Axiomatizing the crisp Gödel modal logic

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  - In Fuzzy logics, distinguished algebra (standard) generating the variety. reasonable to consider the modal logics over that particular evaluation algebra
  - Some modal MV logics have been axiomatised, but most have not.
  - Gödel modal logics can be seen as a hinge
    $\begin{cases} \text{semilinear extension of IL} \\ \text{one of three main FLs (BL + idempotency of \&)} \end{cases}$

The non-modal part

Definition

A Gödel algebra is a semilinear Heyting algebra $=$ idempotent (bounded) residuated lattice. i.e., $\mathbf{A}$ is $\langle A, \wedge, \vee, \rightarrow, 1 \rangle$ such that

- $\langle A, \wedge, \vee, 0, 1 \rangle$ is a bounded distributive lattice,
- For all $x, y \in A$, $x \odot y \leq z \Longleftrightarrow x \leq y \rightarrow z$ (residuation law),
- For all $x, y \in A$, $x \rightarrow y) \vee (y \rightarrow x) = 1$ (semilinearity).

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(semantic) Gödel logics

$\Gamma \models_{\mathcal{C}} \varphi$ iff for any $\mathbf{A} \in \mathcal{C}$ and any $h \in Hom(\mathbf{Fm}, \mathbf{A})$, if $h[\Gamma] \subseteq \{1\}$ then $h(\varphi) = 1$.

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Gödel Propositional Logic

Gödel Logic G is given by the axiomatic system resulting from IPC + $(\varphi \to \psi) \vee (\psi \to \varphi)$ (or $BL + \varphi \to \varphi \& \varphi$).

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Strong Standard Completeness

For any $\Gamma, \varphi \subseteq Fm$ (pos. infinite) the following are equivalent:

- $\Gamma \vdash_{\mathsf{G}} \varphi$,
- $\Gamma \models_{\mathbb{G}} \varphi$,
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$\Gamma \vdash_{\mathsf{G}} \varphi$ iff for any $h \in Hom(\mathbf{Fm}, [\mathbf{0}, \mathbf{1}]_G)$ it holds $inf_{\gamma \in \Gamma} h(\gamma) \leq h(\varphi)$.

Gödel Kripke models

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A (standard) Gödel Kripke model $\mathfrak{M}$ is a $[0,1]$-Kriple frame $\mathfrak{F} = \langle W, R \rangle$ (W set, R: $W^2 \to [0,1]$) with an evaluation e: $W \times V \to [0,1]$.

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$$e(v, \varphi\{\wedge, \vee, \rightarrow\}\psi) = e(v,\varphi)\{\wedge, \vee, \rightarrow\}e(v,\psi)$$

$$e(v, \Box\varphi) = \bigwedge_{w \in W}\{R(v,w) \rightarrow e(w,\varphi)\}, \quad e(v, \Diamond\varphi) = \bigvee_{w \in W}\{R(v,w) \wedge e(w,\varphi)\}$$

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(semantic) Local Gödel modal logics

$\Gamma \Vdash_{\mathcal{C}} \varphi$ (locally) iff for any $\mathfrak{M} \in \mathcal{C}$ and any $v \in W$, if $e(v, [\Gamma]) \subseteq \{1\}$ then $e(v,\varphi) = 1$.

Axiomatized Gödel Modal logics

- $\Box$ and $\Diamond$ fragments over all models are axiomatized (Caicedo, Rodriguez [2010]). The $\Box$ fragment over crisp models coincides with that over all models.

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- Language with both modalities over all models is axiomatized (Caicedo, Rodriguez [2015]). Coincides with Fischer-Servi Modal Intuitionistic Logic plus prelinearity.
- Language with both modalities over crisp models was still not axiomatized (previous proof used heavily the (0,1) values of R).

Axiomatic system

(crisp) Gödel Modal Logic

(crisp) Gödel Modal Logic K_G^c is given by the axiomatic system resulting from G and the following axiom schematas and rules:

$(K_\Box)$	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	$(K_\Diamond)$	$\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$
(P)	$\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$	$(FS2)$	$(\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$
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Some derivable (meta) rules:

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- $\vdash_{\mathsf{K}_\mathsf{G}^c} \varphi \vee (\psi \to \chi)$ implies $\vdash_{\mathsf{K}_\mathsf{G}^c} \Diamond\varphi \vee (\Diamond\psi \to \Diamond\chi)$.

