Axiom T_D and the relation between sublocales and subspaces of a space

Jorge Picado (Univ. Coimbra, Portugal)

— joint work with Aleš Pultr (Prague)

Centre for Mathematics
University of Coimbra

- J. Picado and A. Pultr,
  Axiom T_D and the Simmons sublocale theorem,
  Comment. Math. Univ. Carolinae

  (to appear in the special issue in memory of Věra Trnková)

1. THE SETTING: The category of LOCALES

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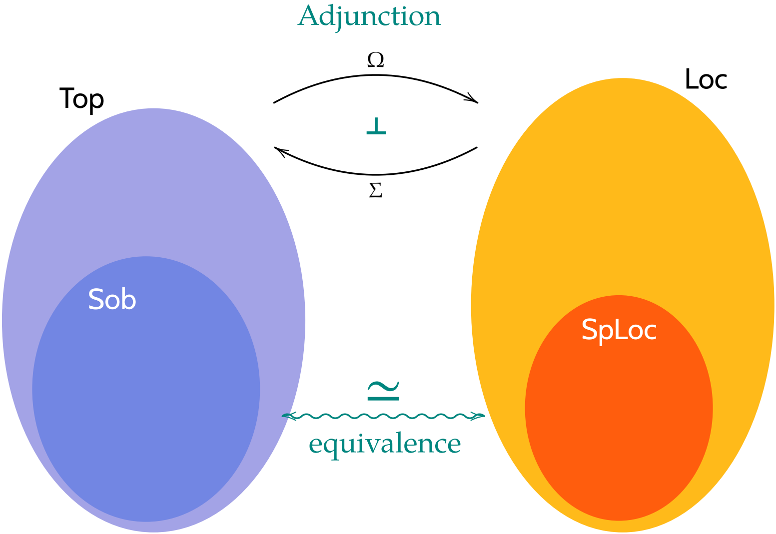

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Top

Sob

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SpLoc

$$\begin{tikzcd}\mathsf{Sob} \arrow[r, leftrightsquigarrow, "\simeq"] & \mathsf{SpLoc}\end{tikzcd}$$

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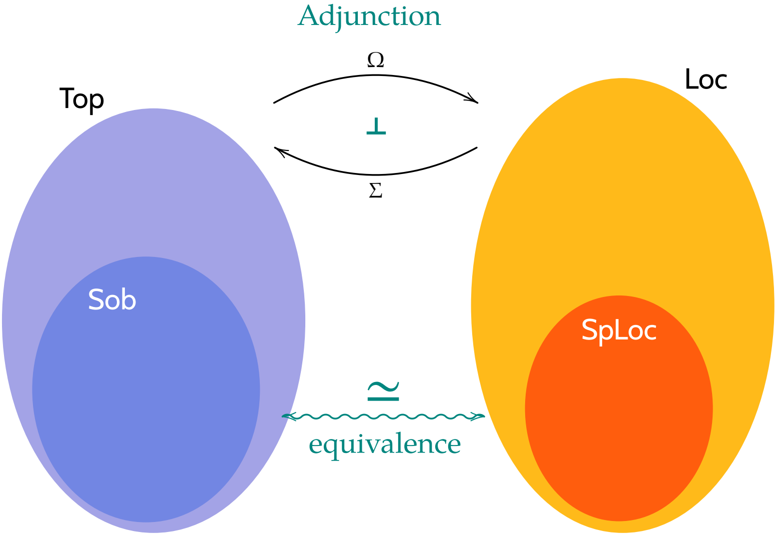

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subspaces: $\mathcal{P}(X)$, a CABool

sublocales: $\mathcal{S}(L)$, a COFRAME!

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«(...) a locale has enough complemented sublocales to compensate for this shortcoming: one simply has to make the sublocales which are complemented do more of the work.»

JOHN ISBELL

[Atomless parts of spaces, *Math. Scand.* (1972)]

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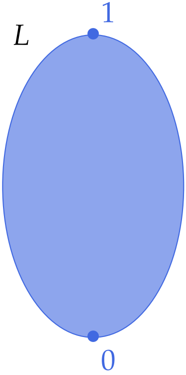

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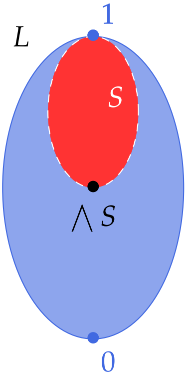

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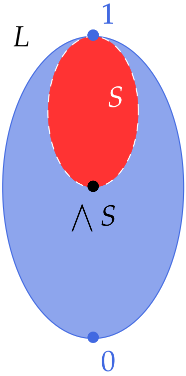

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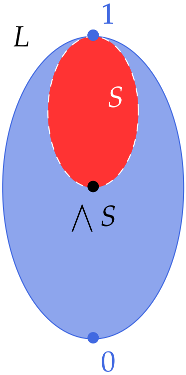

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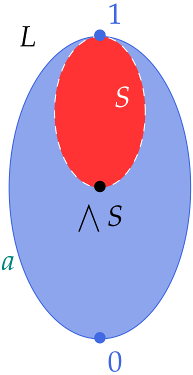

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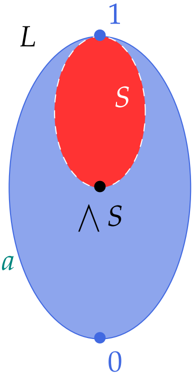

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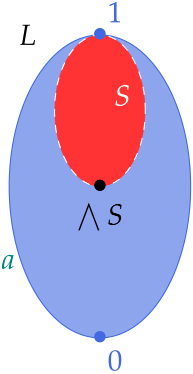

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(there may be more primes in $\Omega(X)$; **sober**: these are the only ones)

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BOTH CASES $\xleftrightarrow{1-1}$ is circumvented; the more complete results of N&R are choice dependable.

