Generic Models for Topological Evidence Logics

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Outline

1. Topological models for epistemic logics
2. Topological evidence logics
3. Generic models
4. Multi-agent topo-e-models

Topological models for epistemic logics

- Let (X, τ) be a topological space, Prop a set of propositional variables and $V : \text{Prop} \to \mathcal{P}(X)$ a valuation.
- Let us start with a language $\mathcal{L}$ defined as follows:

$$\phi ::= p|\phi \wedge \phi|\neg\phi|K\phi,$$

with $p \in \text{Prop}$.

Interior semantics

- $||p|| = V(p)$;
- $||\phi \wedge \psi|| = ||\phi|| \cap ||\psi||$;
- $||\neg\phi|| = X \backslash ||\phi||$;
- $||K\phi|| = \text{Int}\,||\phi||$.

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- Generalisation of Kripke semantics for preordered frames: preorders are Alexandroff topologies.

  From $(X, \leq)$ consider the topology $\tau = \mathsf{Up}(\leq)$.
  We have: $x \in \mathsf{Int}\,\|\phi\|$ iff $y \in \|\phi\|$ for all $y \geq x$.

- Evidential view of knowledge: knowing a proposition amounts to having a piece of evidence (i.e. an open set) than entails it.
- McKinsey & Tarski (1944) proved two important results to this respect:
  - The logic of topological spaces under this semantics is S4;
  - The logic of a single dense-in-itself metrisable space (such as $\mathbb{R}$) under this semantics is S4.

But:

- The interior semantics equates "knowing" to "having evidence".
- Some attempts at introducing belief (Steinsvold, 2006; Baltag et al., 2013) equate true belief to knowledge or confine us to work with really weird spaces.

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Topological evidence models

- Stalnaker (2006) proposes a logic for knowledge and belief.
- Following this and building on evidence models (van Benthem & Pacuit, 2011) a new framework is introduced by Baltag, Bezhanishvili, Özgün & Smets (2016) .
- The dense interior semantics allows us to talk about concepts such as basic and combined evidence, justification, defeasible vs infallible knowledge...
- Sentences are interpreted on topological evidence models.

A topo-e-model is a tuple

$$(X, \tau, E_0, V)$$

where (X, τ) is a topological space, E_0 a subbasis and V a valuation.

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The dense interior semantics

Let (X, τ, E_0, V) be a topo-e-model. We now have a language including the following modalities:

- K: defeasible knowledge;
- B: belief;
- $[\forall]$: infallible knowledge;
- $\Box$: having evidence;
- $\Box_0$: having *basic* evidence.

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- $x \in ||K\phi||$ iff $x \in \text{Int}\ ||\phi||$ and $\text{Int}\ ||\phi||$ is dense;
- $x \in ||B\phi||$ iff $\text{Int}\ ||\phi||$ is dense;
- $x \in ||[\forall]\phi||$ iff $||\phi|| = X$;
- $x \in ||\Box\phi||$ iff $x \in \text{Int}\ ||\phi||$;
- $x \in ||\Box_0\phi||$ iff $x \in e \subseteq ||\phi||$ for some $e \in E_0$.

In this framework, knowing *P* = having an evidence for *P* that can't be defeated by any other evidence (i.e. a dense evidence).

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The logic of some fragments

The knowledge fragment $\mathcal{L}_K$: S4.2.

The knowledge-belief fragment $\mathcal{L}_{KB}$: S4.2 axioms for K, KD45 axioms for B plus:

(PI) $B\phi \to KB\phi$;
(NI) $\neg B\phi \to K\neg B\phi$;
(KB) $K\phi \to B\phi$;
(CB) $B\phi \to \neg B\neg\phi$;
(FB) $B\phi \to BK\phi$.

(This is the logic outlined in Stalnaker, 2006.)

The evidence fragment $\mathcal{L}_{\forall\Box\Box_0}$: S5 for $[\forall]$, plus S4 for $\Box$, plus:

($4_{\Box_0}$) $\Box_0\phi \to \Box_0\Box_0\phi$;
(Universality) $[\forall]\phi \to \Box_0\phi$;
(Factive evidence) $\Box_0\phi \to \Box\phi$;
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Generic models

- Recall:

McKinsey & Tarski (1944)

The logic of topological spaces under the interior semantics is S4, and so is the logic of $\mathbb{R}$ under the interior semantics.

