The undecidability of profiniteness

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Topological algebras

algebra = set + (finitely many) finitary operations

topological algebra = topological space + (finitely many) finitary continuous operations

A topological space is Boolean if it is Hausdorff, compact, totally disconnected.

Examples of Boolean topological spaces.

- 1-point compactification of discrete spaces: $(X \cup \{\infty\}, \mathcal{T})$
  X a set, $\infty \notin X$,
  $O \in \mathcal{T}$ iff $O \subseteq X$ or ($\infty \in \mathcal{T}$ and $X - O$ is finite).
- Cantor space, or more generally
- a closed subspace of $\prod_{i \in I}(X_i, \mathcal{P}(X_i))$, where X_i are finite

Fact: All Boolean topological spaces are as the last one.

Profinite algebras

A topological algebra **A** is profinite iff it is an inverse limit of finite algebras.

Fact

A is profinite iff it is a closed subalgebra of a product of finite algebras $\mathbf{A} \in \mathrm{S_C P}(\text{finite algebras})$

Why profinite algebras?

In language theory (of words or trees):
In profinite algebras we may do implicit limit operations (like Kleene's $*$).
It is crucial for defining varieties of rational languages.

In Galois theory:
Every profinite group is isomorphic to $\mathrm{Gal}(\mathbf{L}/\mathbf{K})$, i.e., to a group of all field automorphisms of $\mathbf{L}$ which fixes elements of $\mathbf{K}$.

Why profinite structures?

In natural dualities:

Schizophrenic object: $\mathbf{A}$ - a finite algebra, $\mathbf{A}_\tau$ a dual, *essentially* the same object.

(Clark, Davey and others)
Sometimes we have a duality

$$\mathsf{SP}^+(\mathbf{A}) \quad \rightleftarrows \quad \mathsf{S_C P}(\mathbf{A}_\tau).$$

Examples:

- Stone duality: $\mathbf{A}$ - 2-element Boolean algebra, $\mathbf{A}_\tau$ - 2-element set.
- Restricted Pontryagin duality: $\mathbf{A} = \mathbb{Z}_m$, $\mathbf{A}_\tau = \mathbb{Z}_m$.
- Priestley duality: $\mathbf{A}$ - 2-element bounded distributive lattice, $\mathbf{A}_\tau$ - 2-element chain (as an ordered set).

A general problem in duality theory

All objects in the dual category $\mathsf{S_C P}(\mathbf{A}_\tau)$ are profinite.
How to describe them?

- Stone duality: Just Boolean topological spaces.
- Restricted Pontryagin duality: Boolean topological abelian groups of exponent m.
- Priestley duality: Priestley spaces - not definable in FO-logic among Boolean topological ordered sets (Stralka and others)!

More examples

- Every Boolean topological group is profinite
- Every Boolean topological semigroup is profinite
- Every Boolean topological ring is profinite
- Every Boolean topological distributive lattice is profinite
- Every Boolean topological Heyting algebra is profinite

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But

- $(\mathbb{N}, x \mapsto \max(x-1, 0))$, with a topology given by one-point compactification of $\mathbb{N} - \{0\}$, is *not* profinite
- Every infinite subdirectly irreducible algebra is not profinite

Why?

Why there are so many profinite algebras?

FDSC

T_x the set of terms $t(x, \bar{p})$ with a distinguished variable x.
For an eqivalence θ on A let
$\mathrm{syn}(\theta)$ be a largest congruence on $\mathbf{A}$ contained in θ.

Definition

A class $\mathcal{K}$ of algebras has finitelly determined syntactic congruences (FDSC) if there is a finite subset F of T_x for every $\mathbf{A} \in \mathcal{K}$ and every eqivalence θ on A we have

$$\mathrm{syn}(\theta) = \{(a, b) \in A^2 \mid (\forall t(x, \bar{p}) \in F,\ \bar{c} \in A^*)\ (t(a, \bar{c}), t(b, \bar{c})) \in \theta\}.$$

Intuition: is FDSC is a form of a restriction on defining principal congruences. It is equivalent to the term finite definability of principal congruences (TFPC).

Standard classes

A class $\mathcal{K}$ of algebras (quasivariety, variety) is standard if every Boolean topological algebra with the algebraic reduct in $\mathcal{K}$ is an inverse limit of finite algebras from $\mathcal{K}$.

Fact

A variety $\mathcal{V}$ is standard iff every Boolean topological algebra with the algebraic reduct in $\mathcal{V}$ is profinite.

Theorem (Clark, Davey, Freese, Jackson, and many others with weaker versions)

Let $\mathcal{K}$ be a class closed under taking homomorphic images. If $\mathcal{K}$ has FDSC, then it is standard.

Examples of varieties with FDSC

- varieties of groups
- varieties of semigroup
- varieties rings
- the variety of distributive lattices
- varieties of Heyting algebras
- finitely generated congruence distributive varieties (Wang)

An even more general problem

Is there a *way* to decide whether a given class of algebras in standard or has FDSC?

Given a finite axiomatization

Theorem (Jackson '08)

There is no algorithm to decide if a given finite set of identities defines a standard variety or a variety with FDSC.

Given a finite generator: our results

Theorem

There is no algorithm to decide if a given finite algebra of finite type generates a standard variety.

