}_\kappa, \mathcal{B}_\kappa$ the subcategories of κ-presentable objects. Suppose we have

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ \uparrow & & \uparrow \\ \mathcal{A}_\kappa & \xrightarrow{F_\kappa} & \mathcal{B}_\kappa \end{array}$$

preserving κ-filtered colimits

...and restricting to $\mathcal{A}_\kappa \xrightarrow{F_\kappa} \mathcal{B}_\kappa$

Then the following hold:

(a) If F_κ is essentially surjective, then $Th_{\kappa\text{-geom}}(F(\mathcal{A})) = Th_{\kappa\text{-geom}}(\mathcal{B})$.

Example

Example 1: (Dilworth's congruence lattice problem)

$$\begin{array}{rcl} \mathbf{Lat} & \to & \mathbf{DistSemLat} \\ L & \mapsto & Con_c(L) \end{array}$$

preserves $\aleph_1$-filtered colimits and $\aleph_1$-presentable objects.

Theorem (Huhn 1985)**:** Every DistSemLat of cardinality $\leq \aleph_1$ is of the form $Con_c(L)$.

Corollary: $\nexists$ an $\aleph_1$-geometric sequent that holds for all $Con_c(L)$ but not for general DistSemLats.

The result

Theorem (A.): Let $\mathcal{A}$, $\mathcal{B}$ be κ-accessible categories, $\mathcal{A} = Mod(\mathbb{T})$, $\mathcal{B} = Mod(\mathbb{S})$ for κ-geometric theories. Denote by $\mathcal{A}_\kappa, \mathcal{B}_\kappa$ the subcategories of κ-presentable objects. Suppose we have

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ \uparrow & & \uparrow \\ \mathcal{A}_\kappa & \xrightarrow{F_\kappa} & \mathcal{B}_\kappa \end{array}$$

preserving κ-filtered colimits

...and restricting to

Then the following hold:

(a) If F_κ is essentially surjective, then $Th_{\kappa\text{-geom}(F(\mathcal{A}))} = Th_{\kappa\text{-geom}(\mathcal{B})}$.

The result

Theorem (A.): Let $\mathcal{A}$, $\mathcal{B}$ be κ-accessible categories, $\mathcal{A} = Mod(\mathbb{T})$, $\mathcal{B} = Mod(\mathbb{S})$ for κ-geometric theories. Denote by $\mathcal{A}_\kappa, \mathcal{B}_\kappa$ the subcategories of κ-presentable objects. Suppose we have

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ \uparrow & & \uparrow \\ \mathcal{A}_\kappa & \xrightarrow{F_\kappa} & \mathcal{B}_\kappa \end{array}$$

preserving κ-filtered colimits

...and restricting to

Then the following hold:

(a) If F_κ is essentially surjective, then $Th_{\kappa\text{-geom}(F(\mathcal{A}))} = Th_{\kappa\text{-geom}(\mathcal{B})}$.

More generally: For different such functors $F\colon \mathcal{A} \to \mathcal{B}$, $F'\colon \mathcal{A}' \to \mathcal{B}$ one has $Th_{\kappa\text{-geom}}(F(\mathcal{A})) = Th_{\kappa\text{-geom}}(F'(\mathcal{A}')) \supseteq \mathbb{S}$ if and only if $F_\kappa(\mathcal{A})$ and $F'_\kappa(\mathcal{A}')$ have equivalent idempotent completions.

Example

Example 2: $\kappa = \aleph_0$, $\mathcal{B} =$ **Groups**, $\mathcal{A} = Mod(Th_{geom}(F(n)))$ where $F(n)$ is the free group on n generators.

Example

Example 2: $\kappa = \aleph_0$, $\mathcal{B} =$ **Groups**, $\mathcal{A} = Mod(Th_{geom}(F(n)))$ where $F(n)$ is the free group on n generators.

Theorem (Kharlampovich-Myasnikov/Sela 2006)**:** The first order theories of all finitely generated free groups coincide.