- This result tells us that $\mathbb{R}$, as a topological space, is generic enough to capture the logic of topological spaces.
- How to apply this idea to our framework? First, let us formalise:

A topological space (X, τ) is a generic model for a language $\mathcal{L}$ if the sound and complete $\mathcal{L}$-logic of topo-e-models is precisely the logic of the class

$$\{(X, \tau, E_0) : E_0 \text{ is a subbasis of } (X, \tau)\}.$$

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Theorems

- Any dense-in-itself metrisable space (X, τ) is a generic model for the knowledge fragment $\mathcal{L}_K$ and the knowledge-belief fragment $\mathcal{L}_{KB}$.
  Examples: $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{I}$, the Baire space, the Cantor space, the binary tree...
  The $\mathcal{L}_K$-logic of any of these spaces is S4.2.
- Any dense-in-itself metrisable space (X, τ) which is idempotent (i.e. homeomorphic to $(X, \tau) \oplus (X, \tau)$) is a generic model for the fragments $\mathcal{L}_{\forall\Box}$, $\mathcal{L}_{\forall K}$ and $\mathcal{L}_{\forall\Box\Box_0}$.
  Examples: all of the above except $\mathbb{R}$ and the binary tree.

So:

- Whatever is true in any of the logics defined earlier is true in any topo-e-model whose topological space is $\mathbb{Q}$, and conversely,
- Whatever is not provable in these logics has a countermodel based on $\mathbb{Q}$.

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Completeness proof (a sketch)

S4.2 is complete with respect to $\mathbb{R}$ as a topo-e-model.

We use this (from Bezhanishvili×2, Lucero-Bryan & van Mill, 2018):

- Completeness wrt finite rooted S4.2 Kripke frames (rooted preorder $B \cup$ final cluster A).
- Partition lemma: for each $n \geq 1$, $\mathbb{R}$ can be partitioned in $\{U_1, \ldots, U_n, G\}$, where G is a dense-in-itself set with dense complement and each U_i is open.
- There exists an open, continuous and surjective map $f : G \to B$.

We can extend f to a surjective map $\bar{f} : \mathbb{R} \to B \cup A$ such that:

- The preimage of an upset is is dense open set (*dense-continuous*);
- The image of a dense open set is an upset (*dense-open*).

If something can be refuted in $B \cup A$, we can construct a valuation using $\bar{f}$ that refutes it in $\mathbb{R}$. Completeness follows.

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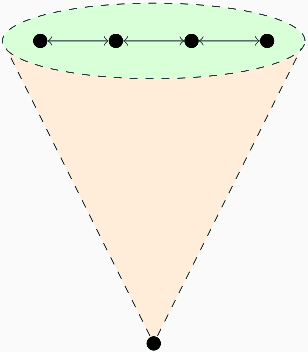

$$\begin{array}{ccc} & & \mathbb{R} \\ & & \shortparallel \\ A & \xleftarrow{x \in U_i \mapsto a_i} & U_1 \cup U_2 \cup U_3 \cup U_4 \\ & & \bigcup \\ B & \xleftarrow{f \text{ open and cont.}} & G \end{array}$$

$$\bar{f} : \mathbb{R} \to A \cup B.$$

Two-agent topo-e-models

The problem of density

- Let us make this framework multi-agent.
- We will consider two epistemic agents (1 and 2) each of them having different sets of evidence on a common space X (τ_1 and τ_2).
- How do we account for defeasibility and infallible knowledge?
- A first naive approach would be to just use density as in the single agent case:

$$x \in \|K_i\phi\| \text{ iff } \exists U \in \tau_i, \text{ dense such that } x \in U \subseteq \|\phi\|.$$

- Two issues with this approach:
  - Same set of worlds is compatible with both agents' information.
  - The logic of this semantics contains theorems like

$$\hat{K}_1 K_1 p \rightarrow K_2 \hat{K}_1 K_1 p.$$

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- To solve this, we make explicit which is the subset of X which is compatible with each agent's information at a world.
- We make this through the use of partitions Π_1 and Π_2.

