General construction of spectra

Overview and perspectives

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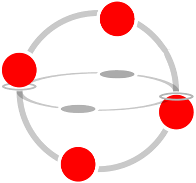


Examples of spectral dualities

Grothendieck duality: commutative rings and locally ringed spaces

$$\mathcal{CR}ing^{op} \underset{Spec}{\overset{\Gamma}{\leftrightarrows}} \mathcal{LRS}paces \quad (\bot)$$

Stone duality: distributive lattices and Stone spaces

$$\mathcal{DL}at^{op} \underset{Spec}{\overset{\mathcal{K}\Omega(-)}{\leftrightarrows}} \mathcal{S}tone \quad (\simeq_{eq})$$

Other examples:

- In algebraic geometry: Pierce spectrum, real spectrum
- Stone-like dualities for boolean algebras, Heyting algebras
- Dubuc & Poveda duality for MV-algebras, dualities for residuated lattices, duality for rigs...

General template

Contravariant adjunction between algebras and spaces:

$$\mathcal{B}^{op} \underset{Spec}{\overset{\Gamma}{\underset{\perp}{\rightleftarrows}}} \mathcal{S}tr\mathcal{S}paces$$

- a category of **algebraic** objects $\mathcal{B} \simeq \mathbb{T}_{\mathcal{B}}[\mathcal{S}et]$
- $\mathcal{S}$et-valued models of an (essentially) algebraic theory
- with a distinguished subcategory of **"local objects"**
- and a **factorization system** $(etale, local)$

- a category of (locally) **structured spaces**
- space-like objects equiped with a sheaf of $\mathcal{B}$-object
- values on opens are in $\mathcal{B}$
- stalks are local objects
- morphisms: underlying continuous maps + morphisms of sheaves with **"local arrows"** at stalks

- *Spec* associates a structured space to each algebra
- Γ reconstructs algebras as global sections of structural sheaves

General template

Geometry is not intrinsic to the category of algebras
Defined relatively to a choice of **local data**:

- **local objects**, models of a geometric theory $\mathbb{T}'$ extending $\mathbb{T}$.
- **local arrows**, behaving as a right class
- **etale arrows**, behaving as a left class

For Grothendieck duality

- $\mathcal{B} = \mathcal{CR}ing$; "structured spaces" = locally ringed spaces
- Local objects = local rings (with unique maximal ideal)
- Local arrows: conservative rings homomorphisms
- Etale arrows: localization of rings

Historic of the construction

- Hakim, 1972: Zariski topos + systematic construction of several geometries for rings
- Johnstone, 1977: first proposal of a general process
- Cole, mid 70' (first published in 2016): admissibility + systematic 2-categorical construction of spectra
- Coste,1979: syntactical interpretation + explicit construction of the spectral site
- Diers, 1981/1984: in term of multiadjonctions
- Taylor, 1998: in term of stable functors
- Dubuc, 2000: axiomatic etale classes
- Lurie, 2009: ∞-categorical synthesis
- Anel, 2009: factorial and topological interpretation

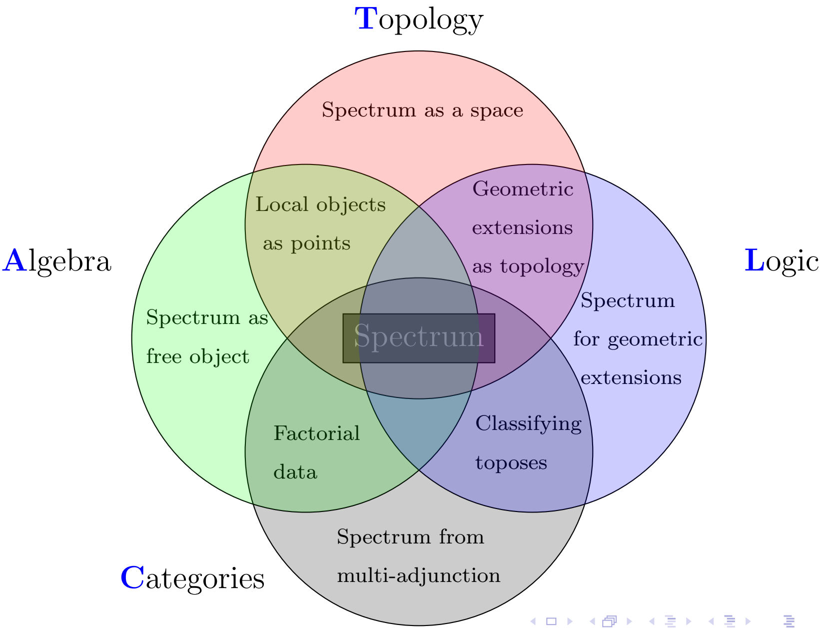
Several methods

Different approaches with unclear links:

- Cole: abstract presentation of admissibility
  Spectrum constructed by 2-limits as a classifying objects
- Coste: syntactical interpretation of Cole's admissibility
  Explicit construction of the spectral site.
- Anel: topological behaviours in the opposite category
- Diers: more divergent, purely categorical approach
  Abstraction of admissibility into multiadjunction
  Spectrum as a space constructed from its points

Our purpose: synthesis and explicit relations of the links between those methods + some additional observations

A multifaceted construction



Context

LFP categories

A locally finitely presentable category $\mathcal{B}$ is a category with:

- small colimits
- a **small** generator $\mathcal{B}_{\aleph_0}$ of **finitely presented objects** generating arbitrary objects under filtered colimits :

$$\mathcal{B} \simeq Ind(\mathcal{B}_{\aleph_0})$$

Category of models of an **essentially algebraic theory** $\mathbb{T}$.
= **cartesian theory** (constructed with $\wedge$ and strict $\exists$)
= **finite-limits theory** (sorts constructed by finite limits)

Syntactic category for $\mathbb{T}$

Syntactic category for $\mathbb{T}$

- Obj: formulas in context $\{\overline{x}, \phi(\overline{x})\}$ in the language of $\mathbb{T}$
- Mor: equivalence classes of functional formulas

$$[\theta(x,y)] : \{\phi, \overline{x}\} \to \{\psi, \overline{y}\} \text{ s.t. } \begin{cases} \theta(\overline{x},\overline{y}) \vdash_{\mathbb{T}} \psi(\overline{y}) \\ \theta(\overline{x},\overline{y}) \wedge \theta(\overline{x},\overline{y'}) \vdash_{\mathbb{T}} \overline{y} = \overline{y'} \\ \phi(\overline{x}) \vdash_{\mathbb{T}} \exists \overline{y} \theta(\overline{x},\overline{y}) \end{cases}$$

$$\mathcal{C}_{\mathbb{T}} \simeq \mathcal{B}_{\aleph_0}^{op} \text{ has finite limits, cf. Gabriel-Ulmer}$$

F.p. objects are determined by presentation formula

- $K = \langle x_1, ..., x_n \rangle / \phi_K(x_1, ..., x_n)$ corresponds to $\{\phi_K, x_1, ..., x_n\}$
- $f : \langle x_1, ..., x_n \rangle / \phi(x_1, ..., x_n) \to \langle y_1, ..., y_m \rangle / \psi(y_1, ..., y_m)$ s.t. $(f(x_i) = \tau_i[y_1, ..., y_m])_{i=1,...,n}$ corresponds to

$$\theta_f(y_1, ..., y_m; x_1, ..., x_n) \Leftrightarrow x_1 = \tau_1[y_1, ..., y_m] \wedge ... \wedge x_n = \tau_n[y_1, ..., y_m]$$

