Stone dualities between étale categories and restriction semigroups

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Joint work with Mark V. Lawson

TACL 2019

Nice, France

June 21, 2019

The talk reports on the research published in the paper:

Ganna Kudryavtseva, Mark V. Lawson,
A perspective on non-commutative frame theory, Adv. Math. **311** (2017), 378 – 468.

which extends and unifies ideas of the papers:

- Pedro Resende, *Etale groupoids and their quantales*, *Adv. Math.* **208** (2007), 117 – 170.
- Mark V. Lawson, Daniel H. Lenz, *Pseudogroups and their étale groupoids*, *Adv. Math.* **244** (2013), 147 – 209.

Pseudogroups as non-commutative frames

- Pseudogroup - inverse semigroup S such that $E(S)$ is a frame and any compatible set of elements in S has a join in S.
- Prototypical example: pseudogroup of homeomorphisms between open sets of a topological space.
- Aim: extend classical dualities (P. Johnstone, Stone spaces) from frames to pseudogroups, with locales (resp. topological spaces) replaced by étale localic (resp. topological) groupoids.
- Pedro Resende 2007 (equivalence between pseudogroups and groupoids, mediated by quantales, at the level of objects).
- Mark Lawson and Daniel Lenz 2013 (equivalence between pseudogroups and groupoids, objects + morphisms).
- GK and Mark Lawson 2017 (all above equivalences made functorial, four natural types of morphisms considered).

Groupoids replaced by categories

- ‘Non-commutative frame’: does one really need the structure of an inverse semigroup? In particular, is the presence of inverses of crucial importance?
- We drop inverses and get a category, rather than a groupoid.
- This generalizes and simplifies at the same time! No inverses - less structure - easier constructions.
- This is connected with non-selfadjoint operator algebras: certian subalgebras of AF C^*-algebras are classified by ‘topological binary relations’ (=principal étale categories), Power (1990); Hopenwasser, Peters, and Power (2008).
- The appropriate replacement of inverse semigroups are restriction semigroups.

Restriction and Ehresmann semigroups

- Restriction semigroups are non-regular generalizations of inverse semigroups.
- An inverse semigroup S is restriction with $s^* = s^{-1}s$ and $s^+ = ss^{-1}$.
- C. Hollings, From right PP monoids to restriction semigroups: a survey (2009).
- Restriction semigroups were first considered (according to the above survey) by A. El-Qallali in 1980.
- Ehresmann semigroups are more general than restriction semigroups and were introduced by Mark Lawson in 1991.
- An important example: X a non-empty set, $A \subseteq X \times X$ a transitive and reflexive relation. Then the powerset $\mathcal{P}(A)$ is an Ehresmann semigroup, and injective maps in $\mathcal{P}(A)$ form a restriction semigroup.

Restriction semigroups: definition

A restriction semigroup is an algebra $(S; \cdot, ^*, ^+)$ of type $(2, 1, 1)$ such that $(S, \cdot)$ is a semigroup and the following axioms hold

$$xx^* = x, x^*y^* = y^*x^*, (xy^*)^* = x^*y^*, x^*y = y(xy)^*; \qquad (1)$$

$$x^+x = x, x^+y^+ = y^+x^+, (x^+y)^+ = x^+y^+, xy^+ = (xy)^+x; \qquad (2)$$

$$(x^+)^* = x^+, (x^*)^+ = x^*. \qquad (3)$$

Semilattice of projections of S:

$$E = \{x^* \colon x \in S\} = \{x^+ \colon x \in S\}.$$

Natural partial order:

$$a \leq b \Leftrightarrow a = eb \text{ for some } e \in E \Leftrightarrow a = bf \text{ for some } f \in E.$$

An Ehresmann semigroup: equations $x^*y = y(xy)^*$ and $xy^+ = (xy)^+x$ are not required to hold.

Complete restriction monoids

- S - restriction semigroup, $a, b \in S$.
- a and b are compatible if $ab^* = ba^*$ and $b^+a = a^+b$. Write $a \sim b$.

- S is called a complete restriction monoid if E is a frame and joins of compatible families of elements exist in S.
- Key example: Let $C = (C_1, C_0)$ be an étale localic (or topological) category. Then the set of all its local bisections forms a complete restriction monoid. Moreover, $O(C_1)$ is an Ehresmann semigroup (with additional structure).
- Corollary: Let $C = (C_1, C_0)$ be an étale localic (or topological) groupoid. Then the set of all its local bisections forms a pseudogroup.

