Goldblatt-Thomason for LE-logics

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TACL 2019
Nice, France

Goldblatt-Thomason theorem for modal logic

Theorem

Let $\mathcal{L}$ be a modal signature and let K be a class of Kripke $\mathcal{L}$-frames that is closed under taking ultrapowers. Then K is $\mathcal{L}$-definable if and only if K is closed under p-morphic images, generated subframes and disjoint unions, and reflects ultrafilter extensions.

LE-logics

The logics algebraically captured by varieties of normal lattice expansions.

$$\phi ::= p \mid \bot \mid \top \mid \phi \wedge \phi \mid \phi \vee \phi \mid f(\overline{\phi}) \mid g(\overline{\phi})$$

where $p \in \mathsf{AtProp}$, $f \in \mathcal{F}$, $g \in \mathcal{G}$.

Normality

- Every $f \in \mathcal{F}$ is finitely join-preserving in positive coordinates and finitely meet-reversing in negative coordinates.
- Every $g \in \mathcal{G}$ is finitely meet-preserving in positive coordinates and finitely join-reversing in negative coordinates.

Examples: substructural, Lambek, Lambek-Grishin, Orthologic...

Goldblatt-Thomason theorem for LE-logics

Theorem

Let $\mathcal{L}$ be an LE signature and let K be a class of $\mathcal{L}$-frames that is closed under taking ultrapowers. Then K is $\mathcal{L}$-definable if and only if K is closed under p-morphic images, generated subframes and co-products, and reflects filter-ideal extensions.

LE frames

Definition

An $\mathcal{L}$-*frame* is a tuple $\mathbb{F} = (\mathbb{W}, \mathcal{R}_{\mathcal{F}}, \mathcal{R}_{\mathcal{G}})$ such that $\mathbb{W} = (W, U, N)$ is a polarity, $\mathcal{R}_{\mathcal{F}} = \{R_f \mid f \in \mathcal{F}\}$, and $\mathcal{R}_{\mathcal{G}} = \{R_g \mid g \in \mathcal{G}\}$ such that for each $f \in \mathcal{F}$ and $g \in \mathcal{G}$, the symbols R_f and R_g respectively denote $(n_f + 1)$-ary and $(n_g + 1)$-ary relations on $\mathbb{W}$,

$$R_f \subseteq U \times W^{\epsilon_f} \text{ and } R_g \subseteq W \times U^{\epsilon_g}, \tag{1}$$

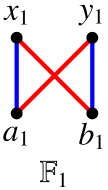
In addition, we assume that the following sets are Galois-stable (from now on abbreviated as *stable*) for all $w_0 \in W$, $u_0 \in U$, $\overline{w} \in W^{\epsilon_f}$, and $\overline{u} \in U^{\epsilon_g}$:

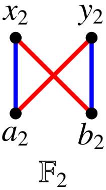
$$R_f^{(0)}[\overline{w}] \text{ and } R_f^{(i)}[u_0, \overline{w}^{\,i}] \tag{2}$$

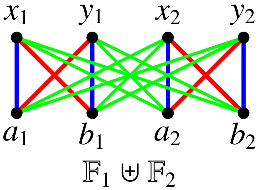
$$R_g^{(0)}[\overline{u}] \text{ and } R_g^{(i)}[w_0, \overline{u}^{\,i}] \tag{3}$$

co-product for LE frames

Let $\mathcal{L} = \{\Box\}$, i.e. $R_\Box \subseteq W \times U$:







p-morphisms for LE logics

Definition

A *p-morphism* of $\mathcal{L}$-frames, $\mathbb{F}_1 = (\mathbb{W}_1, \mathcal{R}^1_{\mathcal{F}}, \mathcal{R}^1_{\mathcal{G}})$ and $\mathbb{F}_2 = (\mathbb{W}_2, \mathcal{R}^2_{\mathcal{F}}, \mathcal{R}^2_{\mathcal{G}})$, is a pair $(S, T) : \mathbb{F}_1 \to \mathbb{F}_2$ such that:

p1. $S \subseteq W_1 \times U_2$ and $T \subseteq U_1 \times W_2$;

p2. $S^{(0)}[u]$, $S^{(1)}[w]$,$T^{(0)}[w]$ and $T^{(1)}[u]$ are Galois stable sets;

p3. $(T^{(0)}[w])^{\downarrow} \subseteq S^{(0)}[w^{\uparrow}]$ for every $w \in W_2$;

p4. $T^{(0)}[(S^{(1)}[w])^{\downarrow}] \subseteq w^{\uparrow}$ for every $w \in W_1$;

p5. $T^{(0)}[((R^2_f)^{(0)}[\overline{w}])^{\downarrow}] = (R^1_f)^{(0)}[\overline{((T^{\epsilon_f})^{(0)}[w])^{\partial}}]$ for every $R^i_f \in \mathcal{R}^i_{\mathcal{F}}$, where $T^1 = T$ and $T^{\partial} = S$;

p6. $S^{(0)}[((R^2_g)^{(0)}[\overline{u}])^{\uparrow}] = (R^1_g)^{(0)}[\overline{((S^{\epsilon_g})^{(0)}[u])^{\partial}}]$ for every $R^i_g \in \mathcal{R}^i_{\mathcal{G}}$, where $S^1 = S$ and $S^{\partial} = T$.