Example

Example 2: $\kappa = \aleph_0$, $\mathcal{B} =$ **Groups**, $\mathcal{A} = Mod(Th_{geom}(F(n)))$ where $F(n)$ is the free group on n generators.

Theorem (Kharlampovich-Myasnikov/Sela 2006)**:** The first order theories of all finitely generated free groups coincide.

Not true in $L_{\infty,\omega}$: $\exists x_1, \ldots, x_n \, \forall y \; \bigvee_{w \in words(x_1,\ldots,x_n)} y = w.$

Example

Example 2: $\kappa = \aleph_0$, $\mathcal{B} =$ **Groups**, $\mathcal{A} = Mod(Th_{geom}(F(n)))$ where $F(n)$ is the free group on n generators.

Theorem (Kharlampovich-Myasnikov/Sela 2006)**:** The first order theories of all finitely generated free groups coincide.

Not true in $L_{\infty,\omega}$: $\exists x_1, \ldots, x_n \, \forall y \, \bigvee_{w \in words(x_1,\ldots,x_n)} y = w.$

Proposition: For $m < n$ we have $Th_{geom}(F(m)) \neq Th_{geom}(F(n))$.

Proof: Idempotent closure of $F(n)$ is $\{F(1), \ldots, F(n))\}$ - these are different for different n. $\square$

Example

Example 2: $\kappa = \aleph_0$, $\mathcal{B} =$ **Groups**, $\mathcal{A} = Mod(Th_{geom}(F(n)))$ where $F(n)$ is the free group on n generators.

Theorem (Kharlampovich-Myasnikov/Sela 2006)**:** The first order theories of all finitely generated free groups coincide.

Not true in $L_{\infty,\omega}$: $\exists x_1, \ldots, x_n \, \forall y \, \bigvee_{w \in words(x_1,\ldots,x_n)} y = w.$

Proposition: For $m < n$ we have $Th_{geom}(F(m)) \neq Th_{geom}(F(n))$.

Proof: Idempotent closure of $F(n)$ is $\{F(1), \ldots, F(n))\}$ - these are different for different n. $\square$

Actually for $m < n$ we have $Th_{geom}(F(n)) \subsetneq Th_{geom}(F(m))$.

The result

Theorem (A.)(continued)**:** Let $\mathcal{A}$, $\mathcal{B}$ be κ-accessible categories, $\mathcal{A} = Mod(\mathbb{T})$, $\mathcal{B} = Mod(\mathbb{S})$ for κ-geometric theories. Denote by $\mathcal{A}_\kappa, \mathcal{B}_\kappa$ the subcategories of κ-presentable objects. Suppose we have $F \colon \mathcal{A} \to \mathcal{B}$ preserving κ-filtered colimits and κ-presentable objects.

The result

Theorem (A.)(continued)**:** Let $\mathcal{A}$, $\mathcal{B}$ be κ-accessible categories, $\mathcal{A} = Mod(\mathbb{T})$, $\mathcal{B} = Mod(\mathbb{S})$ for κ-geometric theories. Denote by $\mathcal{A}_\kappa, \mathcal{B}_\kappa$ the subcategories of κ-presentable objects. Suppose we have $F\colon \mathcal{A} \to \mathcal{B}$ preserving κ-filtered colimits and κ-presentable objects.

Then the following hold:

(b) If $F_\kappa\colon \mathcal{A}_\kappa \to \mathcal{B}_\kappa$ is *fully faithful*, then $F(\mathcal{A}) = Mod(\mathbb{S}')$ for some axiomatic extension $\mathbb{S}' \supseteq \mathbb{S}$ (i.e. the essential image $F(\mathcal{A})$ can be characterized by additional κ-geometric sequents in the language of $\mathbb{S}$).

