SNAPSHOTS OF DUALITY THEORY

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web version of slides

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Overall outline

Snapshot from 2019

Illustrating dualities in action

A report on recent joint work with Leonardo Cabrer, with acknowledgements also to George Metcalfe.

Snapshot from 50 years ago

How it all began ...

Outline: snapshot 2019

Dualities in action

- illustrated by

  Sugihara algebras/monoids,

  complete algebraic semantics for relevant logic R-mingle, with/without truth constant.

- ACL not TACL:

  Finite algebras; finite structures.

  When we restrict dualities to the finite level, we suppress the topology on dual structures since it is discrete.

- Applications to admissible rules (Logic) and free algebras (Algebra), via Categories.

Disclaimer: This talk is not a tutorial on 50 years of duality theory. It's a sales pitch for duality methods.

Preamble: two numbers

$$2^{44} \cdot 3^{36} \cdot (1 + 2^{-7} \cdot 3^{-4})^6$$

and

$$16$$

R-mingle logics and their algebraic semantics

Consider logic R-mingle. This adds the mingle axiom $\varphi \to (\varphi \to \varphi)$ to relevant logic.

Logic	Algebraic semantics
R-mingle, RM	Sugihara algebras
R-mingle with truth constant, $\mathrm{RM}^{\mathbf{t}}$	Sugihara monoids

(J.M. Dunn (1970))

SO: what's a Sugihara algebra/monoid?

Meet Sugihara algebras/monoids

Define $\mathbf{Z}$ to have the integers as universe and operations $\wedge, \vee, \rightarrow, \neg$. The lattice operations are those from the integers with usual order, $\neg \colon a \mapsto -a$ and

$$a \rightarrow b = \begin{cases} (-a) \vee b & \text{if } a \leqslant b, \\ (-a) \wedge b & \text{otherwise.} \end{cases}$$

The variety $\mathcal{SA}$ of Sugihara algebras is

$$\mathcal{SA} := \mathbb{HSP}(\mathbf{Z}) = \mathbb{ISP}(\mathbf{Z}).$$

For Sugihara monoids, let $\mathbf{Z^t}$ be $\mathbf{Z} \setminus \{0\}$ with added constant $\mathbf{t} = 1$. Then

$$\mathcal{SM} = \mathbb{HSP}(\mathbf{Z^t}).$$

Talk focuses on Sugihara algebras, but analogous results can be obtained for the monoid case.

Sugihara algebras/monoids in context

There are forgetful functors from $\mathcal{SA}$ and $\mathcal{SM}$ to

- DLAT (distributive lattices (no bounds))—this will be important
- KlLat (Kleene lattices): forget $\to$

In addition, but outwith this talk:

- There are connections with Heyting algebras and Gödel algebras.
- $\mathcal{SA}$ and $\mathcal{SM}$ come within the ambit of residuated lattices, but they are very special and we treat them directly.

Admissible rules problem, in general

We consider only propositional logics.

Suppose $\mathcal{L}$ is a deductive system.

- A rule for $\mathcal{L}$ consists of a finite set of premises and a conclusion.
- A rule is admissible if adding it to $\mathcal{L}$ introduces no new theorems.

We seek a re-interpretation in terms of algebraic semantics provided by a quasivariety $\mathfrak{B}$ of $\mathcal{L}$-algebras.

Admissible quasi-identities

Assume that $\mathfrak{B}$ provides complete algebraic semantics for the logic $\mathcal{L}$. Admissibility of a quasi-identity

$$\Sigma \Rightarrow \phi \approx \psi$$

is defined so that this is equivalent to its validity in $\mathbf{F}_{\mathfrak{B}}(\omega)$.

- $\Sigma \Rightarrow \phi \approx \psi$ can be verified by showing its validity on every finitely generated free algebra of $\mathfrak{B}$;
- $\Sigma \Rightarrow \phi \approx \psi$ can be refuted by showing it fails on any one of these free algebras.