Theorem

There is no algorithm to decide if a given finite algebra of finite type generates a variety with FDSC.

Theorem

There is no algorithm to decide if a given finite algebra of finite type generates a variety $\mathcal{V}$ such that the class of profinite algebras with the algebraic reducts in $\mathcal{V}$ is FO-axiomatizable.

Challenge

How about quasi-varieties?

It is relevant to duality theory.

Main tool

Theorem (McKenzie)

There is an effective procedure which assigns to each Turing machine $\mathcal{T}$ the algebra $\mathrm{A}(\mathcal{T})$ s.t.

- $\mathrm{HSP}(\mathrm{A}(\mathcal{T}))$ has finite residual bound if $\mathcal{T}$ halts.
- A particular infinite subdirectly irreducible algebra $\mathbf{Q}_\omega$ (up to term equivalence) is in $\mathrm{HSP}(\mathrm{A}(\mathcal{T}))$ if $\mathcal{T}$ does not halt.

Consequently, there is no algorithm to decide if a given finite algebra of a finite type generates a variety with a finite residual bound.

Main tool

Theorem (Moore)

There is an effective procedure which assigns to each Turing machine $\mathcal{T}$ the algebra $\mathrm{A}'(\mathcal{T})$ s.t.

- $\mathrm{HSP}(\mathrm{A}'(\mathcal{T}))$ has DPSC if $\mathcal{T}$ halts.
- $\mathbf{Q}_\omega$ (up to term equivalence) is in $\mathrm{HSP}(\mathrm{A}'(\mathcal{T}))$ if $\mathcal{T}$ does not halt.

Consequently, there is no algorithm to decide if a given finite algebra generates a variety with DPSC.

Fact

$\mathbf{Q}_\omega$ admits a Boolean topology. Thus $\mathrm{HSP}(\mathrm{A}(\mathcal{T}))$ and $\mathrm{HSP}(\mathrm{A}'(\mathcal{T}))$ are *not* standard when $\mathcal{T}$ does not halt.

Defining principal congruences

A congruence formula is a pp-formula (existentially quantified conjunction of atomic formulas) $\pi(u, v, x, y)$ such that

$$\models (\forall u, v, x)\ \pi(u, v, x, x) \to u \approx v$$

$\mathcal{V}$ has definable principal congruences (DPC) if there is a *finite* set Π of congruence formulas such that for every $\mathbf{A} \in \mathcal{V}$ and $a, b, c, d \in A$ we have

$$(c, d) \in \mathrm{cg}(a, b) \quad \text{iff} \quad (\exists \pi \in \Pi)\ \mathbf{A} \models \pi(c, d, a, b).$$

Fact

FDSC is a weakenings of DPC.

There are other weakenings of DPC.

Defining principal subcongruences

Definition (Baker, Wang)

$\mathcal{V}$ has definable principal subcongruences (DPSC) if there is a *finite* set Π of congruence formulas such that for every $\mathbf{A} \in \mathcal{V}$ and $a, b \in A$, $a \neq b$, there are $c, d \in A$, $c \neq d$, s.t.

$$(\exists \pi \in \Pi)\ \mathbf{A} \models \pi(c, d, a, b)$$

and for every $e, f \in A$ we have

$$(e, f) \in \operatorname{cg}(c, d) \quad \text{iff} \quad (\exists \pi \in \Pi)\ \mathbf{A} \models \pi(e, f, c, d).$$

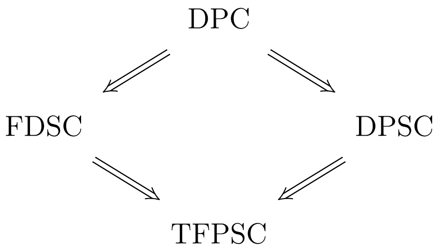
Theorem (Baker, Wang)

Every finitely generated congruence distributive variety has DPSC and, consequently, is finitely axiomatizable.

Question

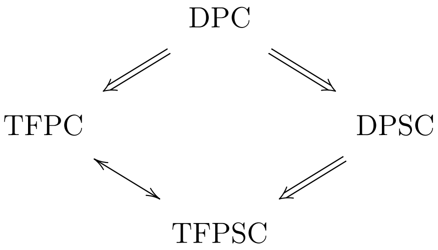
Is there any connection between FDSC and DPSC?

Obviously



TFPCS - obvious generalization od FDSC ans DPSC.

Main new result



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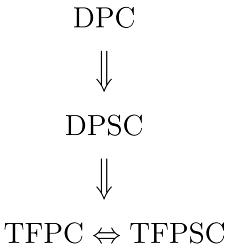

Main corollary

Corollary

For a Turing machine $\mathcal{T}$ let $\mathrm{A}'(\mathcal{T})$ be the algebra from Moore's theorem.

- If $\mathcal{T}$ halts, then $\mathrm{V}(\mathrm{A}'(\mathcal{T}))$ has FDSC.
- If $\mathcal{T}$ does not halt, then the class of profinite algebras with the algebraic reducts in $\mathrm{V}(\mathrm{A}'(\mathcal{T}))$ is not axiomatizable by a set of FO-sentences. Hence $\mathrm{V}(\mathrm{A}'(\mathcal{T}))$ is not standard and does not have FDSC.

The end

This is all Thank you!