From intuitionism to Brouwer's modal logic

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Brouwerian modal logic $\mathbf{KTB} := \mathbf{K} \oplus T \oplus B$

Axioms CL and

$$K := \Box(p \to q) \to (\Box p \to \Box q)$$
$$T := \Box p \to p$$
$$B := p \to \Box\Diamond p,$$

and rules: (MP), (Sub) i (RG).

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[1] G.E. Hughes, M.J. Cresswell, *An Introduction to Modal Logic*, Methuen and Co Ltd, London, (1968).

- Luitzen Brouwer is the founder of the intuitionist school of mathematics.
- The law of double negation does not hold in intuitionistic logic.
- Exactly it holds that
  - (i) $\vdash_{INT} p \to \neg\neg p$ but
  - (ii) $\nvdash_{INT} \neg\neg p \to p$.
- Suppose that negation has a stronger meaning – necessarily negative.
- Hence $\neg p$ may be translated as $\Box\neg p$.

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Motivation (2)

- The corresponding modal formula to (i) $p \to \neg\neg p$ is $p \to \Box\neg\Box\neg p$, which gives us $p \to \Box\Diamond p$ and obviously $\vdash_{KTB} p \to \Box\Diamond p$.
- If we translate (ii) $\neg\neg p \to p$ in this way, we obtain: $\Box\Diamond p \to p$, which is not a thesis even of the system $\mathbf{S5}$ defined below. Hence $\nvdash_{KTB} \Box\Diamond p \to p$.

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Translation from INT into KTB

We define the translation in the following way:

$$t(\bot) = \bot, \qquad t(p) = p, \qquad t(\alpha \to \beta) = \Box(t(\alpha) \to t(\beta)),$$
$$t(\alpha \wedge \beta) = t(\alpha) \wedge t(\beta), \qquad t(\alpha \vee \beta) = t(\alpha) \vee t(\beta).$$

Then we get: $t(\neg p) = \Box\neg p$ (because $\neg p = p \to \bot$) and further $t(\neg\neg p) = \Box\neg\Box\neg p = \Box\Diamond p$.

Formulas written in one variable $Form_p$

$p, \bot \in Form_p$

If $\phi \in Form_p$ then $\neg\phi \in Form_p$

If $\phi \in Form_p$ and $\psi \in Form_p$ then $\phi \circledcirc \psi \in Form_p$

for $\circledcirc \in \{\to, \vee, \wedge, \leftrightarrow\}$

Intuitionistic formulas from $Form_p$

$$\alpha_0 = \bot, \qquad \alpha_1 = p, \qquad \alpha_2 = p \to \bot,$$

$$\alpha_{2n+1} = \alpha_{2n} \vee \alpha_{2n-1}, \qquad \alpha_{2n+2} = \alpha_{2n} \to \alpha_{2n-1} \quad \text{for } n \geq 1$$

$$\alpha_\omega = p \to p \quad \text{for} \quad \omega \notin \mathbb{N}.$$

In $Form_p$ we introduce an equivalence relation $\equiv$ in the standard way: $\varphi \equiv \psi$ if both $\varphi \to \psi$ and $\psi \to \varphi$ are intuitionistic tautologies.

Every formula from $Form_p$ falls into one of the equivalence classes $A_m = [\alpha_m]_\equiv$. The quotient algebra rises to the so-called Rieger - Nishimura lattice $\mathcal{R}$, which is a single-generated free Heyting algebra.

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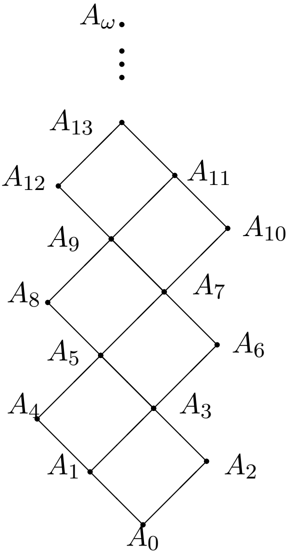
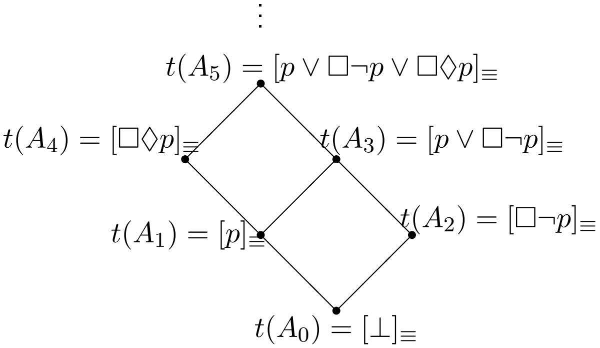