We have converted the problem of admissible rules for $\mathcal{L}$ into an algebraic problem:

in the logic		in the algebraic semantics
admissible rule	$\leftrightsquigarrow$	admissible quasi-identity

Cutting down to the finite level

Work in the algebraic setting. Assume there exists s such that

$$\mathfrak{B} = \mathbb{ISP}(\mathbf{M}),\ \text{where } \mathbf{M} \text{ is } s\text{-generated}.$$

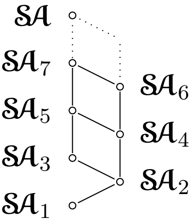
Then, for any quasi-identity $\Sigma \Rightarrow \varphi \approx \psi$,

$$\Sigma \Rightarrow \varphi \approx \psi \text{ is admissible in } \mathfrak{B} \quad \Longleftrightarrow \quad \Sigma \Vdash_{\mathbf{F}_{\mathfrak{B}}(s)} \varphi \approx \psi$$

(Metcalfe & Röthlisberger).

Promising features of $\mathcal{SA}$

- $\mathcal{SA}$ is locally finite.
- The subdirectly irreducible algebras are $\mathbf{Z}_{2n+1} := \mathbf{Z} \cap [-n, n]$ (the odd case) and $\mathbf{Z}_{2n} := (\mathbf{Z} \setminus \{0\}) \cap [-n, n]$ (the even case), for $n = 1, 2, \ldots$.
- Generation: $\mathbf{Z}_k$ is s-generated, where $s = \left[\frac{k+1}{2}\right]$.
- Free algebras: $\mathbf{F}_{\mathcal{SA}}(s) = \mathbf{F}_{\mathcal{SA}_{2n+1}}(s)$ if $n \geqslant s$ (used later).



Quasivarieties $\mathcal{SA}_k$

Admissible quasi-identities, algorithmically

Can we test for admissibility of a quasi-identity by using a smaller algebra than a free algebra on s generators? In theory, YES.

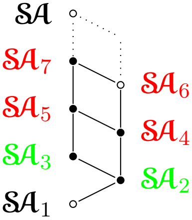
The idea (Metcalfe and Röthlisberger (2013)) is to seek an algebra $\mathbf{A}$ such that $\mathbb{ISP}(\mathbf{F}_{\boldsymbol{\mathcal{B}}}(s)) = \mathbb{ISP}(\mathbf{A})$ and $\mathbf{A}$ is of minimum size.

Can we find such an 'admissibility algebra' in practice?

Computational feasibility?

Metcalfe & Röthlisberger showed that a minimal ‘admissibility algebra’ exists, and implemented an algorithm, TAFA, to compute it. Nice algorithm, but it is not computationally feasible when free algebras on s generators have more than a few million elements.

TAFA succeeds on $\mathcal{SA}_3$ but fails on $\mathcal{SA}_4$ and above.



Limits to TAFA’s capability

Duality to the rescue?

Can we improve computational feasibility by moving from a category $\boldsymbol{\mathcal{A}}$ of algebraic models to a category $\boldsymbol{\mathfrak{X}}$ of relational models?

A dual equivalence set up by functors $\mathsf{D}\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathfrak{X}}$ and $\mathsf{E}\colon \boldsymbol{\mathfrak{X}} \to \boldsymbol{\mathcal{A}}$ will give

- a basis for a dictionary to translate between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathfrak{X}}$.

But for this to be useful it needs

- to do more than formally reverse arrows in diagrams;
- to capture in $\boldsymbol{\mathfrak{X}}$ in a meaningful way the algebraic notions in $\boldsymbol{\mathcal{A}}$ we are interested in;
- $\boldsymbol{\mathfrak{X}}$ to be 'simpler' to work with than $\boldsymbol{\mathcal{A}}$ [eg: pictorial, reduction in computational complexity].

For full functionality we really do want a dual equivalence (not just a dual adjunction).