Quantales

A quantale $(Q, \leq, \cdot)$ is a sup-lattice $(Q, \leq)$ equipped with a binary multiplication operation $\cdot$ such that multiplication distributes over arbitrary suprema:

$$a(\bigvee_{i \in I} b_i) = \bigvee_{i \in I}(ab_i) \text{ and } (\bigvee_{i \in I} b_i)a = \bigvee_{i \in I}(b_i a).$$

A quantale is unital if there is a multiplicative unit e and involutive, if there is an involution $*$ on Q which is a sup-lattice endomorphism.

A quantal frame is a quantale which is also a frame.

Quantale from a localic category

Let $C = (C_1, C_0)$ be a localic (or topological) category and assume that the multiplication map $m\colon C_1 \times_{C_0} C_1 \to C_1$ is semiopen. Then there is the direct image map

$$m_!\colon O(C_1 \times_{C_0} C_1) \to O(C_1)$$

and

$$m_! q\colon O(C_1) \otimes O(C_1) \to O(C_1)$$

is a 'globalization' of multiplication from points to open sets (and similarly for a topological category).

- $O(C_1)$ is a quantale

Quantale from a localic category

Let $C = (C_1, C_0; d, r, u, m)$ be a localic category with maps d, r, u open and m semiopen.

- $a^* = u_! d_!(a)$, $a \in O(C_1)$
- $a^+ = u_! r_!(a)$, $a \in O(C_1)$
- $e = u_!(1_{O(C_0)})$

Theorem (Correspondence Theorem)

1. $(O(C_1), e, {}^+, {}^*)$ is a multiplicative Ehresmann quantal frame.
2. Any multiplicative Ehresmann quantal frame arises in this way.

Partial isometries

- Q – an Ehresmann quantal frame
- $a \in Q$
- a is a partial isometry if $b \leq a$ implies that $b = af = ga$ for some $f, g \leq e$
- Notation: $\mathcal{PI}(Q)$
- Partial isometries are abstract analogues of local bisections.

Example

X a non-empty set, $A \subseteq X \times X$ a transitive and reflexive relation. The partial isometries of the Ehresmann quantal frame $\mathcal{P}(A)$ are precisely partial bijections.

The equivalences

- A localic category $C = (C_1, C_0)$ is étale if u, m are open and d, r are local homeomorphisms.
- An Ehresmann quantal frame Q is a restriction quantal frame if every element is a join of partial isometries and partial isometries are closed under multiplication.

Theorem

The following categories are equivalent:

- Complete restriction monoids
- Restriction quantal frames
- Étale localic categories

Corollary

The following categories are equivalent:

- Pseudogroups
- Inverse quantal frames
- Étale localic groupoids

Morphisms

A morphism $\varphi : Q_1 \to Q_2$ between Ehresmann quantal frames is a quantale map that is also a map of Ehresmann monoids (preserves both $*$ and $+$).

We consider the following four types of morphisms between Ehresmann quantal quantal frames:

- type 1: morphisms;
- type 2: proper morphisms (unital morphism=preserves the top element);
- type 3: $\wedge$-morphisms (preserves non-empty finite meets);
- type 4: proper $\wedge$-morphisms (preserves all finite meets).

Morphisms between respective quantal localic categories are defined as the above morphisms but going in the opposite direction.

Only type 4 morphisms give rise to functors between categories!

Morphisms between complete restriction monoids are restrictions to partial isometries of morphisms between restriction quantal frames.

The adjunction

Theorem

There is an adjunction between the category of étale localic categories and the category of étale topological categories.

This adjunction extends the classical adjunction between locales and topological spaces.

Corollary

There is a dual adjunction between the category restriction quantal frames and the category of étale topological categories.

This adjunction extends the classical dual adjunction between frames and topological spaces.

Relational covering morphisms

Let $C = (C_1, C_0)$ and $D = (D_1, D_0)$ be étale topological categories. A relational covering morphism from C to D as a pair $f = (f_1, f_0)$, where

- $f_0 : C_0 \to D_0$ is a continuous map,
- $f_1 : C_1 \to \mathcal{P}(D_1)$ is a function,

and the following axioms are satisfied:

(RM1) If $b \in f_1(a)$ where $a \in C_1$ then $d(b) = f_0 d(a)$ and $r(b) = f_0 r(a)$.

(RM2) If $(a, b) \in C_1 \times_{C_0} C_1$ and $(c, d) \in D_1 \times_{D_0} D_1$ are such that $c \in f_1(a)$ and $d \in f_1(b)$ then $cd \in f_1(ab)$.

(RM3) If $d(a) = d(b)$ (or $r(a) = r(b)$) where $a, b \in C_1$ and $f_1(a) \cap f_1(b) \neq \varnothing$ then $a = b$.