The result

Theorem (A.)(continued)**:** Let $\mathcal{A}$, $\mathcal{B}$ be κ-accessible categories, $\mathcal{A} = Mod(\mathbb{T})$, $\mathcal{B} = Mod(\mathbb{S})$ for κ-geometric theories. Denote by $\mathcal{A}_\kappa, \mathcal{B}_\kappa$ the subcategories of κ-presentable objects. Suppose we have $F\colon \mathcal{A} \to \mathcal{B}$ preserving κ-filtered colimits and κ-presentable objects.

Then the following hold:

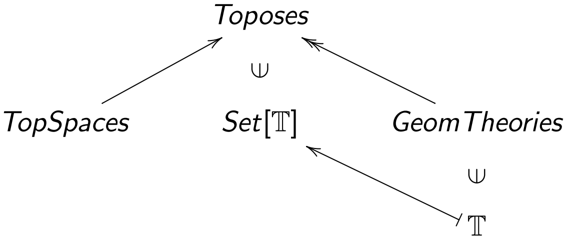
(b) If $F_\kappa\colon \mathcal{A}_\kappa \to \mathcal{B}_\kappa$ is *fully faithful*, then $F(\mathcal{A}) = Mod(\mathbb{S}')$ for some axiomatic extension $\mathbb{S}' \supseteq \mathbb{S}$ (i.e. the essential image $F(\mathcal{A})$ can be characterized by additional κ-geometric sequents in the language of $\mathbb{S}$).

(c) If one has that every $B \in \mathcal{B}_\kappa$ admits a morphism to $F(A)$, for some $A \in \mathcal{A}_\kappa$, then $Th_{\neg\text{-}\kappa\text{-geom}}(F(\mathcal{A})) = Th_{\neg\text{-}\kappa\text{-geom}}(\mathcal{B})$, i.e. the objects in the essential image of F and general objects of $\mathcal{B}$ satisfy exactly the same negations of κ-geometric formulas.

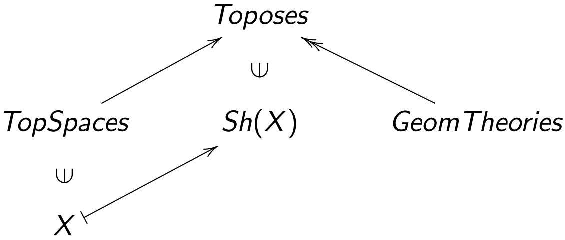
About the proof

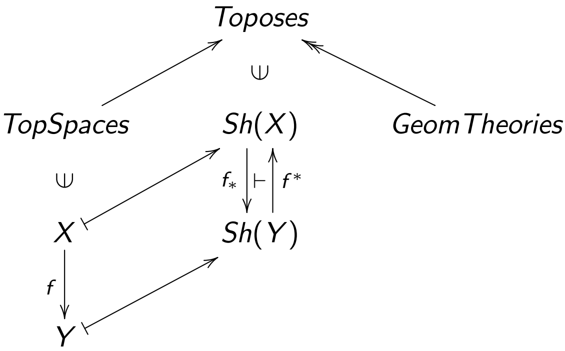
$$\mathit{TopSpaces} \rightarrow \mathit{Toposes} \twoheadleftarrow \mathit{GeomTheories}$$