Stone and Priestley dualities are good guys

To put our general comments in context we view Priestley duality (or Stone duality) as being set up by functors $\mathsf{D}\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathfrak{X}}$ and $\mathsf{E}\colon \boldsymbol{\mathfrak{X}} \to \boldsymbol{\mathcal{A}}$.

Special features

1. Logarithmic property: at the finite level, D acts like a logarithm, so reducing complexity.
2. Products in $\boldsymbol{\mathfrak{X}}$ are concrete (=cartesian), making the duals of free algebras easy to handle.
3. Strong property of morphisms: let f be a $\boldsymbol{\mathcal{A}}$-morphism (ie homomorphism) and $\mathsf{D}(f)$ the dual $\boldsymbol{\mathfrak{X}}$-morphism. Then

$$f \text{ is injective} \iff \mathsf{D}(f) \text{ is surjective}$$

$$f \text{ is surjective} \iff \mathsf{D}(f) \text{ is an embedding.}$$

Why is this special?

Generally, a dual equivalence won't behave this well.

v

- Coproducts will not convert to cartesian products.
- For morphisms on the algebra side, injective/surjective may not correspond to mono/epi.

And the logarithmic property is a bonus. It does not occur for example with Hofmann–Mislove–Stralka duality for semilattices, which is 'trivial' at the finite level.

So: how is good behaviour achieved in Stone and Priestley dualities?

Dualising objects

Assume we have categories $\mathcal{A}$ and $\mathfrak{X}$ and objects $\mathbf{M}$ and $\underset{\sim}{\mathbf{M}}$ in $\mathcal{A}$ and $\mathfrak{X}$, respectively, with the same underlying set M.
We say we have a dualising object [let's denote it by m], if it lives as an object $m = \mathbf{M}$ in $\mathcal{A}$ and as an object $m = \underset{\sim}{\mathbf{M}}$ in $\mathfrak{X}$.

We are aiming for a dual adjunction set up as follows.

$$\begin{array}{llcl} & & \mathcal{A}(-, m) & \\ \mathsf{D}\colon & \mathcal{A} & \longrightarrow & \mathfrak{X} \\ \\ \mathsf{E}\colon & \mathfrak{X} & \longrightarrow & \mathcal{A} \\ & & \mathfrak{X}(-, m) & \end{array}$$

Dualising objects: more detail

The dualising object m lives
in $\mathcal{A}$ (blue form) and in $\mathfrak{X}$ (red form).

To form a hom-set one uses the form in the diagram, and in each case the image objects are then structured pointwise using the other form.

Moreover, because we have hom-functors, duals of morphisms will be given by composition.

For this to work we need compatibility conditions linking $\mathbf{M}$ and $\underset{\sim}{\mathbf{M}}$.

Dualising objects, specialised

Take $\mathbf{M}$ a finite algebra and $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$. Seek $\underset{\sim}{\mathbf{M}}$, an alter ego for $\mathbf{M}$, and a category $\boldsymbol{\mathfrak{X}}$ built from it, to get a dualising object m with $m = \mathbf{M}$ and $m = \underset{\sim}{\mathbf{M}}$.

For a duality: we want sufficient compatibility to give well-defined hom-functors, giving a dual adjunction such that

- the unit and co-unit maps are evaluations, with each unit map an isomorphism;
- $\underset{\sim}{\mathbf{M}} = \mathsf{D}(\mathbf{F}_{\boldsymbol{\mathcal{A}}}(1))$ and $\underset{\sim}{\mathbf{M}}^s = \mathsf{D}(\mathbf{F}_{\boldsymbol{\mathcal{A}}}(s))$;

For a full duality (more ambitious) we also want

- the co-unit maps are isomorphisms.

A full duality gives a dual equivalence between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathfrak{X}}$.

Our prototype good guys

Duality	$\mathcal{A}$	$\mathcal{X}$
Stone	BA	STONE (Stone spaces)
Priestley	BDLat (with bounds)	PRIES (Priestley spaces)
	DLat	pointed Priestley spaces
Birkhoff	$\mathrm{BDLat}_{\mathrm{fin}}$	$\mathrm{POS}_{\mathrm{fin}}$, finite posets

[At the finite level the topology is discrete and plays no role.]