Classifying topos for $\mathbb{T}$

Diaconescu theorem for Lex sites

$$\begin{array}{ccc} \mathcal{C}_{\mathbb{T}} = \mathcal{B}^{op}_{\aleph_0} & \xrightarrow{F\ lex} & Set \\ \downarrow よ & \nearrow F^*\ lex \quad \swarrow F_* & \\ \widehat{\mathcal{B}^{op}_{\aleph_0}} & & \end{array}$$

$$\begin{aligned} \mathcal{B} &\simeq Ind(\mathcal{B}_{\aleph_0}) \\ &\simeq Lex[\mathcal{B}^{op}_{\aleph_0}, Set] \\ &\simeq Geom[Set, \widehat{\mathcal{B}^{op}_{\aleph_0}}] \end{aligned}$$

If $\mathcal{E}$ Grothendieck topos, $\mathbb{T}[\mathcal{E}] = Lex[\mathcal{B}^{op}_{\aleph_0}, \mathcal{E}] \simeq Geom[\mathcal{E}, \widehat{\mathcal{B}^{op}_{\aleph_0}}]$

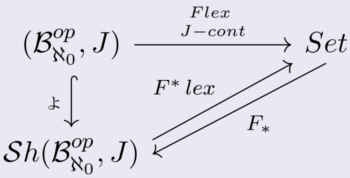
$\mathbb{B} = \widehat{\mathcal{B}^{op}_{\aleph_0}}$ classifies $\mathbb{T}$-models in arbitrary toposes

Geometric extensions

Geometric theory: constructed with $\wedge, \exists, \bigvee$
$\rightarrow$ has a finite-limit part.
A **geometric extension** of $\mathbb{T}$:
$\rightarrow$ a geometric theory $\mathbb{T}'$ whose finite-limit part is $\mathbb{T}$
Corresponds to a topology J on $\mathcal{C}_{\mathbb{T}} = \mathcal{B}_{\aleph_0}^{op}$
Models in $\mathcal{S}et$ are J-continuous Lex functors $F : (\mathcal{B}_{\aleph_0}^{op}, J) \rightarrow \mathcal{S}et$:

$$colim_{i \in I} F(K_i) \overset{\langle F(k_i) \rangle_{i \in I}}{\twoheadrightarrow} F(K)$$

Diaconescu theorem for arbitrary sites



$$\mathbb{T}_J[\mathcal{E}] \simeq Lex_{J-Cont}[(\mathcal{B}_{\aleph_0}^{op}, J), \mathcal{E}]$$
$$\simeq Geom[Set, Sh(\mathcal{B}_{\aleph_0}^{op}, J)]$$

Problem of the free object

Geometric extensions do not have a good notion of free object.
$\rightarrow$ Several locally free $\mathbb{T}'$-models under a given object.

The problem of spectrum

For any B in $\mathcal{B}$ construct:

- a topos $Spec(B)$
- endowed with a free $\mathbb{T}'$-model $\widetilde{B}$ for B

The free model will be

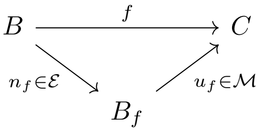
- a sheaf of $\mathbb{B}$-objects in this topos,
- with local objects under B as stalks

Cannot process directly: need to precise **factorization data**
Admissibility relates factorial and geometric data.

Orthogonality and factorization systems

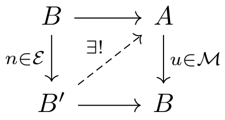
Factorization systems

A pair $(\mathcal{E}, \mathcal{M})$ s.t. any arrows has a unique factorization



Ortogonality structure

A pair $(\mathcal{E}, \mathcal{M})$ s.t. with diagonalization property:



General properties of a factorization system $(\mathcal{E}, \mathcal{M})$

- $\mathcal{E}$ contains iso
- is **post**-absorbant, hence closed by retracts,
- closed by colimits

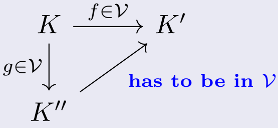
- $\mathcal{M}$ contains iso
- is **pre**-absorbant, hence closed by sections,
- closed by limits

Saturated class

Saturated classes

A saturated class is a $\mathcal{V} \subseteq \overrightarrow{\mathcal{B}_{\aleph_0}}$ closed by:

- composition
- pushouts along f.p. arrows
- post-absorption



In a L.F.P. category

- Orthogonality and factorization systems coincide
- Any factorization system $(\mathcal{E}, \mathcal{M})$ is determined by $\mathcal{E} \cap \mathcal{B}_{\aleph_0}$
- Any saturated class left generates a factorization system $\mathcal{V} \mapsto (Ind(\mathcal{V}), \mathcal{V}^{\perp})$
- Factorization system $\simeq$ saturated classes

Etale and local arrows, admissibility

Coste's admissibility structure

A geometry for $\mathcal{B}$ will be determined by a pair $(\mathcal{V}, J)$ with:

- a $\mathcal{V}$ saturated class determining $(\mathcal{E}t_{\mathcal{V}}, \mathcal{L}oc_{\mathcal{V}})$
- a topology J on $\mathcal{B}^{op}_{\aleph_0}$ **with basic covers in $\mathcal{V}$** (encodding the theory of local objects)

Etale arrows: dual of **open inclusions of the geometry**
$\rightarrow$ will constitute the topological part of spectrum
Local forms in $(\mathcal{V}, J)$: etale arrows toward J-local objects
$\rightarrow$ **points of the geometry**
Etale arrows approximate local forms by filtered colimits
As open neighborhood approximate points
Local arrows: residual, non-topological information
Factorization: separate topological from residual data

Topology on $\mathcal{B}^{op}$

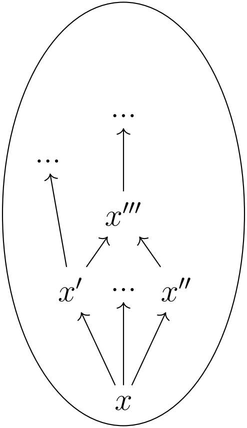
Induced topology

$(\mathcal{V}, J)$ induces a topology on $\mathcal{B}^{op}$
Can transfer J covers under arbitrary objects by pushouts
Define $\widetilde{J}$ whose covers are dual cocones of

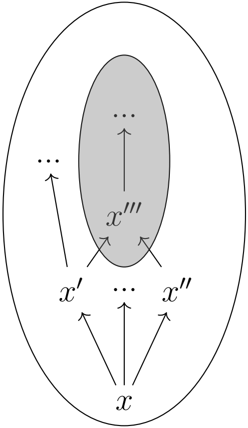
$$(B \xrightarrow{f_i} B_i)_{i \in I} \text{ s.t. } \begin{array}{ccc} B & \longleftarrow & K \\ {\scriptstyle f_i}\downarrow & \lrcorner & \downarrow{\scriptstyle k_i} \\ B_i & \longleftarrow & K_i \end{array}$$