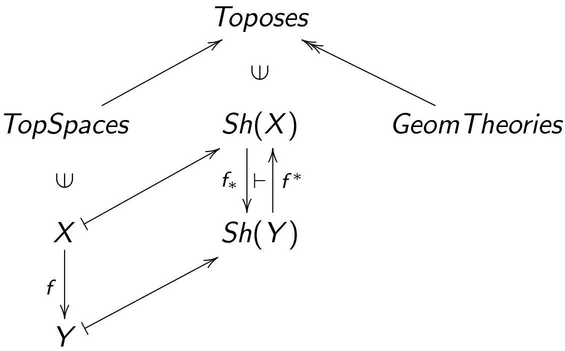
About the proof



About the proof




Toposes
⊎
TopSpaces
Sh(X)
GeomTheories
⊎
f*
⊢
f*
X
Sh(Y)
f
Y



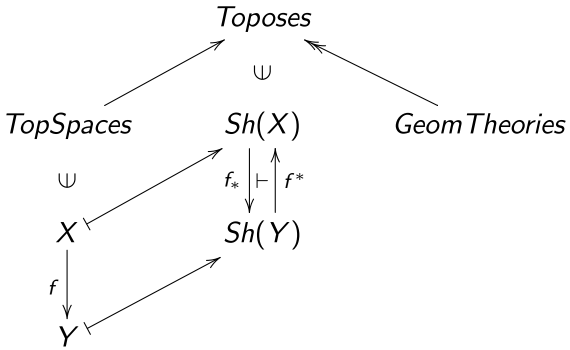
For good enough spaces:

– f surjective $\Leftrightarrow$ f^* faithful

– f embedding $\Leftrightarrow$ f_* fully faithful

– $f(X)$ dense in Y $\Leftrightarrow$ $f_*(0) \cong 0$

– f closed inclusion $\Leftrightarrow$ $f^*(G) \cong G \times U$ for a subterminal object U

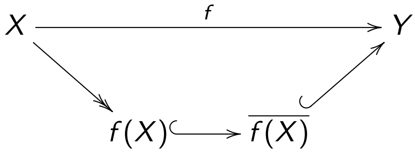


For good enough spaces:

– f surjective $\Leftrightarrow f^*$ faithful $\Leftrightarrow$: (f_*, f^*) is a surjective geometric morphism
– f embedding $\Leftrightarrow f_*$ fully faithful $\Leftrightarrow$: (f_*, f^*) is an inclusion
– $f(X)$ dense in $Y \Leftrightarrow f_*(0) \cong 0$ $\Leftrightarrow$: (f_*, f^*) is dominant
– f closed inclusion $\Leftrightarrow f^*(G) \cong G \times U$ $\Leftrightarrow$: (f_*, f^*) is a closed inclusion

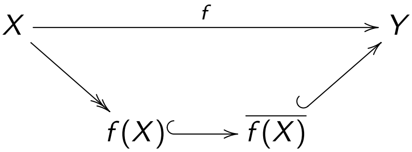
Factorization

In TopSpaces:

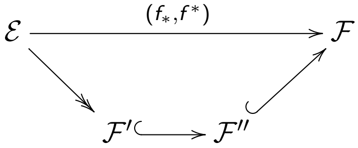


Factorization

In TopSpaces:

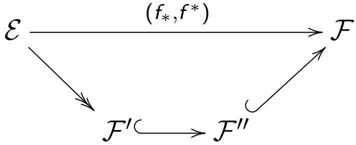


In Toposes:



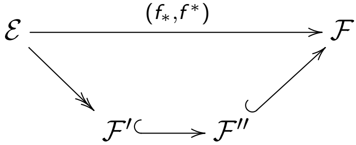
Factorization

In Toposes:

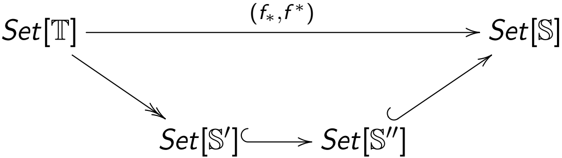


Factorization

In Toposes:



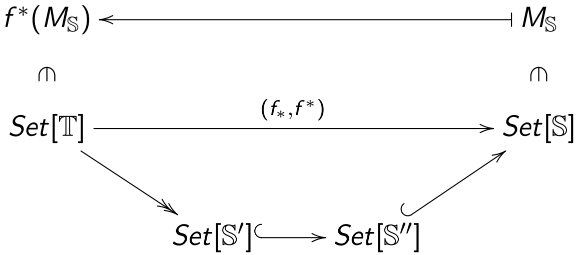
For classifying toposes:



$\mathbb{S}' \supseteq \mathbb{S}'' \supseteq \mathbb{S}$ axiomatic extensions over the same signature.