The bottom of the Rieger-Nishimura lattice after the translation t



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- $\Box(t(A_2) \to t(A_0)) = t(A_4)$, that is $\Box[\Box\neg p \to \bot]_\equiv = [\Box\neg\Box\neg p]_\equiv$
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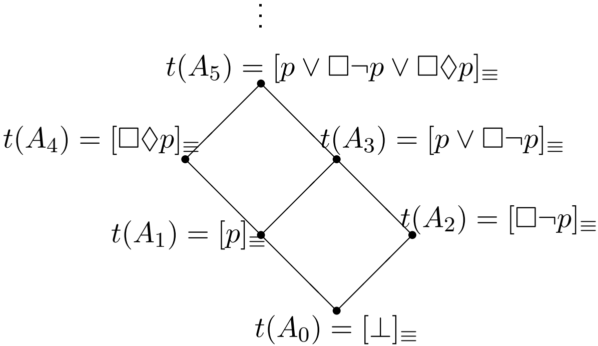

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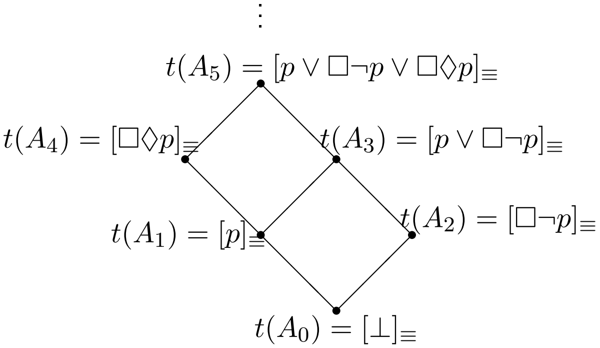

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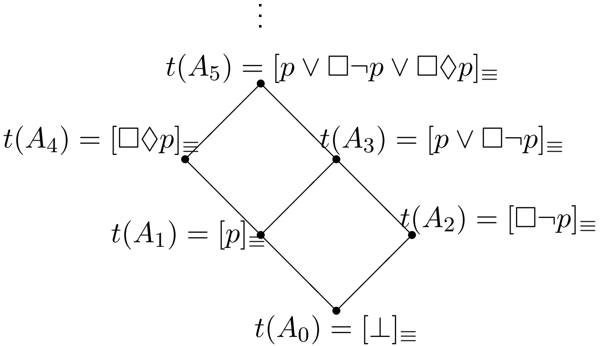

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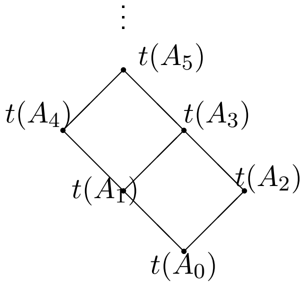

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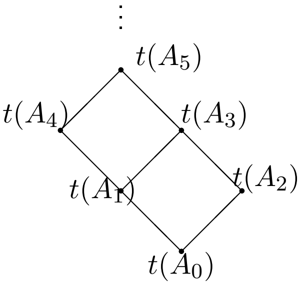

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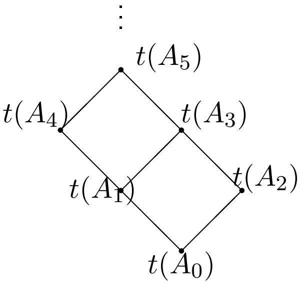

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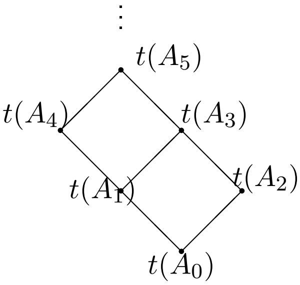

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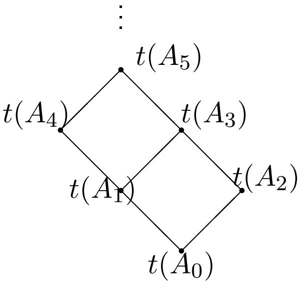

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The translation t does not preserve the conjunction of classes:

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$t(\alpha_{2n-1}) \wedge t(\alpha_{2n}) \leftrightarrow t(\alpha_{2n-3}) \notin \mathbf{KTB}$,

however it will work well on disjunction. By definition we have that

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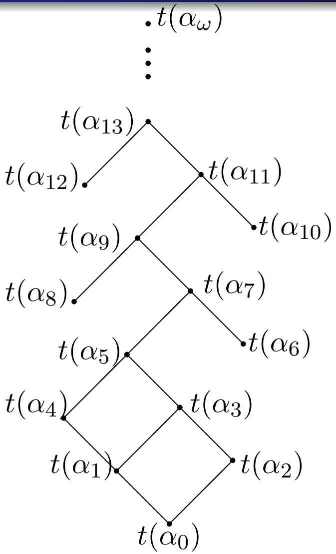

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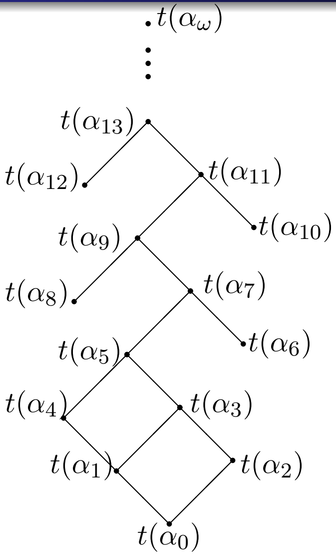

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Answer: yes

$\mathfrak{M}_n := \langle W, R, v\rangle$ where $W = \{x_0, x_1, x_2, ..., x_n\}$, R is reflexive, symmetric and:

$x_0 R x_i$ iff $i \neq 1$

$x_1 R x_i$ iff $i \notin \{0, 3\}$

$x_2 R x_i$ for all i

$x_3 R x_i$ iff $i \notin \{1, 4\}$

for $4 \leq k < n-1$: $x_k R x_i$ iff $i \notin \{k+1, k-1\}$ for $i < n$

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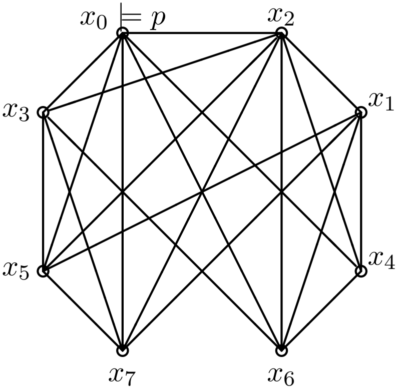

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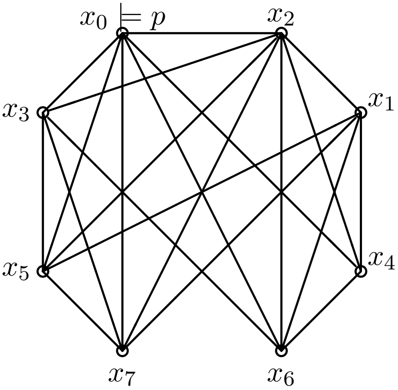

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$x_i \models t(\alpha_7)$ iff $i \in \{0, 1, 3, 4\}$

$x_i \models t(\alpha_8)$ iff $i \in \{3, 5\}$

Then we get:

$x_i \not\models t(\alpha_5) \to t(\alpha_3) \quad \Leftrightarrow \quad i = 3$

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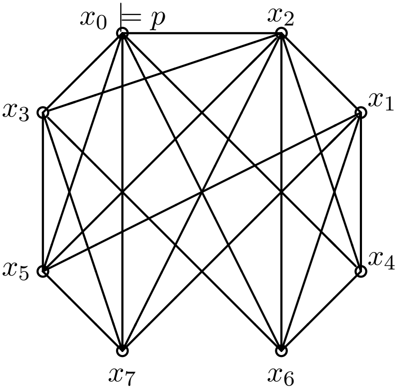

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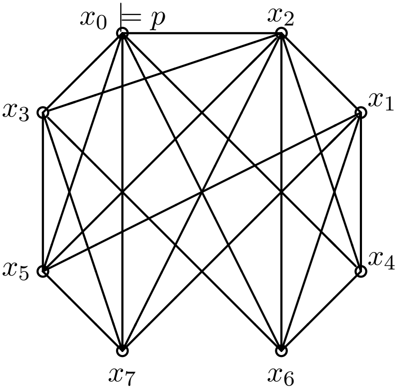

