Anabelian geometry in model theory setting

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Supported by EPSRC program grant “Symmetries and correspondences”

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- To translate Grothendieck anabelian program into model-theoretic language. In particular, to understand $\pi_1^{et}(\mathbb{X}, x)$ the étale fundamental group of a scheme $\mathbb{X}$.

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- excercise in duality theory: schemes (syntax) - geometric models (semantics)
- To translate Grothendieck anabelian program into model-theoretic language. In particular, to understand $\pi_1^{et}(\mathbb{X}, x)$ the étale fundamental group of a scheme $\mathbb{X}$.
- To explain model-theoretically the interaction between structures of algebraic geometry and analytic structures associated with them.

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The semantics of schemes can be given in the form of **structures** $(\mathbb{X}(\mathrm{F}), L_{\mathbb{X},\mathrm{k}})$, where $\mathbb{X}(\mathrm{F}) \subset \mathbf{P}^N(\mathrm{F})$, F an algebraically closed field and $L_{\mathbb{X},\mathrm{k}}$ is the language constructed from the scheme.

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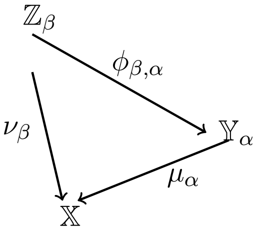
Lemma. *Let* F *be algebraically closed of characteristic 0 and* $\mathrm{k} \subset \mathrm{F}$.
There is a good functorial correspondence between structures $(\mathbb{X}(\mathrm{F}), L_{\mathbb{X},\mathrm{k}})$, *and schemes* $\mathbb{X}$ *of finite type over number fields.*

Finite étale covers of a smooth algebraic variety $\mathbb{X}$

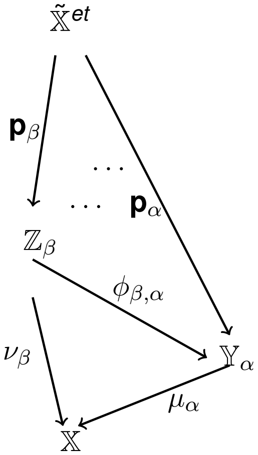
over k, $\mathrm{k}[\alpha]$ and $\mathrm{k}[\beta]$ respectively

finite étale covers

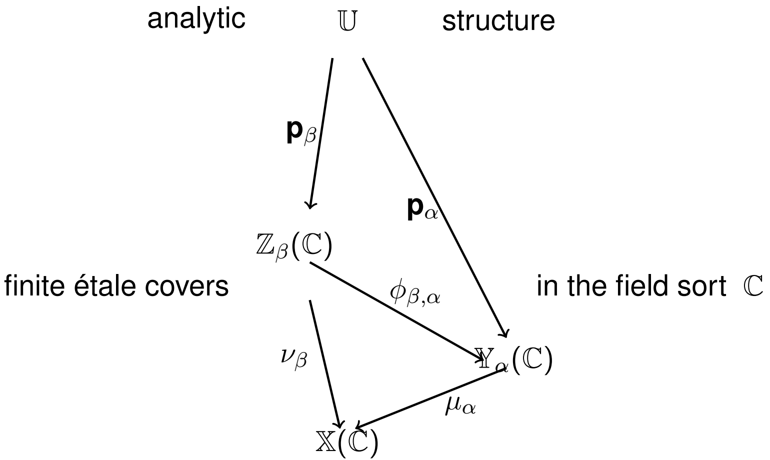


in the field sort $\mathbb{C}$

Finite étale covers of $\mathbb{X}$ and the projective limit $\tilde{\mathbb{X}}^{et}$.



Universal cover $\tilde{\mathbb{X}}^{an}(\mathbb{C})$ of $\mathbb{X}(\mathbb{C})$



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Languages $L_{\mathbb{X},k}$ and $L^{et}_{\mathbb{X},k}$.

A key question, from model theoretic perspective, is the choice of an adequate language to talk about *scheme-theoretic* (and *analytic* aspects of) algebraic geometry.

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A key question, from model theoretic perspective, is the choice of an adequate language to talk about *scheme-theoretic* (and *analytic* aspects of) algebraic geometry.

The answer is a language $L^{et}_{\mathbb{X},\mathrm{k}} \supset L_{\mathbb{X},\mathrm{k}}$.

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Analytic versus algebraic.