Factorization

For classifying toposes:



$\mathbb{S}' \supseteq \mathbb{S}'' \supseteq \mathbb{S}$ axiomatic extensions over the same signature, namely:

$\mathbb{S}' = \{$geometric sequents satisfied by $f^*(M_{\mathbb{S}})\} = Th_{geom}(f^*(M_{\mathbb{S}}))$

$\mathbb{S}'' = \{$negations of geometric formulas satisfied by $f^*(M_{\mathbb{S}})\} \cup \mathbb{S}$
$\quad = Th_{\neg\text{-}geom}(f^*(M_{\mathbb{S}})) \cup \mathbb{S}$

About the proof

Definition: A κ-*geometric morphism* is a geometric morphism (f_*, f^*) such that f^* preserves κ-small limits.

Fact: A κ-accessible category $\mathcal{A}$ is the category of *Set*-valued models of the κ-geometric theory $\mathbb{T}$ classified by the topos $Set^{\mathcal{A}_\kappa^{op}}$:

$$\mathcal{A} \simeq Mod(\mathbb{T}) \simeq \kappa\text{-}geom(Set, Set^{\mathcal{A}_\kappa^{op}})$$

About the proof

$$\mathcal{A} \simeq Mod(\mathbb{T}) \simeq \kappa\text{-}geom(Set, Set^{\mathcal{A}_\kappa^{op}})$$

The hypotheses ensure that the functor $F \colon \mathcal{A} = Mod(\mathbb{T}) \to Mod(\mathbb{S}) = \mathcal{B}$ is induced by composing with a κ-geometric morphism

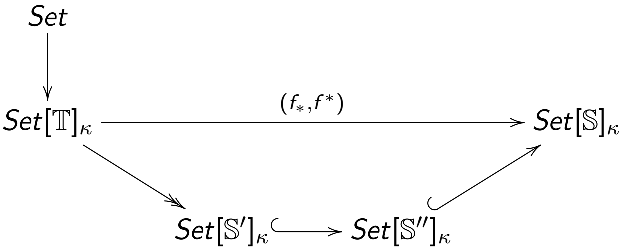
$$Set[\mathbb{T}]_\kappa := Set^{\mathcal{A}_\kappa^{op}} \to Set^{\mathcal{B}_\kappa^{op}} =: Set[\mathbb{S}]_\kappa$$

$$\begin{array}{ccc} Set & & \\ \downarrow & & \\ Set[\mathbb{T}]_\kappa & \xrightarrow{(f_*, f^*)} & Set[\mathbb{S}]_\kappa \end{array}$$

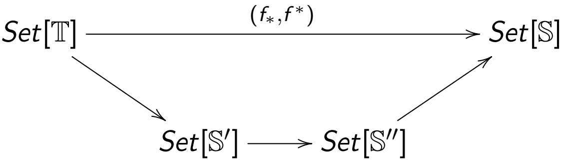
About the proof

$$\mathcal{A} \simeq Mod(\mathbb{T}) \simeq \kappa\text{-}geom(Set, Set^{\mathcal{A}_\kappa^{op}})$$

Factorize this morphism as



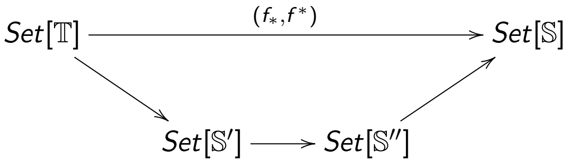
About the proof



where $\mathbb{S}' \supseteq \mathbb{S}'' \supseteq \mathbb{S}$ are axiomatic extensions over the same signature, namely:

$$\mathbb{S}' := Th_{\kappa\text{-}geom}(F(\mathcal{A}))$$
$$\mathbb{S}'' := Th_{\neg\text{-}\kappa\text{-}geom}(F(\mathcal{A})) \cup \mathbb{S}$$