Completeness

For each $\nvdash_{\mathsf{K}^c_{\mathsf{G}}} \chi$ we define a canonical crisp Gödel Kripke model.

- $W := \{h \in Hom(Fm_{\Box,\Diamond}, [\mathbf{0},\mathbf{1}]_G) \colon h(Th(\mathsf{K}^c_{\mathsf{G}})) = \{1\}\}$,
- Rhg iff for all $\psi \in Sub(\chi)$, $h(\Box\psi) \leq g(\psi) \leq h(\Diamond\psi)$,
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$h(\Box\varphi) \leq e(g, \varphi)$ for all Rhg follows from definition of the canonical relation.

Completeness

To see $h(\Box\varphi) = \bigwedge_{Rhg} e(g, \varphi)$ we show for $h(\Box\varphi) = \alpha < 1$ that for any $\epsilon > 0$ there is $g_\epsilon \in W$ such that Rhg_ϵ and $g_\epsilon(\varphi) \in [\alpha, \alpha + \epsilon)$.

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There are three important sets of formulas:

- $\Box^{=1} := \{\psi \in Fm\colon h(\Box\varphi) = 1\}$
- $\Box^{>\alpha} := \{\psi \in Sub(\chi)\colon \alpha < h(\Box\varphi) < 1\}$
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Proposition

There is $u \in Hom(Fm_{\Box,\Diamond}, [\mathbf{0},\mathbf{1}]_G)$ such that

$$u(Th(\mathsf{K}^{\mathsf{c}}_{\mathsf{G}})) = \{1\}, \qquad u(\Box^{=1}) = 1,$$
$$u(\Box^{>\alpha}) > u(\varphi), \qquad u(\Diamond^{<1}) < 1$$

Completeness proof

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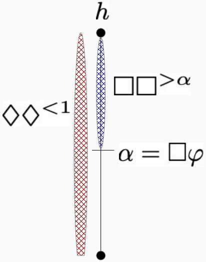
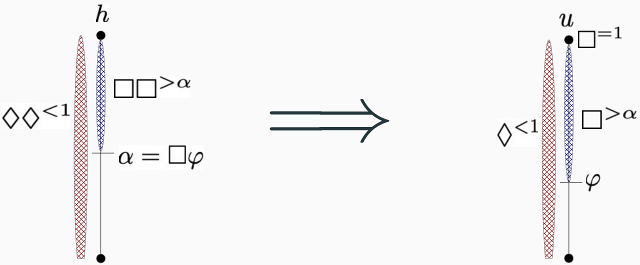


Completeness proof

Let $\delta = (\bigwedge \Box^{>\alpha} \to \varphi) \to \varphi$.

- Either $h(\Diamond(\delta \wedge (\varphi \to \bigvee \Diamond^{<1}))) = 1$
  - we can prove $ThK^c(G), \Box^{=1}, \delta \not\models_{[0,1]_G} (\varphi \to \bigvee \Diamond^{<1}) \to \bigvee \Diamond^{<1}$
- Or $h(\Box(\delta \to (\bigvee \Diamond^{<1} \to \varphi))) = 1$
  - we can prove $Th(K^c_G), \Box^{=1}, \delta, \delta \to (\bigvee \Diamond^{<1} \to \varphi) \not\models_{[0,1]_G} \varphi$



Completeness proof

Proposition

There is an strictly increasing function $\sigma : [0,1] \to [0,1]$ such that $\sigma(u(\psi)) \in [h(\Box\psi), h(\Diamond\psi)]$ for each $\Box\psi, \Diamond\psi \in SFm(\varphi)$ and $\sigma(u(\chi)) \in [\alpha, (\alpha+\epsilon) \wedge u(\Diamond\chi)]$.

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The proof of the $\Diamond$-formulas is similar.

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Theorem

$\Gamma \vdash_{K_G^c} \varphi$ if and only if $\Gamma \Vdash_{\mathbb{K}_G^c} \varphi$.

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$\Gamma \vdash_{K_G^c} \varphi$ if and only if $\Gamma \Vdash_{\mathbb{K}_G^c} \varphi$.

This can be extended also to infinite sets of formulas.

Miscellanea

Lemma

Global deduction over $\mathbb{K}^c_G$ is axiomatized by K^c_G plus $\frac{\varphi}{\Box\varphi}$.

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$\vdash_{K^c_G}$ is decidable (Caicedo, Metacalfe et. al, 2013).

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decidability of global deduction/ $4\mathsf{K}^c_G$?

Merçi beaucoup!