(RM4) If $p = f_0(q)$ and $d(s) = p$ (resp. $r(s) = p$) where $q \in C_0$ and $s \in D_1$ then there is $t \in C_1$ such that $d(t) = q$ (resp. $r(t) = q$) and $s \in f_1(t)$.

(RM5) For any $A \in O(D_1)$: $f_1^{-1}(A) = \{x \in C_1 : f_1(x) \cap A \neq \varnothing\} \in O(C_1)$.

(RM6) $uf_0(t) \in f_1 u(t)$ for any $t \in C_0$.

(RM2) - weak form of preservation of multiplication;
(RM3) and (RM4) – f_1 is star-injective and star-surjective;

(RM5) – f_1 is a lower-semicontinuous relation.

From quantale to topological morphisms

Let $C = (C_1, C_0)$ and $D = (D_1, D_0)$ be étale localic categories and $f_1^*\colon \mathcal{O}(D) \to \mathcal{O}(C)$ a morphism of restriction quantal frames.

Theorem

- If f_1^* is of type 1 then $\mathrm{Pt}(f_1)$ is a relational covering morphism.
- If f_1^* is of type 2 (=proper=unital) then $\mathrm{Pt}(f_1)$ is at least single-valued relational covering morphism.
- If f_1^* is of type 3 (preserves non-empty finite meets) then $\mathrm{Pt}(f_1)$ is at most single valued relational covering morphism.
- If f_1^* is of type 4 (preserves finite meets) then $\mathrm{Pt}(f_1)$ is a single-valued relational covering morphism.

Sober-spatial equivalences

- Let $C = (C_1, C_0)$ be an étale localic category. Then the locale C_1 is spatial iff the locale C_0 is spatial. If these hold C is called spatial.
- Let $C = (C_1, C_0)$ be an étale topological category. Then the space C_1 is sober iff the space C_0 is sober. If these hold C is called sober.

Theorem

The category of spatial étale localic categories is equivalent to the category of sober étale topological categories.

Spectral, coherent and Boolean categories

- An étale localic category $C = (C_1, C_0)$ is called coherent (resp. strongly coherent) if the locale C_0 (resp. C_1) is coherent.
- There is an equivalence of categories between coherent (resp. strongly coherent) étale localic categories and distributive restriction semigroups (resp. distributive restriction $\wedge$-semigroups).
- An étale topological category $C = (C_1, C_0)$ is called spectral (resp. strongly spectral) if the space C_0 (resp. C_1) is spectral.
- An étale topological category $C = (C_1, C_0)$ is called Boolean (resp. strongly Boolean) if the space C_0 (resp. C_1) is Boolean (locally compact).

The topological duality theorem

Topological duality theorems

- The category of distributive restriction semigroups (resp. $\wedge$-semigroups) is dual to the category of spectral (resp. strongly spectral) étale topological categories.
- The category of Boolean restriction semigroups (resp. $\wedge$-semigroups) is dual to the category of Boolean (resp. strongly Boolean) étale topological categories.

Remark. All results above have corollaries with

- restriction semigroups replaced with inverse semigroups and categories replaced by groupoids.

Summary of topological dualities (inverse semigroups)

Algebraic object	Topological étale groupoid $C = (C_1, C_0)$
Spatial pseudogroup	C_0 or C_1 (and then both) sober
Coherent pseudogroup	C_0 coherent
Strongly coherent pseudogroup	C_1 coherent
Distributive inv. sem.	C_0 – spectral
Distributive $\wedge$ inv. sem.	C_1 (and thus also C_0) spectral
Boolean inv. sem.	C_0 – Boolean
Boolean $\wedge$ inv. sem.	C_1 (and thus also C_0) Boolean

The results remain valid also in a wider setting: if ‘inverse semigroup’ is appropriately replaced by ‘restriction semigoup’ and ‘groupoid’ by ‘category’.

More non-commutative dualities (and adjunctions)

- Étale spaces (=sheaves) over Boolean spaces are dual to skew Boolean algebras (GK, 2012)
- Hausdorff étale spaces over Boolean spaces are dual to skew Boolean $\wedge$-algebras (GK, 2012; Bauer and Cvetko Vah, 2013)
- Étale spaces over Priestley spaces are dual to distributive skew lattices (Bauer, Cvetko Vah, Gehrke, van Gool and GK, 2013)
- There are dual adjunctions between skew Boolean algebras and Boolean spaces induced by dualizing objects $\{0, \dots n\}$, for each $n \geq 1$ (GK, 2013) (for $n = 1$ this is the classical Stone duality)