Local objects are $\widetilde{J}$-irreducible $\rightarrow$ lift their own covers

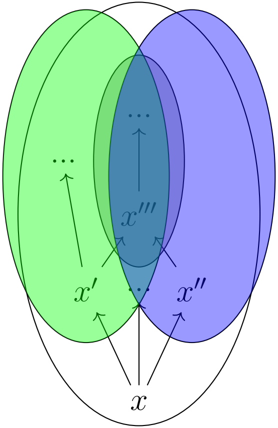
Local objects as focal spaces



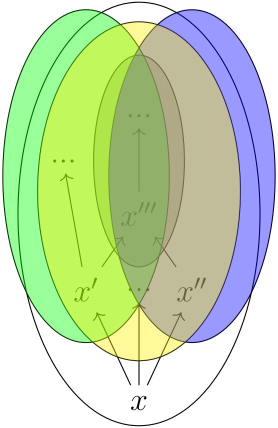
Local objects as focal spaces



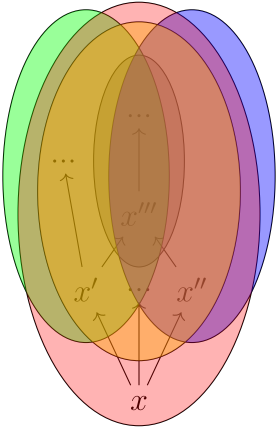
Local objects as focal spaces



Local objects as focal spaces



Local objects as focal spaces



Topological interpretation (Anel)

In $\mathcal{B}^{op}$ etale maps behave as open inclusions

In $\mathcal{B}$ (algebraic side)

Etale arrow $B \xrightarrow{l} C$

Local objects

$\rightarrow$ Lift their own cover:

$$\begin{array}{ccc} A & = & A \\ \downarrow l_i & \overset{\exists r}{\nearrow} & \\ B_i & & \end{array}$$

Cone of local units

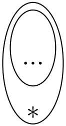
$$\begin{array}{ccccc} & & B & & \\ & \swarrow^{\eta_i} & & \searrow^{\eta_j} & \\ A_i & & \dots & & A_j \end{array}$$

In $\mathcal{B}^{op}$ (spatial site)

Etale open inclusion $C \xrightarrow{l^{op}} B$

Focal spaces

$\rightarrow$ Has a minimal point:



Cocone of focal components

$$\begin{array}{ccccc} & & \text{“}Spec(B)\text{”} & & \\ & \nearrow & & \nwarrow & \\ \uparrow_{\sqsubseteq} \{x_i\} & & \dots & & \uparrow_{\sqsubseteq} \{x_j\} \end{array}$$

Syntactic aspects: etale and local arrows (Coste)

- Arrows in $\mathcal{V}$ "create witnesses of codomain formulas from witnesses of domain formula"
- Local arrows "reflect witnesses of codomain formulas"

$$\left\{\begin{array}{l} \forall f \in \mathcal{V} \\ \forall \overline{a} \in A \text{ s.t. } A \models \phi_f(\overline{a}) \\ \forall \overline{b} \in B \text{ s.t. } B \models \psi_f(\overline{b}) \wedge \theta_f(\overline{g(a)}, \overline{b}) \end{array}\right. \Rightarrow \exists! \overline{c} \in A \left\{\begin{array}{l} A \models \psi_f(\overline{c}) \\ A \models \theta_f(\overline{a}, \overline{c}) \\ \overline{g(c)} = \overline{b} \end{array}\right.$$

$$\begin{array}{ccc} \langle x_1, \ldots, x_n \rangle_\Sigma / \phi_f(x_1, \ldots, x_n) & \xrightarrow{\ulcorner a \urcorner} & A \\ {\scriptstyle f} \downarrow & \overset{\exists! \ulcorner c \urcorner}{\nearrow} & \downarrow {\scriptstyle g} \\ \langle y_1, \ldots, y_m \rangle_\Sigma / \psi_f(y_1, \ldots, y_n) & \xrightarrow[\ulcorner b \urcorner]{} & B \end{array}$$

For Grothendieck duality

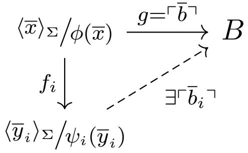
Etale arrows = localizations: create invertible from nonzero
Local arrows = conservative morphisms: reflect invertibility

Syntactic aspects: local objects

$$\mathcal{L}oc_J \simeq \mathbb{T}_J[\mathcal{S}et] \simeq pt(\mathcal{S}h(\mathcal{B}^{op}_{\aleph_0}, J))$$

Covers = disjunctions of cases for witnesses of domain formulas

$$\mathbb{T}_J = \mathbb{T}_{\mathcal{B}} \cup \left\{ \phi(\overline{x}) \vdash \bigvee_{i \in I} \exists \overline{y}_i (\psi_i(\overline{y}_i) \wedge \theta_{f_i}(\overline{y}_i, \overline{x})) \right\}_{(f_i)_{i \in I} \in J(\langle \overline{x} \rangle / \phi(\overline{x}))}$$



If $\overline{b} \in B$ such that $B \models \phi(\overline{b})$ then $\exists i \in I$ and $\overline{b}_i \in B$ s.t. $B \models \psi_i(\overline{b}_i) \wedge \theta_{f_i}(\overline{b}_i, \overline{b}))$

Example of local rings

$$\mathbb{T}_{LocRing} = \mathbb{T}_{CRing} \cup \{x \neq 0 \vdash \exists y(xy = 1) \vee \exists y'((1-x)y' = 1)\}$$

Admissibility for local objects + local arrows

Relates factorial and geometric data

Cole's admissibility

An admissibility structure is the data of:

- a (finite-limits) theory $\mathbb{T}$
- a geometric extension $\mathbb{T}'$
- a class of arrows *Loc* in $\mathbb{T}[Set]$ closed by inverse image, composition and pre-absorption containing iso

such that any arrows from a $\mathbb{T}$ model toward a $\mathbb{T}'$ model admits an **initial** factorization through $\mathbb{T}'$ model **with a local arrow on the right**.

Local and multi right adjoints

Local right adjoint (aka Stable functors)

Let $U : \mathcal{A} \to \mathcal{B}$ a functor:

- U local RAdj if each slice is RAdj: $\mathcal{A}/_A \underset{U/_A}{\overset{L_A}{\underset{\perp}{\leftrightarrows}}} \mathcal{B}/_{U(A)}$
- U is multi-RAdj if any B in $\mathcal{B}$ has a **small** cone of local units

$$(B \overset{\eta_i}{\to} U(A_i))_{i \in I_B}$$

initial in the comma $B \downarrow U$

Multireflection

(Non-full) faithful multi RAdj are (non-full) multireflections.

Multireflection induced by admissibility

Admissibility is encoded by stability

Multireflection from admissibility

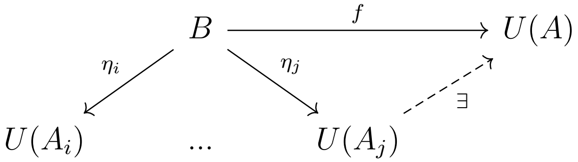
For an admissibility structure $(\mathcal{V}, J)$:

- Local objects are downclosed for local maps: if $u : A \to L$ a local map with L local, then A is local.
- $\mathbb{T}_J[\mathcal{S}et]^{\mathcal{L}oc} \hookrightarrow \mathcal{B}$ is multireflective.