A topological partitional model is a tuple

$$(X, \tau_1, \tau_2, \Pi_1, \Pi_2, V)$$

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Our density condition is not global (relative to the whole space X) but local (relative to the cell of the partition):

For $U \subseteq X$ and $\pi \in \Pi_i$ we say that U is π-locally dense whenever $U \cap \pi$ is dense in the subspace topology

$$\tau_i|_\pi = \{V \cap \pi : V \in \tau_i\}.$$

Semantics for knowledge in topological-partitional models

For $i = 1, 2$ and $x \in X$, let $\pi \in \Pi_i$ be the unique cell with $x \in \pi$. We have:

$$x \in \|K_i\phi\| \text{ iff } \exists U \in \tau_i,\ \pi\text{-locally dense such that } x \in U \subseteq \|\phi\|.$$

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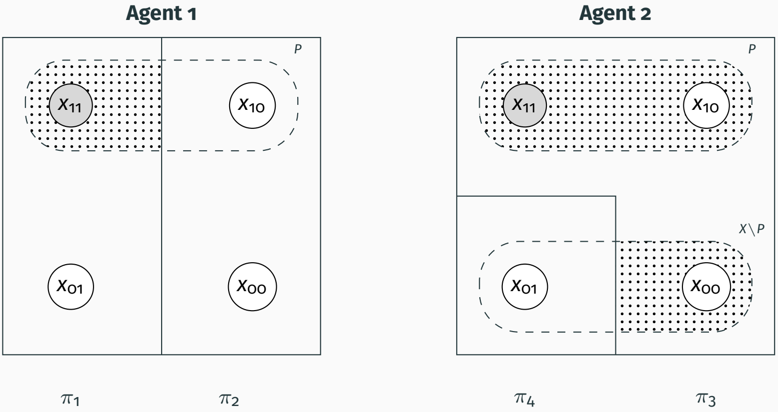

$$x_{11} \models K_1 p \wedge \neg K_2 p$$

Theorem

The logic of the knowledge-only fragment of topological parititonal models is the fusion logic $\mathsf{S4.2}_{K_1} + \mathsf{S4.2}_{K_1}$.

The infinite branching quaternary tree is an example of a generic model for this fragment.

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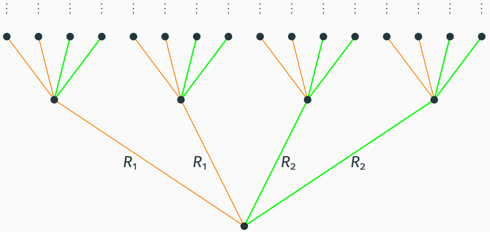

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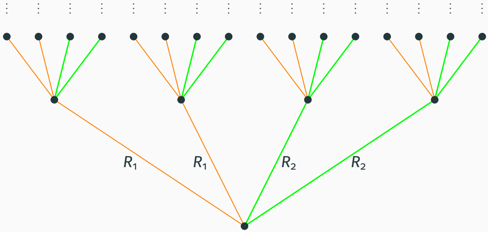

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Generic models for two-agent topological evidence logics

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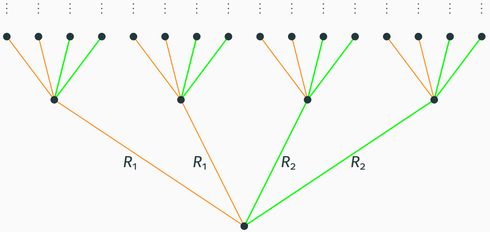

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- The horizontal topology τ_H is originated by the family of sets

$$\{U \times \{x\} : U \text{ open }, x \in \mathbb{Q}\}.$$

  Similarly, the vertical topology τ_V is originated by the sets $\{x\} \times U$.

- We prove that there exists an open partition on $\mathbb{Q} \times \mathbb{Q}$ making it into a generic model for the knowledge fragment.

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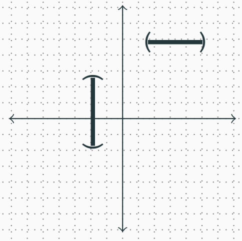

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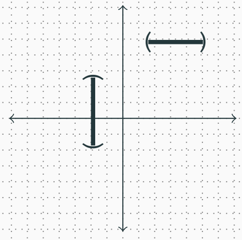

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What have we seen?

- Certain topological spaces are generic enough to capture the logic of topo-e-models.
- A framework for multi-agent topological evidence logics that generalises the one agent case.

Some directions for future work

- Finding generic models with a designated subbasis.
- Strong completeness?
- Characterising a class of generic models for the two-agent logic.
- Complete logic of common knowledge for topological-partitional models.
- Dynamic two-agent topological-partitional models.

I never know how to end talks. Please clap now.

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