Stone and Priestley dualities are full dualities. They arise from dualising objects, given by the 2-element objects in the respective pairs of categories.

The earlier special properties were well known before the category formalism was in widespread use.

Not-so-good guys

Most dualities for classes of DLE's (distributive lattice expansions):
Start from Stone or Priestley duality;

- add extra algebraic operations and capture these on the dual side, and
- characterise dual morphisms.

[how easy or otherwise these tasks may be does not concern us hete.]

For such restricted Stone or Priestley dualities:

GOOD Retain set-based representations. Close tie-up with canonical extensions.

NOT GOOD for admissible rules problem or describing free algebras.
Duals of free algebras very seldom given by concrete products.
[Alternative of calculating left adjoint to forgetful functor may be challenging.]

More good guys—pulled out of a Black Box

Assume we are interested in $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$ where

- $\mathbf{M}$ is a finite algebra and has a reduct in BDLat or DLat.

Then there exists an alter ego $\underset{\sim}{\mathbf{M}}$ for $\mathbf{M}$ and a category $\boldsymbol{\mathfrak{X}}$ of topological relational structures, generated by $\underset{\sim}{\mathbf{M}}$, such that

- the hom-functors into $\mathbf{M}$ and $\underset{\sim}{\mathbf{M}}$ set up a dual equivalence between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathfrak{X}}$;
- the strong properties for morphisms hold;
- we get logarithmic behaviour at the finite level.

Theory supplies systematic ways to find such an $\underset{\sim}{\mathbf{M}}$ and to simplify it insofar as is possible.

Comments (staying in setting of previous slide)

The hard part in getting that a dualising object gives a dual equivalence is ensuring the dual category is not too big.
The theory of strong dualities addresses this.

Key points

- Any strong duality is full, and so gives a dual equivalence.
- Strongness is closely linked to having the strong properties for morphisms.
- We can achieve a strong duality from a non-strong one by adding extra structure to the alter ego.
  Theory tells us how to do this.

Many of our claims above rely on our assumption that $\mathbf{M}$ is finite. The class $\mathcal{SA} = \mathbb{ISP}(\mathbf{Z})$ does not come within the scope of natural duality theory. All is not lost: for the problems in this snapshot, having dualities for all finitely generated subquasivarieties is all we need.

Sugihara algebra quasivarieties: strong dualities

Consider $\mathcal{SA}_k = \mathbb{ISP}(\mathbf{Z}_k)$. Here k can be even or odd.

Let $\mathbf{M} = \mathbf{Z}_k$. From general theory, we should put into $\underset{\sim}{\mathbf{M}}$

- (certain) binary relations, which are universes of subalgebras of $\mathbf{M}^2$;
- operations, which are endomorphisms of $\mathbf{M}$;
- (certain) partial operations, which are partial endomorphisms of $\mathbf{M}$ (to get strongness);
- [technical] 1-element subalgebras (because we're working over Dlat not BDlat).

Total and partial endomorphisms are the key ingredient.

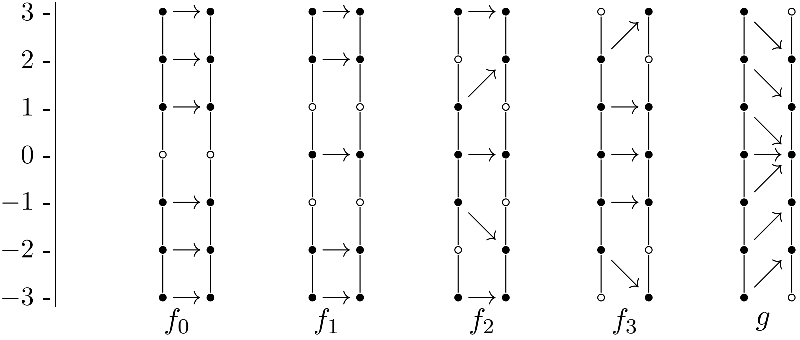
Strong dualities: Sugihara algebras, odd case

