# When is the frame of nuclei spatial: A new approach

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# Frames

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Let $\mathrm{pt}(L)$ be the set of points of $L$. For $a \in L$, we set

$$\eta(a) = \{x \in \mathrm{pt}(L) \mid a \in x\}.$$

Then $\{\eta(a) \mid a \in L\}$ is a topology on $\mathrm{pt}(L)$.

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$\mathcal{O}$ and pt yield a contravariant adjunction which restricts to a dual equivalence between the category of spatial frames and the category of sober spaces.

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Some examples of nuclei.

$u_a(x) = a \vee x; \qquad v_a(x) = a \to x; \qquad w_a(x) = (x \to a) \to a.$

# Some Important Results about Nuclei

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- Isbell proved that if $S$ is sober, then $N(\mathcal{O}S)$ is spatial iff $S$ is weakly scattered.

- Niefield and Rosenthal gave necessary and sufficient conditions for $N(L)$ to be spatial, and derived that if $N(L)$ is spatial, then so is $L$.

# Priestley Spaces

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If $x \not\leq y$, then there is a clopen upset $U$ containing $x$ and missing $y$.

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The Priestley space $X_L$ of a bounded distributive lattice $L$ is the set $X_L$ of prime filters of $L$ ordered by inclusion. The topology $\pi$ on $X_L$ is given by the basis

$$\{\varphi(a) \setminus \varphi(b) \mid a, b \in L\}$$

where

$$\varphi(a) = \{x \in X_L \mid a \in x\}.$$

# Esakia Spaces

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An Esakia space is **extremally order-disconnected** if the closure of each open upset is clopen.

**Theorem**. (Pultr-Sichler) If $L$ is a bounded distributive lattice and $X_L$ its Priestley space, then $L$ is a frame iff $X_L$ is an extremally order-disconnected Esakia space.

# Nuclear Subsets of an Esakia Space

**Definition**. Let $X$ be an extremally order-disconnected Esakia space. A closed subset $F$ of $X$ is called a **nuclear subset** provided for each clopen set $U$ in $X$, the set $\downarrow(U \cap F)$ is clopen in $X$.

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Let $N(X)$ be the set of all nuclear subsets of $X$. It is a coframe with the inclusion order.

**Theorem**. (B., Ghilardi) Let $L$ be a frame and $X_L$ its Esakia space. Then $N(L)$ is dually isomorphic to $N(X_L)$.

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The join-prime elements of $N(X_L)$ are precisely the singletons $\{y\}$ with $y \in Y_L$. From this we can see if $N(L)$ is spatial then $L$ is spatial.

# When are $L$ and $N(L)$ spatial?

**Theorem**. A frame $L$ is spatial iff $Y_L$ is dense in $(X_L, \pi)$. When this happens, $L$ is isomorphic to the frame of opens of $(Y_L, \pi_u)$.

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**Theorem**. The following conditions are equivalent.

- $N(L)$ is spatial.
- If $N \in N(X_L)$ is nonempty, then so is $N \cap Y_L$.
- $N(L)$ is isomorphic to $\mathcal{O}_\pi(Y_L)$.

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Thus, when $L$ is spatial, it is isomorphic to the frame of opens of $(Y_L, \pi_u)$, while when $N(L)$ is spatial, it is isomorphic to the frame of opens of $(Y_L, \pi)$.

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**Theorem**. Let $L$ be a frame and $X_L$ its Esakia space. Then the following conditions are equivalent.

- $N(L)$ is boolean;
- $N(X_L) = \mathrm{RC}(X_L)$;
- $\max(D)$ is clopen for each clopen downset $D$ of $X_L$.

# A Theorem of Beazer and Macnab

Recall that $d \in L$ is **dense** if $\neg d = 0$. If $a \in L$, then ${\uparrow}a$ is a frame, and $d \geq a$ is dense in ${\uparrow}a$ iff $d \to a = a$.

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As a consequence of the previous theorem, we obtain the following.

**Theorem**. (Beazer, Macnab) Let $L$ be a frame. Then $N(L)$ is boolean iff for each $a \in L$ the principal upset $\uparrow a$ has a smallest dense element.