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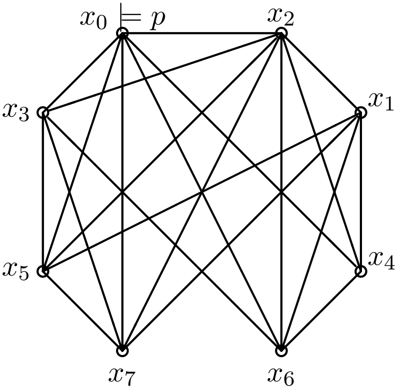

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$x_i \models t(\alpha_5)$ iff $i \in \{0, 1, 3\}$

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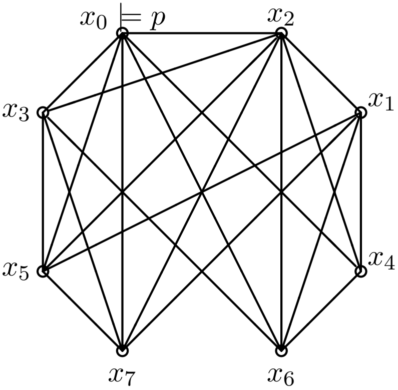
$x_i \not\models t(\alpha_9) \to t(\alpha_7) \quad \Leftrightarrow \quad i = 5 \ldots$



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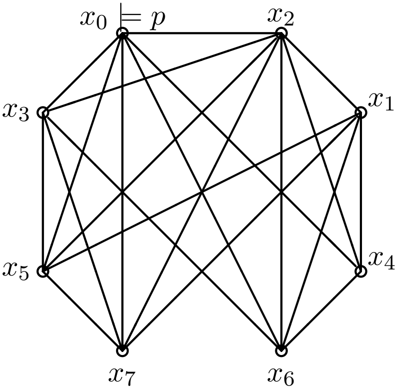

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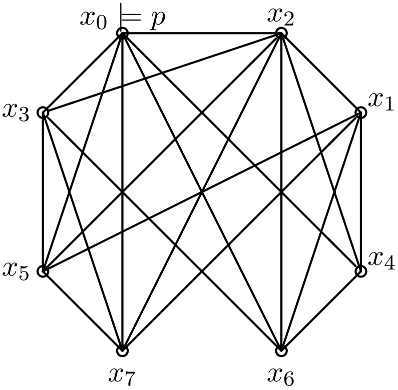

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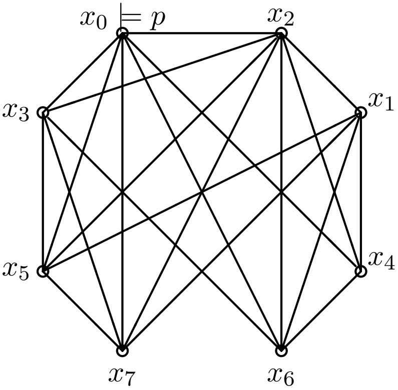

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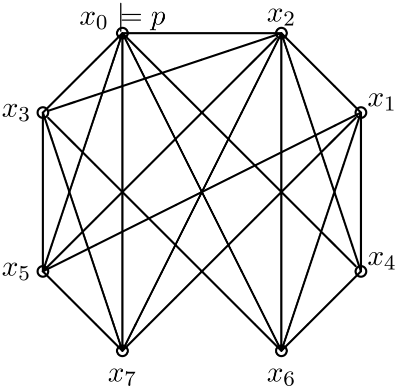

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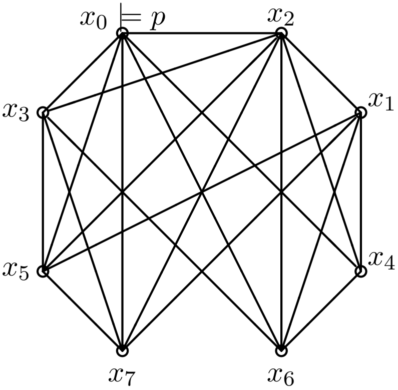

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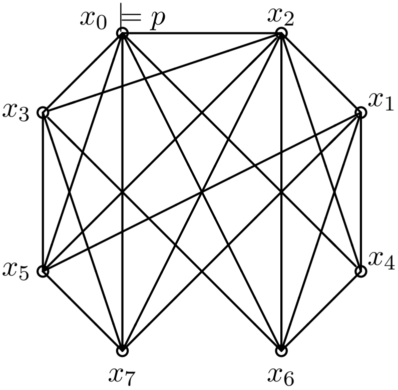

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Theorem

- $t(\alpha_n) \notin \mathbf{KTB}$ *for any* $n \geq 1$.