Local units correspond to local forms = points

Multireflection and admissibility

Conversely: multiadjunctions produce admissibility.
Defect of uniqueness of the unit
Universal property of reflection **jointly** assumed by the universal cone.



Taking as local maps the right class generated by $U(\overrightarrow{\mathcal{A}})$:
Stability says that one of the factorization is admissible
Initial amongst those with an arrow in $U(\overrightarrow{\mathcal{A}})$ on the right

Admissibility structure for stable functor

Factorization system for a stable functor

$$(\underbrace{{}^{\perp}U(\overrightarrow{A})}_{\mathcal{D}iag_U}, \underbrace{({}^{\perp}U(\overrightarrow{\mathcal{A}}))^{\perp}}_{\mathcal{L}oc_U})$$

On the right: class generated by arrows in the range of U

On the left: Diers' "diagonally universal morphisms"

Right generated factorization system

Topology of U-localizing families

$$J_U(B) = \left\{ (B \overset{\delta_i}{\to} B_i)^{op}_{i \in I} \mid \forall j \in I_B,\ \exists i \in I \quad \begin{array}{ccc} B & \xrightarrow{\eta_j} & U(A_j) \\ & {\scriptstyle \delta_i}\searrow & \nearrow{\scriptstyle \exists} \\ & B_i & \end{array} \right\}$$

$(J_U \mid_{\aleph_0}, \mathcal{D}iag_U \cap \mathcal{B}_{\aleph_0})$ admissibility structure

Coste and Diers contexts

Comparison of contexts

$$Coste_{\mathcal{B}} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} Diers_{\mathcal{B}}$$

$$(\mathcal{V}, J) \longmapsto U_{\mathcal{V},J} : \mathbb{T}_J[Set]^{Loc_{\mathcal{V}}} \hookrightarrow \mathcal{B}$$

$$(Diag_U \cap \overrightarrow{\mathcal{B}_{\aleph_0}}, J_U \mid_{\aleph_0}) \longleftarrow U : \mathcal{A} \to \mathcal{B}$$

- Coste contexts on $\mathcal{B}$: saturated class $\mathcal{V}$ + topology J generated in $\mathcal{V}$
- $(\mathcal{V}_1, J_1) \leq (\mathcal{V}_2, J_2)$ if
  $J_1 \leq J_2$ and $\mathcal{V}_2 \subseteq \mathcal{V}_1$
  $Et_{\mathcal{V}_2} \subseteq Et_{\mathcal{V}_1}$
  $Loc_{\mathcal{V}_1} \subseteq Loc_{\mathcal{V}_2}$
- Diers contexts on $\mathcal{B}$: $U : \mathcal{A} \to \mathcal{B}$ multiRAdj s.t. local forms are filt.colim of etale arrows
- $U_1 \leq U_2$ if

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{U_1} & \mathcal{B} \\ \big\downarrow & \nearrow_{U_2} & \\ \mathcal{A}_2 & & \end{array}$$

Coste and Diers contexts

Comparison of contexts

$$Coste_{\mathcal{B}} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} Diers_{\mathcal{B}}$$

$$(\mathcal{V}, J) \longmapsto U_{\mathcal{V},J} : \mathbb{T}_J[Set]^{Loc_{\mathcal{V}}} \hookrightarrow \mathcal{B}$$

$$(Diag_U \cap \overrightarrow{\mathcal{B}_{\aleph_0}}, J_U) \longleftarrow U : \mathcal{A} \to \mathcal{B}$$

- Closure of Diers contexts:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{U} & \mathcal{B} \\ {\scriptstyle eso}\downarrow & \nearrow & \\ \mathbb{T}_{J_U|\aleph_0}[Set]^{Loc_U} & & \end{array}$$

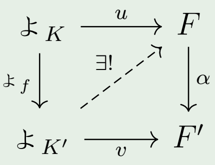
- Same local objects but new etale maps

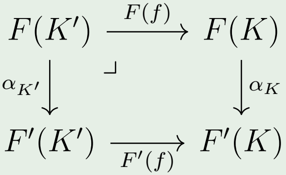
$$(DiagU_{\mathcal{V},J} \cap \mathcal{B}^{op}_{\aleph_0}, J_{U_{\mathcal{V},J}} = J) \leq (\mathcal{V}, J)$$

Local maps in arbitrary toposes

Localness can be expressed by pullback

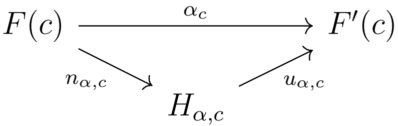
In $\mathbb{T}[\mathcal{S}et] \simeq Ind(\mathcal{B}_{\aleph_0})$





Generalizes to arbitrary toposes

Pointwise factorization in $\mathbb{T}[\mathcal{E}]$: $(\mathcal{E}t_{\mathcal{E}}, \mathcal{L}oc_{\mathcal{E}})$



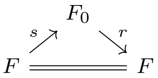
$$H_\alpha = a_{J_{\mathcal{E}}}(c \mapsto H_{\alpha,c})$$

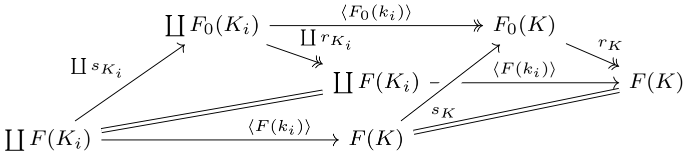
Admissibility in arbitrary toposes

Admissibility is inherited in any arbitrary topos $\mathcal{E}$

- Local objects in $\mathcal{E}$ are "absorbant right to local maps"
- The inclusion $\mathbb{T}_J[\mathcal{E}]^{\mathcal{L}oc} \hookrightarrow \mathbb{T}[\mathcal{E}]$ is multireflective
- In any topos $\mathcal{E}$, a retract of a local object is local

For $F, F_0 : \mathcal{B}^{op}_{\aleph_0} \rightarrow \mathcal{E}$ s.t. F_0 is J-local and





(Locally) modelled topose

Œcumene for $\mathbb{T}$-models

$\mathbb{T}_{\mathcal{B}}\mathcal{T}opos$: $\mathbb{T}_{\mathcal{B}}$-modeled toposes

- Obj: $(\mathcal{E}, E)$ with E in $\mathbb{T}[\mathcal{E}]$
- Arr: $(f, f^\sharp) : (\mathcal{E}, E) \to (\mathcal{F}, F)$ with: $\left\{ \begin{array}{ll} \mathrm{F} \xrightarrow{f} \mathcal{E} & \text{geom.} \\ \mathrm{f}^* E \xrightarrow{f^\sharp} F & \mathbb{T}\text{-morph.} \end{array} \right.$

$\mathbb{T}_{J,\mathcal{V}}\mathcal{L}oc\mathcal{T}opos$: $\mathbb{T}_J$-locally modelled toposes:

- Obj: $(\mathcal{E}, E)$ with each E_x local, $x \in pt(\mathcal{E})$
- Arr: $(f, f^\sharp)$ with $f^\sharp$ in $\mathbb{T}_J$ transformation

$$f^\sharp_x : E_{fx} \to F_x \text{ a local arrow in } \mathbb{T}_J[Set]$$

$$\left.\begin{array}{l} \mathbb{T}_{\mathcal{B}}\mathcal{T}opos = \int \mathbb{T}[-] \\ \mathbb{T}_{J,\mathcal{V}}\mathcal{L}oc\mathcal{T}opos = \int \mathbb{T}_J[-]^{Loc} \end{array}\right\} \text{Indexed categories over } \mathcal{GT}op^{op}$$

Turning admissibility into reflection

The fundamental adjunction

One wants to construct a left adjoint *Spec* to the inclusion

$$\mathbb{T}_{J,\mathcal{V}}\mathcal{L}oc\mathcal{T}opos \underset{w}{\overset{Spec_{\mathcal{V},J}}{\leftrightarrows}} \mathbb{T}_{\mathcal{B}}\mathcal{T}opos \quad (\bot)$$

Consider models jointly, regardless of their base topos
Then admissibility turns into proper reflection
One can construct a free local object under a given $\mathbb{T}$-model
If allowed to change of topos

For models in $\mathcal{S}et$

Adjunction for models in $\mathcal{S}et$

In particular if restricting to models over $\mathcal{S}et$:

$$\mathcal{B}^{op} \underset{Spec_{\mathcal{V},J}}{\overset{\Gamma}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbb{T}_{J,\mathcal{V}}\mathcal{L}oc\mathcal{T}opos$$

Here Γ applies the direct image part of

$$! : Spec(F) \to \mathcal{S}et$$

to the structure sheaf $\widetilde{F}$

Coste's spectrum of a $\mathcal{S}et$-valued model

Spectral site of $B \in \mathcal{B}$

$$\mathcal{V}_B = \left\{ l : B \rightarrow C \;\middle|\; \begin{array}{ccc} B & \xleftarrow{\;f\;} & K \\ {\scriptstyle l}\downarrow & \ulcorner & \downarrow{\scriptstyle k} \\ C & \xleftarrow[\;k_* f\;]{} & K' \end{array} \quad \begin{array}{l} \text{for some } k \in \mathcal{V} \\ \text{and } f : K \rightarrow B \end{array} \right\}$$

$$\underbrace{J_B(l)}_{on\, \mathcal{V}_B^{op}} = \left\{ \left(\begin{array}{ccc} B & \xrightarrow{\;l\;} & A_l \\ & \searrow^{n_i} & \downarrow{\scriptstyle m_i} \\ & & A_{n_i} \end{array} \right)_{i \in I} \;\middle|\; \begin{array}{ccc} B_l & \xleftarrow{\;u\;} & K \\ {\scriptstyle m_i}\downarrow & \ulcorner & \downarrow{\scriptstyle k_i} \\ B_{n_i} & \longleftarrow & K_i \end{array} \right\}$$

Gathers etale arrows under B with relative topology

Coste's spectrum of a $\mathcal{S}et$-valued model

Spectrum of $B \in \mathcal{B}$

$$Spec_{\mathcal{V},J}(B) = \mathcal{S}h(\mathcal{V}_B^{op}, J_B)$$

$\mathcal{V}_B^{op}$ is a Lex site coding for "basic open inclusions"
Etale arrow $\delta : B \to C$ correspond to etale geometric morphisms

$$Spec(\delta) : Spec(C) \simeq Spec(B)/a_{J_B}(よ_\delta) \to Spec(B)$$

Ind-etale maps live in $Spec(B)$ as Ind-objects on $\mathcal{V}_B$

Points of the spectrum

Points

- Points of spectral site of B are local forms under B
- If $B \xrightarrow{l} C$ etale, any point of $Spec(C)$ is a point of $Spec(B)$
- Etale arrows between local objects = specialization order
- If A $\mathcal{S}$et-valued local, $Spec(A)$ local topos

Structural sheaf of $\mathcal{S}et$-valued model

Structural sheaf of B in $\mathcal{B}$

- $\widetilde{B}$ is a distinguished sheaf of $\mathcal{B}$-objects in $Spec(B)$:

$$\widetilde{B} = a_{J_B}((B \xrightarrow{l} C) \longmapsto C)$$

  Sheafification of the Codomain functor
- At stalks: $\widetilde{B}$ returns local objects under B
  Hence $\widetilde{B}$ is a $\mathbb{T}_J$-model in $Spec(B)$
  $\rightarrow$ This is **the free local object under B**

$Spec(B)$ is the good topos over which one can define the free local object for B
$\widetilde{B}$ gathers local forms of B as its stalks.

Coste's spectral site: general case

Definition of $(\mathcal{V}_F^{op}, J_F)$ for $\mathcal{F} = \mathcal{S}h(\mathcal{C}_\mathcal{F}, J_\mathcal{F})$ and F in $\mathbb{T}[\mathcal{F}]$

- Obj: (c, l) with $c \in \mathcal{C}_\mathcal{F}$ and l a morphism in $\mathcal{V}_F(c)$
- Arr: $(s, h) : (c, f) \to (c', f')$ with $\begin{array}{ccc} F(c) & \xrightarrow{f} & B_{c,f} \\ {\scriptstyle F(s)}\downarrow & & \downarrow{\scriptstyle h} \\ F(c') & \xrightarrow{f'} & B_{c',f'} \end{array}$
- J_F jointly generated by:

$$((c, 1_c) \xrightarrow{(s_i, F(s_i))} (c_i, 1_{c_i}))_{i \in I} \text{ with } (c \xrightarrow{s_i} c_i)_{i \in I} \in J_\mathcal{F}^{op}(c)$$

$$((c, l) \xrightarrow{(1_c, h_i)} (c, l_i))_{i \in I} \text{ with } \left(\begin{array}{ccc} F(c) & \xrightarrow{l} & B_{c,l} \\ & \searrow^{f_i} & \downarrow{\scriptstyle h_i} \\ & & B_{c,l_i} \end{array} \right)_{i \in I} \in J_{F(c)}^{op}(l)$$

Relation between the sites

Gluing relation

The site for F is the gluing of the sites for its values:

$$(\mathcal{V}_F^{op}, J_F) = \underset{c \in \mathcal{C}_\mathcal{F}}{colim}(\mathcal{V}_{F(c)}^{op}, J_{F(c)})$$

$$Spec(F) \simeq \underset{c \in \mathcal{C}_\mathcal{F}}{lim} Spec(F(c))$$

Spectral sites for $\mathcal{S}$et-valued models are the building blocks for spectral sites of arbitrary models

Diers construction

Diers quotients $\mathcal{V}_B$ by factorization relation
Diers contexts have enough points : the spectrum is spatial

Spectral space of Diers for a B in $\mathcal{B}$

- Point are the local units $n_i : B \to U(A_i)$ indexed by I_B
- Topology is generated by f.p. etale maps:
  for $l : B \to C \in \mathcal{V}_B$ defines a subset of I_B

$$D(l) = \{n \in I_B \mid l \leq n\}$$

$$D(l : B \to C) \cap D(l' : B \to C') = D(B \to C +_B^{l,l'} C')$$

$(D(l))_{l \in \mathcal{V}_B}$ basis for a topology on I_B

Gets a posite $(\mathcal{V}_B / \sim_{\leq}, J_B)$
Localic reflection of Coste Spectrum