# Scattered and Weakly Scattered Spaces

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**Theorem**. Let $S$ be a topological space. Then $N(\mathcal{O}S)$ is spatial iff the soberification of $S$ is weakly scattered.

**Corollary**. (Isbell) If $S$ is $T_0$, then $S$ is sober and $N(\mathcal{O}S)$ is spatial iff $S$ is weakly scattered.

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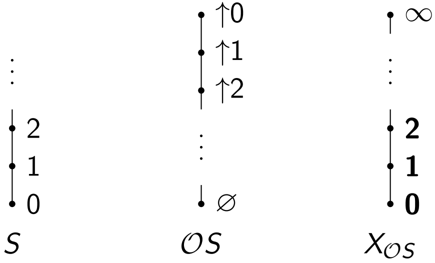

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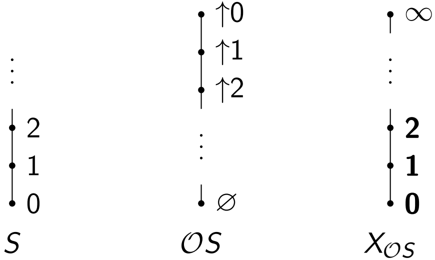


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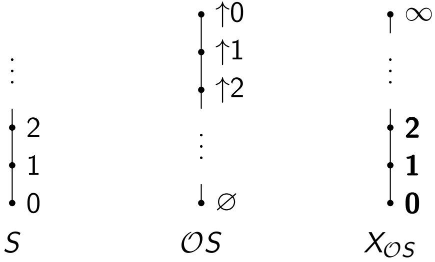


$S$ is not weakly scattered since it has no weakly isolated points.

$Y_{\mathcal{O}S} = X_{\mathcal{O}S}$ and is weakly scattered. Therefore, $N(\mathcal{O}S)$ is spatial.

# Simmons's Theorem about Weakly Scattered Spaces

Define $\sigma$ from $N(\mathcal{O}S)$ to the opens of the front topology of $S$ by

$$\sigma(j) = \bigcup\{j(U) \setminus U \mid U \in \mathcal{O}S\}$$

for each $j \in N(\mathcal{O}S)$. The map $\sigma$ is an onto frame homomorphism.

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As a consequence of our results, we obtain the following.

**Theorem**. (Simmons) A topological space $S$ is weakly scattered iff $\sigma : N(\mathcal{O}S) \to \mathcal{O}_F(S)$ is an isomorphism.

# Simmons's Theorem about Scattered Spaces

**Theorem**. For a spatial frame $L$, the following conditions are equivalent.

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Since scattered spaces are sober and subspaces of scattered spaces are scattered, we recover Simmons's result for $T_0$-spaces.

**Theorem**. (Simmons) Let $S$ be a $T_0$-space. Then $N(\mathcal{O}S)$ is boolean iff $S$ is scattered.

By making use of the $T_0$-reflection, we can drop the $T_0$ assumption in the previous theorem and obtain the full version of Simmons's result.

# When is $N(\mathcal{O}S)$ spatial for an Alexandroff Space $S$

Recently we have used these results to show that if $S$ is a preorder with the topology of upsets, $N(\mathcal{O}S)$ is spatial iff the infinite binary tree does not embed in $S$.

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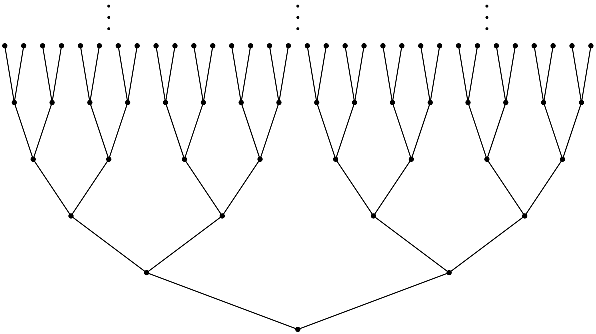

Thanks to the organizers for the invitation to speak at this conference and thanks for your attention.