Towards completeness of logics of common belief and information

Marta Bílková

Institute of Computer Science of the Czech Academy of Sciences

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Flat fixpoint modalities

- In PDL:

$$\langle\alpha^*\rangle a \equiv \mu x.a \vee \langle\alpha\rangle x \qquad [\alpha^*]a \equiv \nu x.a \wedge [\alpha]x$$

- in CTL

$$AFa \equiv \mu x.a \vee \Box x$$

- In logics of common knowledge (belief):

$$Ca \equiv \nu x. \bigwedge_{i\in I} \Box_i(a \wedge x)$$

$$Ca \equiv \nu x. \bigwedge_{i\in I} \Diamond_i(a \wedge x)$$

(the latter in case of a diamond-like notion of confirmed belief)

Flat fixpoint modalities

We start with a (multi)modal propositional language

and, for a modal scheme $c(p, x)$ we add a modality $\flat_c(p)$ expressing $\nu x.c(p, x)$

Semantically

$$\|\flat_c(p)\| = \bigcup\{Y \in \mathbb{U}X \mid Y \subseteq \|c(x, p)\|_{x:Y}\}$$

Two axiomatizations

For a modal scheme $c(p, x)$ and a modality $\flat_c(p)$

- Kozen's axiomatization of $\flat_c$ as the greatest fixed point:

$$\flat_c(p) \vdash c(p, \flat_c(p)) \qquad \frac{q \vdash c(p, q)}{q \vdash \flat_c(p)}$$

- An infinitary axiomatization, with the following rule replacing the induction rule and using finite approximations of $\flat_c(p)$
$c^0(p) = \top$ and $c^{n+1}(p) = c(p, c^n(p))$:

$$\{c^n(p) \mid n \in N\} \vdash_\omega \flat_c(p).$$

(Strong) completeness?

In many interesting cases

- Kozen's axiomatization known to be complete (over multimodal K plus some syntactical restrictions on c, covering clasical PDL, and common knowledge logic).
- Infinitary axiomatization known strongly complete (classical PDL and common knowledge logic)

Our goal: to advance in both directions for modal logics with non-classical base such as Dunn-Belnap logic BD, or (distributive) substructural logics (i.e. non-classical versions of PDL or logics of belief based on information).

Running example

Syntax of BD extended with individual belief modalities $\{\Diamond_i \mid i \in I\}$ and a common belief modality $\flat_c$ for $c(p, x) = \bigwedge_i \Diamond_i(p \wedge x)$

$$a ::= p \mid t \mid f \mid a \vee a \mid a \wedge a \mid \neg a \mid \Diamond_i a \mid \flat_c a$$

Plus a suitable axiomatization of the purely modal part (DL, de Morgan and involutive negation, normal diamonds, ...), and of the $\flat_c$ modality.

Frames

Frames for BD are based on involutive posets $(X, \leq, *)$, equipped with monotone relations $\{S_i \mid i \in I\}$

$$S_i : X^{op} \times X \longrightarrow 2$$

Valuation of atoms by uppersets in X are extended in the obvious way to constants and $\wedge, \vee$.

$$\begin{array}{lll} x \Vdash \neg\alpha & \equiv & *x \nVdash \alpha \\ x \Vdash \Diamond_i\alpha & \equiv & \exists s(sS_ix \wedge s \Vdash \alpha) \\ x \Vdash \Box_i\alpha & \equiv & \forall s(*sS_i * x \longrightarrow s \Vdash \alpha) \end{array}$$

S-frames can be seen as poset coalgebras for the lowerset functor $\mathbb{L}$, or, $S, *S*$-frames as $\mathbb{L} \times \mathbb{U}$ coalgebras.

Cover modalities over poset coalgebras

For a polynomial poset (locally monotone) endofunctor T

$$T ::= E \mid Id \mid T + T \mid T \times T \mid T^E \mid T^\partial \mid \mathbb{L}T \quad (\mathbb{U} = \mathbb{L}^\partial)$$

an alternative modal language is available, based on cover modalities $\nabla^T : T\mathcal{L} \longrightarrow \mathcal{L}$ and $\Delta^T : T\mathcal{L} \longrightarrow \mathcal{L}$ to reason about T^∂-coalgebras.

- DNF based on ∇^T is available (and CNF based on Δ^T)
- they are mutually definable, and often inter-definable with usual modalities
- for example,

$$\nabla^{\mathbb{U}_\omega} A \equiv \bigwedge_{a \in A} \Diamond a \text{ and } \Diamond a \equiv \nabla^{\mathbb{U}_\omega} a{\uparrow}$$

$$\nabla^{\mathbb{U}_\omega \times \mathbb{L}_\omega}(A, B) \equiv \bigwedge \Diamond A \wedge \Box \bigvee B \text{ and } \Diamond a \equiv \nabla^{\mathbb{U}_\omega \times \mathbb{L}_\omega}(a{\uparrow}, T{\downarrow})$$

Bílková, M. and Dostál, M., Moss' logic for ordered coalgebras, to appear in LMCS. https://arxiv.org/abs/1901.06547.