p-morphisms for LE logics

Definition

A *p-morphism* of $\mathcal{L}$-frames, $\mathbb{F}_1 = (\mathbb{W}_1, R^1_\Diamond, R^1_\Box)$ and $\mathbb{F}_2 = (\mathbb{W}_2, R^2_\Diamond, R^2_\Box)$, is a pair $(S, T) : \mathbb{F}_1 \to \mathbb{F}_2$ such that:

- p1. $S \subseteq W_1 \times U_2$ and $T \subseteq U_1 \times W_2$;
- p2. $S^{(0)}[u]$, $S^{(1)}[w]$,$T^{(0)}[w]$ and $T^{(1)}[u]$ are Galois stable sets;
- p3. $(T^{(0)}[w])^\downarrow \subseteq S^{(0)}[w^\uparrow]$ for every $w \in W_2$;
- p4. $T^{(0)}[(S^{(1)}[w])^\downarrow] \subseteq w^\uparrow$ for every $w \in W_1$;
- p5. $T^{(0)}[((R^2_\Diamond)^{(0)}[w])^\downarrow] = (R^1_\Diamond)^{(0)}[((T)^{(0)}[w])^\downarrow]$;
- p6. $S^{(0)}[((R^2_\Box)^{(0)}[u])^\uparrow] = (R^1_\Box)^{(0)}[((S)^{(0)}[u])^\uparrow]$.

Injective and surjective p-morphisms

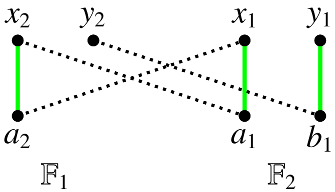
Definition

For every p-morphism $(S, T) : \mathbb{F}_1 \to \mathbb{F}_2$,

1. $(S, T) : \mathbb{F}_1 \twoheadrightarrow \mathbb{F}_2$, if $a \neq b$ implies $S^{(0)}[(\!| a |\!)] \neq S^{(0)}[(\!| b |\!)]$, for every $a, b \in (\mathbb{F}_2)^+$. In this case we say that $\mathbb{F}_2$ is a *p-morphic image* of $\mathbb{F}_1$.
2. $(S, T) : \mathbb{F}_1 \hookrightarrow \mathbb{F}_2$, if for every $a \in (\mathbb{F}_1)^+$ there exists $b \in (\mathbb{F}_2)^+$ such that $S^{(0)}[(\!| b |\!)] = [\![a]\!]$. In this case we say that $\mathbb{F}_1$ is a *generated subframe* of $\mathbb{F}_2$.

Example: generated subframe

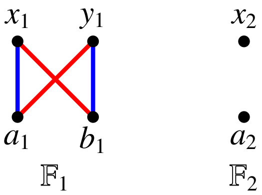
$(S,T) : \mathbb{F}_1 \hookrightarrow \mathbb{F}_2$, if for every $a \in (\mathbb{F}_1)^+$ there exists $b \in (\mathbb{F}_2)^+$ such that $S^{(0)}[(\!|b|\!)] = [\![a]\!]$. In this case we say that $\mathbb{F}_1$ is a *generated subframe* of $\mathbb{F}_2$.



$\mathbb{F}_2$ is a generated subframe of $\mathbb{F}_1$.

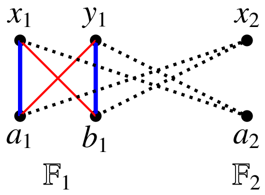
Example: p-morphic image

$(S, T) : \mathbb{F}_1 \to \mathbb{F}_2$ is *surjective*, in symbols $(S, T) : \mathbb{F}_1 \twoheadrightarrow \mathbb{F}_2$, if $a \neq b$ implies $S^{(0)}[(\!|a|\!)] \neq S^{(0)}[(\!|b|\!)]$ (or equivalently $T^{(0)}[[\![a]\!]] \neq T^{(0)}[[\![b]\!]]$), for every $a, b \in (\mathbb{F}_2)^+$. In this case we say that $\mathbb{F}_2$ is a *p-morphic image* of $\mathbb{F}_1$.



$(\varnothing, \varnothing) = (S, T) : \mathbb{F}_1 \to \mathbb{F}_2$.
$\mathbb{F}_2$ is a p-morphic image of $\mathbb{F}_1$.

(Counter)example



Indeed, $(T^{(0)}[a_2])^{\downarrow} = \varnothing \neq \{a_1, b_1\} = S^{(0)}[(a_2)^{\uparrow}]$ violating a Lemma.

