Pierce stalks in preprimal varieties

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(BFC) ***V** has Boolean factor congruences, i.e., the set of factor congruences of any algebra in $\mathcal{V}$ is a Boolean sublattice of its congruence lattice.*

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$\mathbf{Z}(A) = (Z(A), \vee_A, \wedge_A, ^{c_A}, \vec{0}, \vec{1})$ is a Boolean algebra which is isomorphic to $(FC(A), \vee, \cap, ^*, \Delta^A, \nabla^A)$.

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The Fraser-Horn Property

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We say that a variety has the Fraser-Horn property (FHP) [4] if every congruence on a (finite) direct product of algebras factorizes.

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Lemma

Let $\mathcal{V}$ be a variety with $\vec{0}$ and $\vec{1}$ with the FHP such that $\mathbb{P}_u(\mathcal{V}_{SI}) \subseteq \mathcal{V}_{DI}$. Then, the property "$\vec{e} \in Z(A)$" is definable in $\mathcal{V}$ with a single first order formula.

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2. *The homomorphisms in $\mathcal{V}$ preserve central elements.*

Theorem

Let $\mathcal{L}$ be a language of algebras with at least a constant symbol. Let $\mathcal{V}$ be a variety of $\mathcal{L}$-algebras with the FHP. Suppose that there is a universal class $\mathcal{F} \subseteq \mathcal{V}_{DI}$ such that every member of $\mathcal{V}$ is isomorphic to a global subdirect product with factors in $\mathcal{F}$. Then there exists a $(n+2)$-ary term $u(x, y, \vec{z})$ and 0-ary terms $0_1, \ldots, 0_n, 1_1, \ldots, 1_n$ such that

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4. *$\sigma \neq P^h$.*

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Pierce stalks: Proper equivalence relations

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