Theorem. *There is an explicit complete first order theory $T^{et}_{\mathbb{X}}$ in the language $L^{et}_{\mathbb{X},\mathrm{k}}$. Both analytic structure $\tilde{\mathbb{X}}^{an}(\mathbb{C})$ and the algebraic structure $\tilde{\mathbb{X}}^{et}(\mathrm{F})$ are models of $T^{et}_{\mathbb{X}}$. More precisely,*

$$\begin{array}{ccccccc} \tilde{\mathbb{X}}^{et}(\bar{\mathrm{k}}) & \longrightarrow & \tilde{\mathbb{X}}^{et}(\mathrm{F}) & \longrightarrow & \tilde{\mathbb{X}}^{et}(\mathbb{C}) & \longrightarrow \\ \uparrow & & \uparrow & & \uparrow & \\ \tilde{\mathbb{X}}^{an}(\bar{\mathrm{k}}) & \longrightarrow & \tilde{\mathbb{X}}^{an}(\mathrm{F}) & \longrightarrow & \tilde{\mathbb{X}}^{an}(\mathbb{C}) & \longrightarrow \end{array}$$

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Note that the analysis for $\tilde{\mathbb{X}}^{et}(\bar{\bar{\mathrm{k}}})$ is **adelic** (or *p*-adic) while that for $\tilde{\mathbb{X}}^{an}(\mathbb{C})$ is **complex analytic.**

Étale fundamental group.

Theorem (joint with R.Abdolahzade). For any smooth quasi-projective variety $\mathbb{X}$ over k

$$\pi_1^{et}(\mathbb{X}, x) \cong \mathrm{Aut}\, \tilde{\mathbb{X}}^{et}(\bar{\mathrm{k}}) \cong \mathrm{Las}(T_{\mathbb{X}}^{et}),$$

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Anabelian conjectures of Grothendieck and some known facts can be reformulated and re-interpreted in this setting.

Model theory classification around categoricity

A few interesting cases of analytic structures $\tilde{\mathbb{X}}^{an}(\mathbb{C})$ have been studied in the context of logical **categoricity**:

Categoricity problem

Problem. Consider $\tilde{\mathbb{X}}^{an}(\mathbb{C})$ as an abstract $L^{et}_{\mathbb{X},\mathrm{k}}$-structure. Find a natural set of possibly non-elementary (say $\mathcal{L}_{\omega_1,\omega}$) axioms $\Sigma_{\mathbb{X}}$ extending $T^{et}_{\mathbb{X}}$

- $\tilde{\mathbb{X}}^{an}(\mathbb{C}) \vDash \Sigma_{\mathbb{X}}$
- For any uncountable cardinal number κ there is **unique**, up to isomorphism, model

$$\tilde{\mathbb{X}}(\mathrm{F}) \vDash \Sigma_{\mathbb{X}}, \quad |\mathrm{F}| = \kappa.$$

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The problem brings us into the context of categoricity and stability theory for Shelah's AEC (abstract elementary classes).

Results on categoricity

Categoricity problem has positive answer over number fields k for:

- $\mathbb{X} = \mathbf{P}^1 \setminus \{0, \infty\}$, the algebraic torus (B.Z. 2004, B.Z and M.Bays 2010)
- Elliptic curves (M.Bays, 2011, M.Bays, B.Hart, A.Pillay, 2017)
- Abelian varieties (special cases, M.Bays, B.Hart, A.Pillay, 2017)
- Modular curves and Shimura varieties (partial answers, A.Harris 2013, A.Harris and C.Daw, 2014, S.Eterovic, 2019)

Results and Proofs

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1. Use Kiesler-Shelah categoricity theory to establish the **equivalence:** *categoricity holds for* $\Sigma_{\mathbb{X}}$ *iff (i) and (ii) hold:*

(i) $\mathrm{Gal_k}$ acts on $\hat{\Gamma}$ as a certain subgroup $\mathrm{Out}_S\,\hat{\Gamma} \subseteq \mathrm{Out}\,\hat{\Gamma}$

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(ii) An arithmetic statement known as Kummer theory in abelian cases.

2. Learn what is known on (i) and (ii) from your number theory colleagues.

Results and Proofs cont.

2. Establishing (i) and (ii):
(i) For 'abelian' curves it holds by Dedekind (algebraic torus) and Serre (open image theorem for elliptic curves).

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(ii) For algebraic torus is classical Kummer theory. In case of elliptic curves it is due to Bashmakov, Ribet. For abelian varieties, K.Ribet and M.Larsen.
In case of modular curves (anabelian case) Serre open image theorem for product of no-CM elliptic curves suffices.

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In case of modular curves (anabelian case) Serre open image theorem for product of no-CM elliptic curves suffices. In the more general cases of Shimura varieties this is a conjecture about Mumford-Tate groups.

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It is known

- $\mathrm{Gal}(\bar{\mathbb{Q}} : \mathbb{Q})$ is isomorphic to a subgroup of GT (Belyi)
- $\mathrm{GT} \cong \mathrm{Gal}(\bar{\mathbb{Q}} : \mathbb{Q})$ (conjectured by Grothendieck)
- GT has been described conjecturally (Drinfeld, Ihara)
- very recent work by Mochizuki and collaborators challenges Drinfeld's conjecture

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Model-theoretic classification analysis of $\tilde{\mathbb{X}}^{an}$ establishes a strong interaction of:

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Why is it that the logical assumption of categoricity leads to *correct* conjectures of arithmetic and geometric nature?