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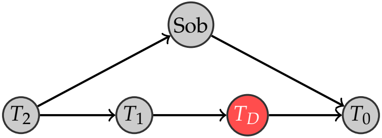

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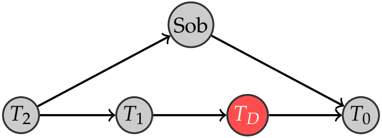

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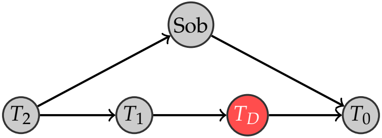

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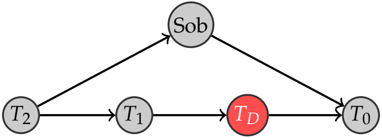

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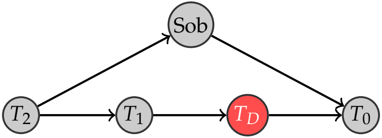

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- The primes $p = X \smallsetminus \overline{\{x\}}$ are covered: $p = \bigwedge U_i \implies p = U_K$ for some k.

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3 A sublocale $S \subseteq \Omega(X)$ is induced iff $S = \bigvee\{\mathfrak{b}(p_{X,x}) \mid p_{X,x} \in S\}$.

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(5) $\begin{aligned} \mu\colon \mathcal{S}(\Omega(X)) &\to \mathcal{P}(X) \\ S &\mapsto \{x \in X : p_{X,x} \in S\} \end{aligned}$ is a poset isomorphism.

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(extension of the usual concept: $(x \to 0) \to 0 = x^{**}$)

Theorem

TFAE for a T_D-space X:

(1) $\mathcal{S}(\Omega(X))$ is Boolean.

(2) Each Boolean sublocale is complemented.

(3) For every open $U \neq X$, there is a U-regular element of the form $p_{X,x}$.

(4) All sublocales are induced and precisely represent subspaces of X.

(5) $$\begin{array}{rcl} \mu\colon \mathcal{S}(\Omega(X)) & \to & \mathcal{P}(X) \\ S & \mapsto & \{x \in X\colon p_{X,x} \in S\} \end{array}$$ is a poset isomorphism.

$$\begin{tikzcd} \bigvee\{\mathfrak{b}(p_{X,y})\colon y \in Y\} & Y \subseteq X \arrow[l, bend left, "\mu^{-1}"'] \end{tikzcd}$$

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[main point: $(p_{X,x} \to U) \to U = \mathrm{int}\big(\overline{(X \smallsetminus U) \smallsetminus \overline{\{x\}}} \cup U\big)$]

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— this is a further example of the importance of axiom T_D in fitting together spatial and pointfree facts.

7. MORE (joint work with D. Baboolal, P. Pillay and A. Pultr)

QUESTION When is every induced sublocale complemented?

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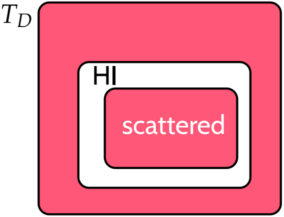
scattered

$\neq$ (even within T_D)

[G. Bezhanishvili, Mines, Morandi, *Topology Appl.*, 2003]

8. CONCLUSIONS for T_D-spaces

$\mathcal{H}$-spaces either scattered or non-HI



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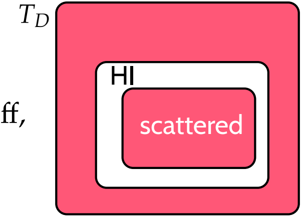

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include: metrizable, locally compact Hausdorff, Alexandroff, 1st countable, spectral, etc. [G. Bezhanishvili, Mines, Morandi, 2003]

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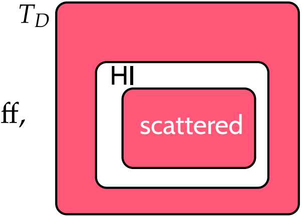

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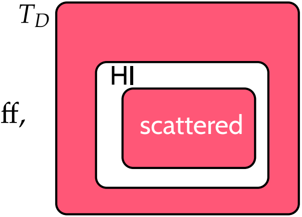

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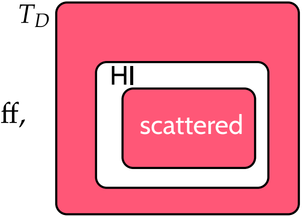

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non $\mathcal{H}$-spaces exist!

- each of their subspaces is complemented in $\mathcal{S}(\Omega(X))$ while this coframe contains also non-complemented elements.