Structural sheaf of Diers Spectrum

Structural sheaf

Defined by left Kan extension + sheafification

$$\begin{array}{ccc} \mathcal{K}\Delta_B & \xrightarrow{Cod} & \mathcal{B} \\ {\scriptstyle D}\big\downarrow & \nearrow_{\overline{B}=Lan_D Cod} & \\ \Omega(Spec_U B)^{op} & & \end{array}$$

$$\overline{B}(U) = \underset{U \subseteq D(\delta)}{colim}\ Cod(\delta)$$

$$\widetilde{B} = (\overline{B})^{++}$$

Stalks are colimits of values on neighborhoods

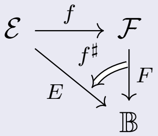
Hence Diers condition of approximability

Modelled toposes as an over 2-category

Models as geometric morphisms

$\mathbb{T}_{\mathcal{B}}\mathcal{T}opos \simeq \mathcal{GT}op/\mathbb{B}$ where $\mathbb{B} = \widehat{\mathcal{B}^{op}_{\aleph_0}}$

- Structural sheaves are geometric morphisms toward the classifier: $(\mathcal{F}, F)$ is a $\mathcal{F} \stackrel{F}{\rightarrow} \mathbb{B}$ in $\mathcal{GT}op$
- A morphism $(\mathcal{F}, F) \stackrel{(f, f^\sharp)}{\rightarrow} (\mathcal{E}, E)$ is a 2-cell in $\mathcal{GT}op$:

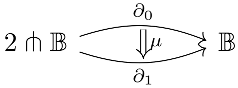


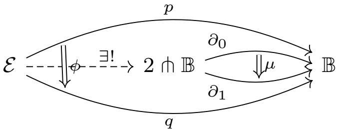
2-cells in $\mathbb{T}_{\mathcal{B}}\mathcal{T}opos$ are inessential, can be seen as 1-cells: $\mathbb{T}_{\mathcal{B}}\mathcal{T}opos$ must be seen as a 1-category.

Usefull 2-limits of toposes

Use 2-limits in $\mathcal{GT}op$ to construct classifying objects

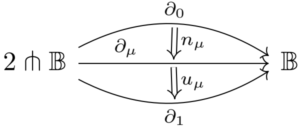
Classifyer of natural transformations:





Can do the same over $\mathbb{B}_J$

Universal factorization of the universal map:



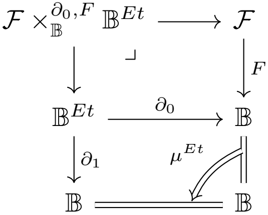
$$\mathbb{B}^{Et} = Inv(u_\mu)$$

$$\mathbb{B}^{Loc} = Inv(n_\mu)$$

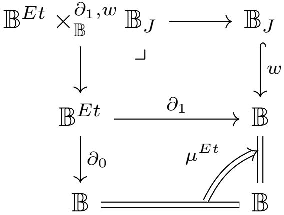
Usefull 2-limits of toposes

Use 2-limits in $\mathcal{GT}op$ to construct classifying objects

Classifier of etale maps under F:



For etale maps to local objects:

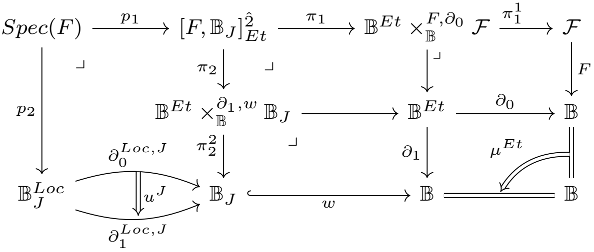


Compose by pullback the classifier of etale map from F to local objects

$$[F, \mathbb{B}_J]^{\hat{2}}_{Et} = \mathcal{F} \times_{\mathbb{B}}^{\partial_0, F} \mathbb{B}^{Et} \times_{\mathbb{B}^{Et}} (\mathbb{B}^{Et} \times_{\mathbb{B}}^{\partial_1, w} \mathbb{B}_J)$$

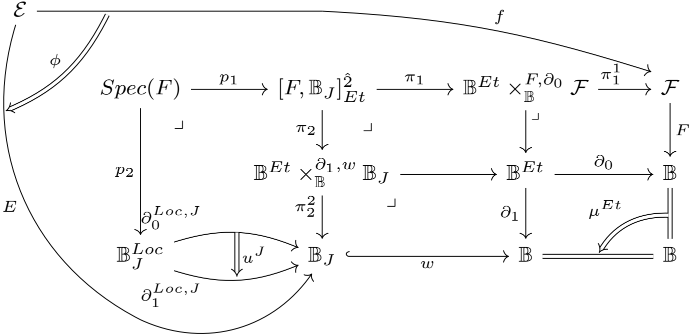
Cole's spectrum

Cole spectrum is constructed by comma and pullbacks
Exhibits $Spec(F)$ as the classifier of admissible factorizations of arrows from F toward a local object



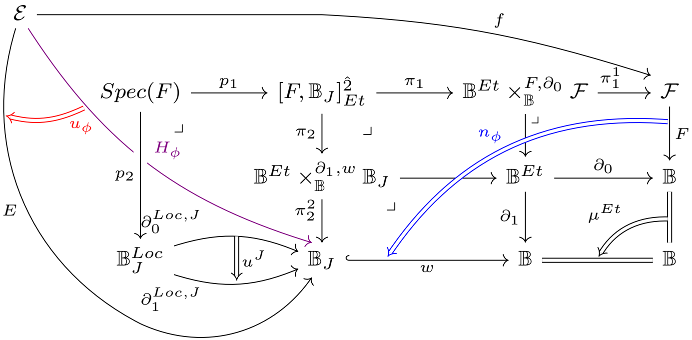
Cole's spectrum

$$\begin{array}{ccc} \mathbb{T}_{J,\mathcal{V}}\mathcal{L}oc\mathcal{T}opos[(Spec(F),\widetilde{F}),(\mathcal{E},E)] & \simeq & \mathbb{T}_{\mathcal{B}}\mathcal{T}opos[(\mathcal{F},F),(\mathcal{E},wE)] \\ (?,?) & \hookleftarrow & (f,\phi) \end{array}$$



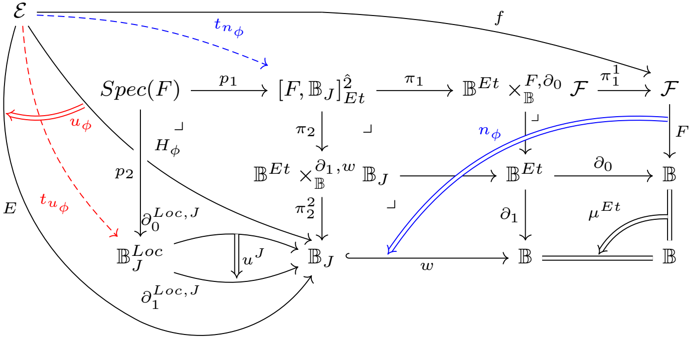
Cole's spectrum

$$\mathbb{T}_{J,\mathcal{V}}\mathcal{L}oc\mathcal{T}opos[(Spec(F),\widetilde{F}),(\mathcal{E},E)] \simeq \mathbb{T}_{\mathcal{B}}\mathcal{T}opos[(\mathcal{F},F),(\mathcal{E},wE)]$$
$$(?,?) \leftarrow\!\shortmid (f,\phi)$$