Relative finitary adjoints

For a polynomial finitary functor T, and a L.T. ($\flat$) modal algebra $\mathcal{L}$,

- $\nabla^T : T\mathcal{L} \longrightarrow \mathcal{L}$ is left relative adjoint: there is $r : \mathcal{L} \longrightarrow \mathbb{L}_\omega T\mathcal{L}$

  $$\nabla^T(A) \leq b \text{ iff } A \mathrel{\overline{\leq}} C \text{ for some } C \in r(b).$$

  (using the T-lifting of relation $\leq$)

- $\Delta^T : T\mathcal{L} \longrightarrow \mathcal{L}$ is right relative adjoint: there is $l : \mathcal{L} \longrightarrow \mathbb{U}_\omega T\mathcal{L}$

  $$a \leq \Delta^T(B) \text{ iff } C \mathrel{\overline{\leq}} B \text{ for some} C \in l(a).$$

Constructivity of fixed points

We know that whenever each c is a right relative adjoint, the $\flat$ modal algebra $\mathcal{L}$ is constructive: for each c, a:

$$[\flat_c(a)] = \bigwedge_{n \in N} [c^n(a)]$$

Thus we can show that the common belief modality over BD, and some implication-free fragments of distributive substructural logics is constructive.

Over classical (multi)modal logic K and all c harmless w.r.t. x, this consequently yields completeness of Kozen's axiomatization.

L. Santocanale: Completions of μ-algebras, LICS 2005.

L. Santocanale, Y. Venema: Completeness for flat modal fixpoint logics, APAL 2010.

Strong completeness of $\vdash_\omega$

From the constructivity we see that

- the infinitary rule
$$\{c^n(p) \mid n \in N\} \vdash_\omega \flat_c(p)$$
is (globally) sound,
- $\vdash_\omega$ is a conservative expansion of $\vdash$
$$\alpha \nvdash \beta \ \text{ then } \alpha \nvdash_\omega \beta,$$
- and that $\mathcal{L}$ embeds into the complex algebra of the canonical model of $\vdash_\omega$ (yet to be constructed).

Pair extension lemma

For a logic $\vdash_\omega$ we define a relation $\Vdash$:

$$\Gamma \Vdash \Delta \quad \text{iff} \quad \text{there is a finite } \Delta' \subseteq \Delta \text{ and } \Gamma \vdash_\omega \bigvee \Delta'.$$

A tuple $\langle \Gamma, \Delta \rangle$ is a pair if $\Gamma \nVdash \Delta$, it is full if $\Gamma \cup \Delta = Fm_{\mathcal{L}}$ (iff Γ is a prime theory)

Proposition (Pair extension property)

Every pair of $\Vdash$ with finite Δ can be extended in a full pair, provided $\vdash_\omega$ be a countably axiomatizable logic with a strong disjunction.

M. Bílková, P. Cintula, and T. Lávička. Lindenbaum and Pair Extension Lemma in Infinitary Logics. WOLLIC 2018.

- Pair Extension Property for finite Δs suffices to obtain a separation by prime theories,
- for a canonical model construction we moreover need to prove valuation lemma for normal diamond-like operators (diamonds or fusion):

$$\Diamond a \in \Gamma \quad \text{implies} \quad \langle \{a\}, \{b \mid \Diamond b \notin \Gamma\} \rangle$$

  is a pair that can be extended to a full one.
- for the argument to work we need to start with a countable axiomatization of $\vdash_\omega$, with a strong disjunction, and all infinitary rule instances closed under boxes (meet preserving definable modalities).

Canonical model of $\vdash_\omega$

We use pair-extension property to built saturated theories:

- $\Gamma \vdash_\omega a$ and $\Gamma \subseteq T$ implies $a \in T$,
- $\Gamma \vdash_\omega a$ and $\neg a \in T$ implies $T \cap \neg\Gamma \neq \emptyset$ (case of BD),
- T is prime.

Canonical frame is defined on the poset $(ST, \subseteq)$ by

- $*T = \{a \mid \neg a \notin T\}$
- $TS_i T'$ if and only if $\forall a(a \in T \longrightarrow \Diamond_i a \in T')$

Lemma

For all formulas a and all saturated theories T,

$$T \Vdash a \quad \textit{iff}\ a \in T.$$

To conclude

We have provided:

- completeness for a logic of confirmed common belief over BD (both fintary and strong infinitary completeness),
- over some substructural logics (dFL) we only understand the infinitary part of the story (for implication-free fragments, finitary part also works)

Further challenges:

- adding modal axioms (e.g. expressing factivity or consistency of belief)
- extending language with e.g. implication(s)