About the proof



where $\mathbb{S}' \supseteq \mathbb{S}'' \supseteq \mathbb{S}$ are axiomatic extensions over the same signature, namely: $\mathbb{S}' := Th_{\kappa\text{-}geom}(F(\mathcal{A}))$, $\mathbb{S}'' := Th_{\neg\text{-}\kappa\text{-}geom}(F(\mathcal{A})) \cup \mathbb{S}$

The conditions on F_κ ensure

– in case **(a)**: that the 2nd and 3rd morphisms are equivalences
– in case **(b)**: that the 1st morphism is an equivalence
– in case **(c)**: that the 3rd morphism is an equivalence

About the proof

The factorization is a κ-geometric variant of the

surjection - dense inclusion - closed inclusion

factorization from topos theory.

About the proof

The factorization is a κ-geometric variant of the

surjection - dense inclusion - closed inclusion

factorization from topos theory.

This factorization becomes computable, and the conditions on F_κ exploitable, because

1. the toposes are *presheaf toposes*
2. the geometric morphisms are "essential"
   (i.e. induced by functors between the index categories)

(using joint work with Eduardo Ochs)

Both 1. and 2. are made possible by passage from geometric to κ-geometric morphisms!

κ-geometric morphisms

Recall: A *κ-geometric morphism* is a geometric morphism (f_*, f^*) such that f^* preserves κ-small limits.

Proposition(A.)**:** For a κ-geometric morphism (f_*, f^*) the above factorization yields κ-geometric morphisms.

κ-geometric morphisms

Recall: A *κ-geometric morphism* is a geometric morphism (f_*, f^*) such that f^* preserves κ-small limits.

Proposition(A.)**:** For a κ-geometric morphism (f_*, f^*) the above factorization yields κ-geometric morphisms.

How to compute the factorization?

A functor between small categories $f\colon C \to D$ yields a κ-geometric morphism

$$Set^C \underset{-\circ f}{\overset{Ran_f}{\rightleftarrows}} Set^D$$

A functor between small categories $f\colon C \to D$ yields a κ-geometric morphism

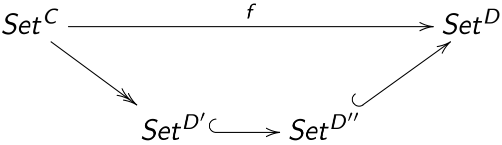
$$Set^C \underset{-\circ f}{\overset{Ran_f}{\rightleftarrows}} Set^D$$

Proposition (joint w/ Eduardo Ochs)**:** For this geometric morphism the factorization is induced by a factorization of f:

$$\begin{array}{ccccc} C & & \xrightarrow{\quad f \quad} & & D \\ & \searrow & & \nearrow & \\ & & D' \hookrightarrow D'' & & \end{array}$$

where
– D' is the full subcategory of D whose objects are in the image of f
– D'' is the full subcategory of D whose objects admit a morphism into the image of f

Thus we get:



where
– D' is the full subcategory of D whose objects are in the image of f
– D'' is the full subcategory of D whose objects admit a morphism into the image of f

When proving the theorem we *are* in this situation:
– $Set[\mathbb{T}]_\kappa \simeq Set^{\mathcal{A}_\kappa}$ is a presheaf category
– $F(\mathcal{A}_\kappa) \subseteq \mathcal{B}_\kappa$ ensures that $Set^{\mathcal{A}_\kappa^{op}} \to Set^{\mathcal{B}_\kappa^{op}}$ is induced by $F_\kappa \colon (\mathcal{A}_\kappa)^{op} \to \mathcal{B}_\kappa^{op}$

Afterthoughts

Advantages of being able to choose $\kappa > \aleph_0$:

– includes all accessible categories into the scope

– ensures that our categories are models of κ-geometric theories *of presheaf type* $\Rightarrow$ can apply the easy factorization theorem.

– often makes sure that κ-presentable objects are preserved

Afterthoughts

Continuations:

– applications

– exploit other factorizations

– ∞-categorical version

– relation to Wehrung's work?