One may notice that for the formulas of one variable the modal equivalent of the Glivienko theorem holds:

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From INT into $KTB.Alt_n$

Logics determined by reflexive and symmetric Kripke frames with a bounded degree of branching:

$\mathbf{KTB.Alt_n} := \mathbf{KTB} \oplus (alt_n)$, $n \geq 2$, where:

$$alt_n := \Box p_1 \vee \Box(p_1 \to p_2) \vee ... \vee \Box((p_1 \wedge ... \wedge p_n) \to p_{n+1}).$$

Logics $\mathbf{KTB.Alt_n}$ are characterized by Kripke frames in which one point has at most n successors (including itself).

Lemma

For each $n \in N$, $t(\alpha_{2n+1}) \in \mathbf{KTB.Alt_n}$.

Proof. By the model $\mathfrak{M}_n$

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Symmetric closure of intuitionistics models

Definition

Let $\mathfrak{M} := \langle W, R, v\rangle$ *be an intuitionistic model. Its symmetric closure is defined as follows:* $\mathfrak{M}^* := \langle W, R^*, v\rangle$ *where* xR^*y *iff* $(xRy$ *or* $yRx)$.

Example

The weak law of the excluded middle: $\neg p \vee \neg\neg p \notin INT$. *After translation it is:* $\Box\neg p \vee \Box\Diamond p$. *We show that it is not a theorem of* $\mathbf{KTB}$ *using the symmetric closure of the appropriate intuitionstic model.*

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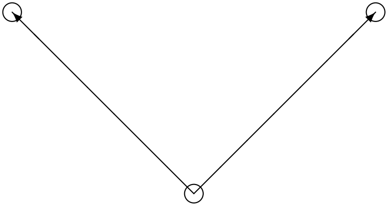
INT-model $\mathfrak{M}$

$x_2 \models p,$ $x_3 \models \neg p,$ that is $x_3 \not\models p$



$x_1 \not\models \neg p \vee \neg\neg p$

$x_1 \not\models \neg p, \; x_1 \not\models \neg\neg p$

KTB-model $\mathfrak{M}^*$

$x_2 \models p,$ $x_1 \not\models \Box\neg p \vee \Box\Diamond p$ $x_3 \not\models \Diamond p,$

$x_1 \not\models \Box\neg p, \;\; x_1 \not\models \Box\Diamond p$

$x_1 \models \Diamond p, \;\; x_1 \not\models p$

INT-model $\mathfrak{M}$

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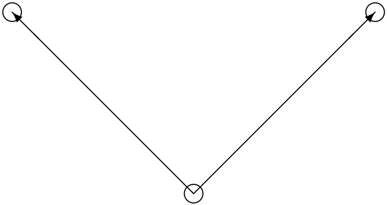

$x_1 \not\models \neg p \vee \neg\neg p$

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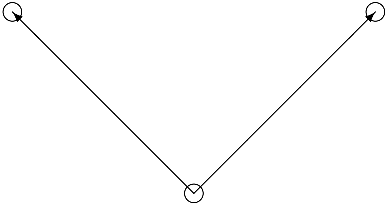

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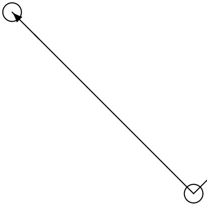

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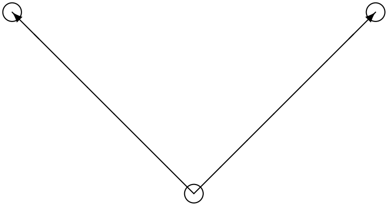

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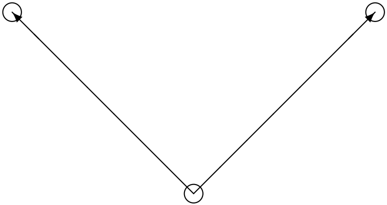

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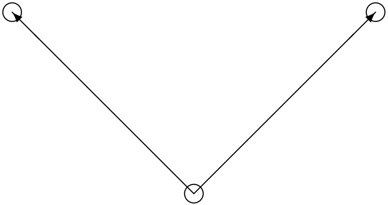

