Order-enriched solid functors

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on joint work with Walter Tholen

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Nice

Setting: ordered ($=$ order-enriched) categories and functors

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Examples: Pos, SLat, Frm, ..., ordered varieties, Top_0, ...

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(i) $f \cdot i \leq g \cdot i$

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Dually: coinserters, weighted colimits

Construction of weighted colimits in some categories:

$$\begin{tikzcd}
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$P : \mathcal{A} \to \mathcal{X}$ is a (strongly) order-solid functor if, for every family $\xi = (\xi_i : PA_i \to X)_{i \in I}$, there is $\alpha = (\alpha_i : A_i \to A)_{i \in I}$, and $q : X \to PA$

$$\begin{tikzcd} PA_i \arrow[r, "\xi_i"] \arrow[rr, bend left, dotted, "P\alpha_i"] & X \arrow[r, "q"] & PA \end{tikzcd} \qquad (i \in I)$$

with

(i) $P\alpha = q \cdot \xi$

(ii) (α, A, q) universal with respect to property (i)

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Walter Tholen, Semi-topological functors I, JPAA, in 1979

Tholen and Wischnewsky, Semi-topological functors II, JPAA, 1979

Street, Tholen, Wischnewsky and Wolff, Semi-topological functors III, JPAA, in 1980

In the book of Adámek, Herrlich and Strecker, and in subsequent papers, they are called solid

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$$P \text{ strongly order-solid} \Leftrightarrow P \text{ order-solid.}$$

Open: Do the two notions agree independently of order-faithfulness ?

Theorem

Every strongly order-solid functor $P : \mathcal{A} \to \mathcal{X}$

(a) *is order-faithful;*

(b) *is an order-right adjoint (i.e., r.a. and units are order-P-epic)*

(c) *detects weighted colimits.*

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Theorem

Let $\mathcal{X}$ have inserters. An ordered functor $P : \mathcal{A} \to \mathcal{X}$ is strongly order-solid if and only if

(a) P is solid as an ordinary functor;

(b) $\mathcal{A}$ has inserters and P preserves them;

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Open: ordinary solid + order-right adjoint + order-faithful

$$\Downarrow$$

strongly order-solid ?

Theorem

An ordered functor $P : \mathcal{A} \to \mathcal{X}$ is strongly order-solid iff P is order-right adjoint, and there exists a class $\mathcal{E}$ of order-epimorphisms in $\mathcal{A}$ such that:

(a) *All adjunction co-units lie in $\mathcal{E}$;*

(b) *The pushout of a morphism of $\mathcal{E}$ along any morphism exists in $\mathcal{A}$ and belongs to $\mathcal{E}$;*

(c) *The wide pushout (that is, the cointersection) of any (possibly large) family of morphisms in $\mathcal{E}$ with common domain exists in $\mathcal{A}$ and belongs to $\mathcal{E}$.*

Examples of strongly order-solid functors

$$\mathsf{Top}_0 \xrightarrow{\quad S \quad} \mathsf{Pos}$$

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$$SA_i \xrightarrow{\ \xi_i\ } X \xrightarrow{\mathrm{id}} (X, \tau) \xrightarrow{T_0\text{-reflection}} \overline{X}$$

$\tau =$ down sets U whose pre-image by ξ_i is open in A_i for all $i \in I$

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$$UA_i \xrightarrow{\quad \xi_i \quad} X \xrightarrow[\text{}]{\substack{\text{reflection}\\ \text{in Frm}\\ r}} DX \xrightarrow{\quad q \quad} \frac{DX}{\sim}$$

$\sim$ is the least congruence in DX with which we obtain frame homomorphisms $q \cdot r \cdot \xi_i$, for all i.

Examples of strongly order-solid functors

$$\mathsf{SLat} \xrightarrow{V} \mathsf{Pos}$$

$$VA_i \xrightarrow{\xi_i} X \xrightarrow[\text{in SLat}]{\text{reflection}} EX \xrightarrow{q} \frac{EX}{\sim}$$

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In particular,

$$\begin{tikzcd} \mathsf{Frm} \arrow[r] \arrow[rr, bend left] & \mathsf{SLat} \arrow[r] & \mathsf{Pos} \end{tikzcd}$$

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The ordered algebraic functor induced by any morphism of order-varietal algebraic theories

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Example: $\mathsf{Frm} \to \mathsf{SLat}$

Preordered setting: hom-sets equipped with a preorder

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Proposition

In the commutative diagram of preordered functors

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, "H"] \arrow[d, "P"'] & \mathcal{A}' \arrow[d, "P'"] \\ \mathcal{X} \arrow[r, hook, "J"'] & \mathcal{X}' \end{tikzcd}$$

with H and J full emdeddings and H a preorder-right adjoint

$$\begin{aligned} P' \textit{ strongly preorder-solid } &\Rightarrow \; P \textit{ strongly preorder-solid} \\ &\Rightarrow \; P \textit{ strongly order-solid, if ordered} \end{aligned}$$

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Example:

$$\begin{tikzcd} \mathsf{Top}_0 \arrow[r, hook] \arrow[d, "S"'] & \mathsf{Top} \arrow[d, "S'"] \\ \mathsf{Pos} \arrow[r, hook] & \mathsf{PrOrd} \end{tikzcd}$$

The example of ordered vector spaces

Ordered vector space: $+ : V \times V$ and $\lambda - : V \to V$, $\lambda \geq 0$, are monotone

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Two possible orders:

$$
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V
\end{tikzcd}
$$

- if $f(x) \leq g(x)$, for all $x \in PV$
- if $f(x) \leq g(x)$, for all $x \in V$ (equivalently, $f = g$,
  since $f(x) \leq g(x) \Rightarrow f(-x) \geq g(-x)$)

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The functor

$$P : \mathcal{OV}ec \longrightarrow \mathsf{Pos}$$
$$V \ \mapsto \ PV = \text{ positive cone}$$

is strongly order-solid.

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It fails to preserve inserters: for $\mathbb{R} \underset{\mathrm{id}}{\overset{2-}{\rightrightarrows}} \mathbb{R}$, the inserter in Pos is $\mathbb{R}_0^+$,

but in $\mathcal{OV}ec_=$ it is just $\{0\}$.