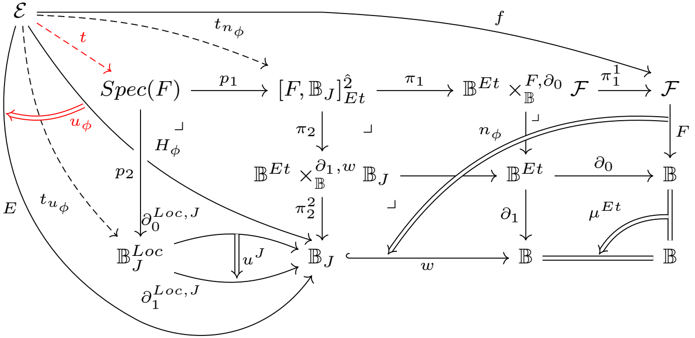
Cole's spectrum

$$\begin{array}{ccc} \mathbb{T}_{J,\mathcal{V}}\mathcal{L}oc\mathcal{T}opos[(Spec(F),\widetilde{F}),(\mathcal{E},E)] & \simeq & \mathbb{T}_{\mathcal{B}}\mathcal{T}opos[(\mathcal{F},F),(\mathcal{E},wE)] \\ (?,?) & \leftarrow\!\!\shortmid & (f,\phi) \end{array}$$



Cole's spectrum

$$\begin{array}{ccc} \mathbb{T}_{J,\mathcal{V}}\mathcal{L}oc\mathcal{T}opos[(Spec(F),\widetilde{F}),(\mathcal{E},E)] & \simeq & \mathbb{T}_{\mathcal{B}}\mathcal{T}opos[(\mathcal{F},F),(\mathcal{E},wE)] \\ ({\color{red}t},{\color{red}u_\phi}) & \leftarrow\!\shortmid & (f,\phi) \end{array}$$



Unit and canonical map

$$(SpecF, \widetilde{F}) \xrightarrow{(t, t^\sharp)} (\mathcal{E}, E)$$

$$t^\sharp = u_\phi : \widetilde{F}t = \partial_0^{Loc,J} \circ p_2 \circ t \Rightarrow E = \partial_1^{Loc,J} \circ t$$

$$\begin{array}{ccc}
(\mathcal{F}, F) & \xrightarrow{(f,\phi)} & (E, wE) \\
{\scriptstyle (\eta_{(\mathcal{F},F)}, \eta^\sharp_{(\mathcal{F},F)})} \big\downarrow & \nearrow {\scriptstyle (wt, w^* u_\phi)} & \\
(SpecF, w\widetilde{F}) & &
\end{array}$$

$$\eta_{(\mathcal{F},F)} = \pi_1^1 \circ \pi_1 \circ p_1 : SpecF \to \mathcal{F}$$

$$\eta^\sharp_{(\mathcal{F},F)} = (p \circ p_1)^* \iota_{Et}^* \mu : F \circ \eta_{(\mathcal{F},F)} \Rightarrow w \circ \partial_0^{Loc,J} \circ p_2 = w\widetilde{F}$$

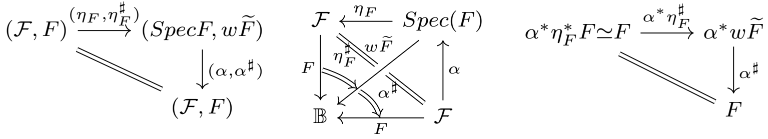
Locally modelled toposes as algebra for $wSpec$

Theorem of algebraicity

The category of modelled toposes coincides with the category of algebras of the monad $wSpec$

$$\mathbb{T}_{J,\mathcal{V}}\mathcal{L}oc\mathcal{T}opos \simeq (\mathbb{T}_B\mathcal{T}opos)^{wSpec}$$

Locally modelled toposes are automatically algebras via their reflections maps
By naturality, local morphisms are morphisms of algebras
Conversely: an algebra is endowed with a retraction of its unit



Hence the structural sheaf is a retraction of a local object, hence local

Locally modelled toposes as algebra for $wSpec$

Theorem of algebraicity

The category of modelled toposes coincides with the category of algebras of the monad $wSpec$

$$\mathbb{T}_{J,\mathcal{V}}\mathcal{L}oc\mathcal{T}opos \simeq (\mathbb{T}_B\mathcal{T}opos)^{wSpec}$$

Corollary

$\mathbb{T}_{J,\mathcal{V}}\mathcal{L}oc\mathcal{T}opos$ is complete and w creates limits
$\mathbb{T}_{J,\mathcal{V}}\mathcal{L}oc\mathcal{T}opos$ has coproducts.

Example

Locally rings spaces are complete and cocomplete

$wSpec$ is not idempotent by non-fullness

Being locally modelled is more a structure than a property.

Ongoing works and perspectives

- Condition of redundancy: when is the topological reduct sufficient for faithful dualization ?
- Conditions for representability and existence of dualizing object ?
- Criterion for a best factorization system associated to a class of local object ...

Spectrum for monadic categories

Problem of the Pt functor for frames:

$$\mathcal{F}rm^{op} \underset{pt}{\overset{\Omega}{\underset{\bot}{\leftrightarrows}}} \mathcal{T}op$$

This functor should be the prototypical spectrum
Corresponds to the stable inclusion

$$\underbrace{\mathcal{F}oc\mathcal{F}rm^{0-cons}} \hookrightarrow \mathcal{F}rm$$

- Obj: focal frames: where $\{0\}$ is prime ideal
- Mor: 0-conservative morphisms f s.t. $f^{-1}(\{0\} = \{0\}$

But $\mathcal{F}rm$ **is not L.F.P.**, not even accessible ! However it is monadic:
$\rightarrow$ geometry for monadic categories ?

Semantics as a 2-categorical geometry ?

Stone-like dualities = propositional syntax-semantic dualities
And they are (topological reduct of) spectral dualities

Propositional dualities

- Lindenbaum algebras closed by operations coding connectors
- Models =morphisms $f \in \mathcal{B}[B, 2]$ = identified with $f^{-1}(1)$ = points of the spectrum
- 2 is dualizing object
- Frame of ideals = topology on the spectrum

First order dualities

- Syntactic site = categories in corresponding doctrine (KZ-monadic)
- Models are functors in the doctrine F in $\mathbb{D}[C_{\mathbb{T}}, \mathcal{S}et]$ = identified with $\int F$ = pts of classifying topos
- $\mathcal{S}et$ as dualizing object
- Classifying topos = topology on the category of models

Semantics as a 2-categorical geometry ?

Example of correspondences

- Jipsen-Moshier
  $\wedge - \mathcal{SL}at_1^{op} \simeq \mathcal{HMS}$
- Stone
  $\mathcal{DL}at^{op} \simeq \mathcal{S}tone$
- Esakia
  $\mathcal{H}eyt^{op} \simeq \mathcal{E}sa$
- Duality for frames
- Gabriel-Ulmer
  $\mathcal{L}ex^{op} \simeq \mathcal{LFP}$
- Awodey-Forsell, Makkai
  For coherent theories
- Duality for small ccc ?
- Duality for geometric theories ?