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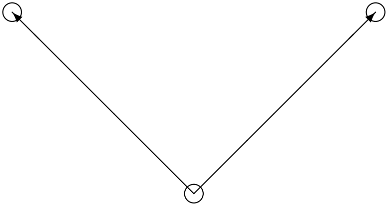

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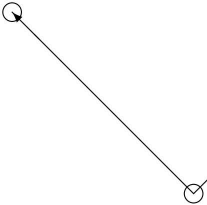

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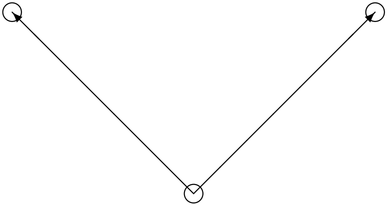

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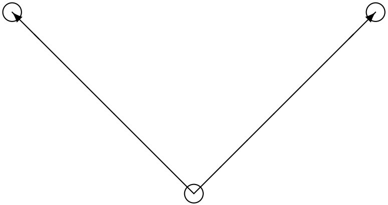

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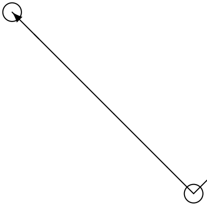

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For $\alpha \in Form_p$: if $\alpha \notin \mathbf{INT}$, then $t(\alpha) \notin \mathbf{KTB}$.

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- *If $\alpha_{2k+1} \to \alpha_{2n+1} \in \mathbf{INT}$, then $t(\alpha_{2k+1} \to \alpha_{2n+1}) \in \mathbf{KTB}$, for $k, n \in \mathbb{N}$,*
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- $(t(\alpha_6) \wedge t(\alpha_5)) \to t(\alpha_3) \in \mathbf{KTB}$
- This is $\{\Box[\Box\Diamond p \to (p \vee \Box\neg p)] \wedge (\Box\Diamond p \vee p \vee \Box\neg p)\} \to (p \vee \Box\neg p) \in \mathbf{KTB}$
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Lemma

Let $\mathfrak{M} := \langle W, R, v \rangle$ *be an intuitionistic model falsifying some* $\alpha \in Form$. *Then its symmetric closure* $\mathfrak{M}^*$ *will falsify* $t(\alpha)$.

Theorem

Let $\alpha \to \beta \in \mathbf{INT}$. *Then*

1. *if* $md(t(\alpha)) > md(t(\beta))$ *then* $t(\alpha) \to t(\beta) \in \mathbf{KTB}$
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Proof. First, we shall prove that all axioms of $\mathbf{INT}$ after applying the translation t and operation of $\Box^{k+1}$ (if it is necessary) will be theorems of $\mathbf{KTB}$, after that we shall prove that the rules of (MP), (Sub) after t are admissible. Below, we list the axioms of $\mathbf{INT}$:

A1. $p \to (q \to p)$,

A2. $(p \to (q \to s)) \to ((p \to q) \to (p \to s))$,

A3. $(p \vee q) \to (q \vee p)$,

A4. $(p \wedge q) \to (q \wedge p)$,

A5. $p \to (q \vee p)$,

A6. $(p \wedge q) \to p$,

A7. $p \to (q \to (p \wedge q))$,

A8. $(p \to s) \to ((q \to s) \to ((p \vee q) \to s))$,

A9. $((p \to q) \wedge \neg q) \to \neg p$,

A10. $(p \wedge \neg p) \to q$.

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Proof for A1. We see that $t(p) \to t(q \to p) = p \to \Box(q \to p)$ and $md(p) = 0$ and $md(\Box(q \to p)) = 1$. We will prove that $\Box^2 p \to \Box(q \to p) \in \mathbf{KTB}$.
Suppose that $\Box^2 p \to \Box(q \to p) \notin \mathbf{KTB}$. Then there exists a model $\mathfrak{M} = \langle W, R, v \rangle$ and a point $x \in W$ such that

$$x \models \Box^2 p \tag{1}$$

$$x \not\models \Box(q \to p); \tag{2}$$

From (2) we know that x sees another point, say x_2 such that $x_2 \not\models (q \to p)$ what means that $x_2 \models q$ and $x_2 \not\models p$. The last condition is in contradiction with (1).

Question and problems

- We do not known why the Glivienko theorem does not hold in language with n variables, (but sometimes it holds)
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Thank you for your attention.