Filter-ideal extensions

Definition

The *filter-ideal frame* of an $\mathcal{L}$-algebra $\mathbb{A}$ is $\mathbb{A}_\star = (\mathfrak{F}_\mathbb{A}, \mathfrak{I}_\mathbb{A}, N^\star, \mathcal{R}^\star_\mathcal{F}, \mathcal{R}^\star_\mathcal{G})$ defined as follows:

1. $\mathfrak{F}_\mathbb{A} = \{F \subseteq \mathbb{A} \mid F \text{ is a filter}\}$;
2. $\mathfrak{I}_\mathbb{A} = \{I \subseteq \mathbb{A} \mid I \text{ is an ideal}\}$;
3. $F N^\star I$ if and only if $F \cap I \neq \varnothing$;
4. for any $f \in \mathcal{F}$ and any $\overline{F} \in \overline{\mathfrak{F}}^{\epsilon_f}$, $R^\star_f(I, \overline{F})$ if and only $f(\overline{a}) \in I$ for some $\overline{a} \in \overline{F}$;
5. for any $g \in \mathcal{G}$ and any $\overline{I} \in \overline{\mathfrak{I}}^{\epsilon_g}$, $R^\star_g(F, \overline{I})$ if and only if $g(\overline{a}) \in F$ for some $\overline{a} \in \overline{I}$.

Definition

Let $\mathbb{F}$ be an $\mathcal{L}$-frame. The *filter-ideal extension* of $\mathbb{F}$ is the $\mathcal{L}$-frame $(\mathbb{F}^+)_\star$.

Ultraproducts of LE-frames

- $\mathcal{L}$-frames as (multi-sorted) first-order structures.
- Given a family $\{\mathbb{F}_i \mid j \in J\}$ of $\mathcal{L}$-frames and an ultrafilter $\mathcal{U}$ over J, the ultraproduct $(\prod_{i \in I} \mathbb{F}_i)/\mathcal{U}$ is defined as usual.
- $(\prod_{i \in I} \mathbb{F}_i)/\mathcal{U}$ is an $\mathcal{L}$-frame, by Łos Theorem.
- Let $\mathbb{F}^J/\mathcal{U}$ be the ultrapower of $\mathbb{F}$.

Enlargement property

Theorem (Enlargement property)

There exists a surjective p-morphism $(S, T) : \mathbb{F}^J/\mathcal{U} \twoheadrightarrow (\mathbb{F}^+)_\star$ for some set J and some ultrafilter $\mathcal{U}$ over J.

$$sSI \iff s^{-1}[[\![c]\!]] \in \mathcal{U} \text{ for some } c \in I \tag{4}$$

$$tTF \iff t^{-1}[(\!|c|\!)] \in \mathcal{U} \text{ for some } c \in F. \tag{5}$$

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Theorem

Let $\mathcal{L}$ be an LE signature and let K be a class of $\mathcal{L}$-frames that is closed under taking ultrapowers. Then K is $\mathcal{L}$-definable if and only if K is closed under p-morphic images, generated subframes and co-products, and reflects filter-ideal extensions.

Proof.

Let $\mathbb{F}$ be an $\mathcal{L}$-frame validating the $\mathcal{L}$-theory of K. By Birkhoff's Theorem:

$$\mathbb{F}^+ \twoheadleftarrow \mathbb{A} \hookrightarrow (\coprod_{i \in I} \mathbb{F}_i)^+.$$

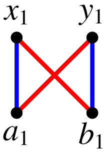
This gives

$$(\mathbb{F}^+)_\star \hookrightarrow \mathbb{A}_\star \twoheadleftarrow ((\coprod_{i \in I} \mathbb{F}_i)^+)_\star \twoheadleftarrow (\coprod_{i \in I} \mathbb{F}_i)^J / \mathcal{U}.$$

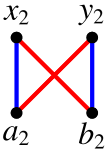
□

Examples revisited: Difference

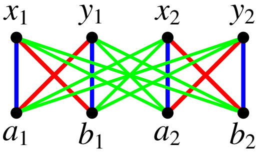
The first-order condition $R_\Box = N^c$ is not $\mathcal{L}$-definable:



$\mathbb{F}_1$



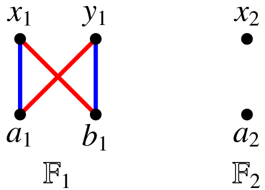
$\mathbb{F}_2$



$\mathbb{F}_1 \uplus \mathbb{F}_2$

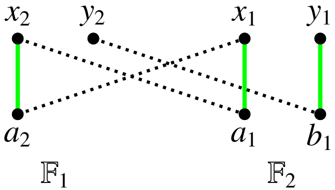
Examples revisited: Irreflexivity

The first-order condition $R^c \subseteq N$ is not $\mathcal{L}$-definable:



Examples revisited: Every point has a predecessor

The following first-order condition $\forall u \exists w(\neg wRu)$ is not $\mathcal{L}$-definable:



Thank you!