Using a 2-spectrum of models to characterize categories of models for presheaf types, regular, coherent, geometric theories ?

Thanks for your attention !

Structured Stone duality

Usually Stone duality is defined without structural sheaf:
The underlying spectral space is sufficient for reconstructing a DLat D with $\mathcal{K}\Omega$
$\rightarrow$ situation of redundancy

Stable inclusion for Stone

Define the category $\mathcal{F}oc\mathcal{D}\mathcal{L}at^{0-cons}$ having:

- Obj: focal DLat, where $\{0\}$ is prime ideal
- Mor: 0-conservative morphisms f s.t. $f^{-1}(\{0\} = \{0\}$

Then $\mathcal{L}oc\mathcal{D}\mathcal{L}at^{0-cons} \hookrightarrow \mathcal{D}\mathcal{L}at$ is a multireflection

Structured Stone duality

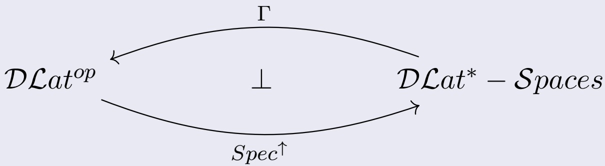
If x prime ideal of D, then D/x is local

Structured Stone spectrum

The associated spectrum for D is

$$(Spec(D) = (\mathcal{I}_D^{Prime}, \tau_D^{CoZariski}), \widetilde{D})$$

with $\widetilde{D}$ defined on the basis as $\widetilde{D}(U_a^{coZar}) = D/\theta_{(a,0)}$ for any $a \in D$



Relation between the sites

A canonical bifibration

$\mathcal{V}_F^{op} \to \mathcal{C}_{\mathcal{F}}$ is a cloven bifibration
Hence there is a geometric surjection $Spec(F) \twoheadrightarrow \mathcal{F}$

For an arrow in the site $s : c \to c'$

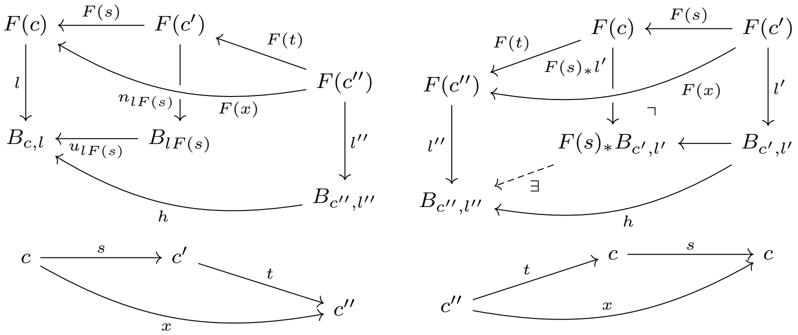
$$\mathcal{V}_{F(c')} \underset{n_{(-)\circ F(s)}}{\overset{F(s)_*(-)}{\rightleftarrows}} \mathcal{V}_{F(c)} \qquad (\bot)$$

$$\begin{array}{ccc} F(c) & \xleftarrow{F(s)} & F(c') \\ {\scriptstyle F(f)_* l'}\downarrow & \lrcorner & \downarrow{\scriptstyle l'} \\ F(s)_* B_{c',l'} & \longleftarrow & B_{c',l'} \end{array} \qquad \begin{array}{ccc} F(c) & \xleftarrow{F(s)} & F(c') \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle n_{lF(s)}} \\ B_{c,l} & \xleftarrow[u_{lF(s)}]{} & B_{lF(s)} \end{array}$$

Relation between the sites

A canonical bifibration

$\mathcal{V}_F^{op} \to \mathcal{C}_{\mathcal{F}}$ is a cloven bifibration
Hence there is a geometric surjection $Spec(F) \twoheadrightarrow \mathcal{F}$



Gabriel-Ulmer & Jipsen-Moshier

$Gabriel - Ulmer$	$Jipsen - Moshier$
$\underline{\mathcal{L}ex^{op}} \quad \simeq_{eq} \quad \underline{\mathcal{LFP}}$ $\mathcal{C} \quad \mapsto \quad \mathcal{L}ex[\mathcal{C}, \mathcal{S}et]$ $\mathcal{C} \xrightarrow{F} \mathcal{D} \quad \mapsto \quad (-) \circ F$ $(\mathcal{A}_{fp})^{op} \quad \mapsfrom \quad \mathcal{A}$ $(\mathcal{B}_{fp})^{op} \xrightarrow{G_{fp}^{op}} (\mathcal{A}_{fp})^{op} \quad \mapsfrom \quad \mathcal{A} \overset{G}{\leftrightarrows} \mathcal{B}$ finitary	$\underline{\bigwedge - \mathcal{SL}at_1}^{op} \quad \simeq \quad \mathcal{HMS}$ $L \quad \mapsto \quad \mathcal{F}_L$ $f : L \to M \quad \mapsto \quad f^{-1} : \mathcal{F}_M \to \mathcal{F}_L$ $\mathcal{KOF}_X \quad \mapsfrom \quad X$ h^{-1}
$\mathcal{L}ex[\mathcal{C}, \mathcal{S}et] \quad \simeq \quad \mathcal{C} - Mod_{\mathcal{S}et}$ $F \quad \mapsto \quad \int F$ $\alpha : F \to G \quad \mapsto \quad \int \alpha(????)$ $F_M \quad \mapsfrom \quad M = (M_c)_{c \in \mathcal{C}}$	$\bigwedge - \mathcal{SL}at_1[L, 2] \quad \simeq \quad \mathcal{F}_L$ $f \quad \mapsto \quad f^{-1}(1) \simeq \int f$ $\chi_F \quad \mapsfrom \quad F$
$\mathcal{LFP}$ categories are complete and cocomplete	$X \in \mathcal{HMS} \Rightarrow (X, \sqsubseteq) \in \mathcal{CDL}at$
$K \in \mathcal{A}_{fp} \Leftrightarrow \mathcal{A}[K, -]$is finitary : $\forall f : K \to colim^{\uparrow} X_i,\ \exists i, g : K \to X_i,\ f : q_i \circ g$	$\uparrow_{\sqsubseteq} x \in \mathcal{KOF}_X \Leftrightarrow \uparrow_{\sqsubseteq} x$ open so $x \sqsubseteq \bigsqcup^{\uparrow} x_i \Leftrightarrow \bigsqcap_{\downarrow} \uparrow_{\sqsubseteq} x_i \subseteq \uparrow_{\sqsubseteq} x$ $\Rightarrow \exists i\ x \sqsubseteq x_i$ because HMS spaces are well filtered
$\mathcal{A}_{fp} \downarrow X$ is filtered	$\uparrow_{\mathcal{KOF}_X} F$ is directed
$\mathcal{A}_{fp} \downarrow X, X \downarrow \mathcal{A}_{fp}$ are LFP	$\uparrow_{\sqsubseteq} x, \downarrow_{\sqsubseteq} x$ are HMS