THEOREM

Fix n $(n = 1, 2, \ldots$. Let $\mathbf{M} = \mathbf{Z}_{2n+1}$. Then

$$\underset{\sim}{\mathbf{M}} = (\mathbf{Z}_{2n+1}; f_0, f_1, f_2, \ldots, f_n, g, \mathbf{0}, \mathcal{T})$$

yields a strong duality on $\mathcal{SA}_{2n+1}$. ($\mathcal{T}$ is the discrete topology.)



The case of $\mathcal{SA}_7$

Back to admissibility: TAFA in a dual form

Assume that $\mathbf{M}$ is a finite s-generated algebra in a quasivariety $\boldsymbol{\mathcal{A}}$ and that we have a strong duality between $\boldsymbol{\mathcal{A}}$ and a dual category $\boldsymbol{\mathcal{X}}$.

The Test Spaces Method (Cabrer *et al.* (2018)) provides an algorithm which allows us to find the admissibility algebra for $\boldsymbol{\mathcal{A}}$ by first identifying its dual space.

A triple $(\mathbf{X}, \gamma, \eta)$ is a Test Space configuration if

$$\mathsf{D}(\mathbf{M}) \overset{\eta}{\hookrightarrow} \mathbf{X} \overset{\gamma}{\twoheadleftarrow} \mathsf{D}(\mathbf{F}_{\boldsymbol{\mathcal{A}}}(s)) = \underset{\sim}{\mathbf{M}}^s.$$

Such a configuration supplies $\mathbf{A} := \mathsf{E}(\mathbf{X})$ in $\boldsymbol{\mathcal{A}}$ which is in $\mathbb{S}(\mathbf{F}_{\boldsymbol{\mathcal{A}}}(s))$ and such that $\mathbf{M}$ is a quotient of $\mathbf{A}$. The algorithm tells us how to choose $\mathbf{X}$ to get $|\mathbf{A}|$ minimal: the admissibility algebra.

We can apply this with $\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{SA}}_k = \mathbb{ISP}(\mathbf{Z}_k)$, for any k.

Two numbers, again

The algebra $\mathbf{Z}_5$ is 3-generated. Take $2n + 1 = 5$ and $s = 3$. Then the Test Spaces Method implies that the dual of the admissibility algebra has size 7.

The algebra itself has size

$$16$$

Two numbers, again

To find the admissibility algebra for $\mathcal{SA}_5$ algebraically, TAFA would need to hunt for $\mathbf{A}$ of minimum size such that

$$\mathbf{Z}_5 \hookrightarrow \mathbf{A} \twoheadleftarrow F_{\mathcal{SA}_5}(s).$$

But $\mathbf{F}_{\mathcal{SA}_5}(3)$ has size

$$2^{44} \cdot 3^{36} \cdot (1 + 2^{-3} \cdot 3^{-4})^6.$$

Admissibility algebras as a tool

With the aid of the Test Spaces Method, the admissibility algebra can be determined for every $\mathcal{SA}_k$.

For $\mathcal{SA}_{2n+1}$ it is an explicitly described subalgebra of $\mathbf{Z}_2 \times \mathbf{Z}_4 \times \cdots \times \mathbf{Z}_{2n} \times \mathbf{Z}_{2n+1}$ of size $5 \cdot 2^n - 4$.

Taster applications

Testing Sugihara algebra quasi-identities for admissibility:–

$p \leftrightarrow \neg p \vdash q \leftrightarrow r$ is not admissible

$p, (p \to |q|) \to (p \to q) \vdash p \to q$ is admissible.

An invitation to admissible rules buffs to take this topic forward!

Free Sugihara algebras

Now move away from logic and towards algebra. As before, results are available for Sugihara algebras and monoids, in odd and even cases. To illustrate we consider odd Sugihara algebras.
The free algebras in Sugihara quasivarieties are BIG.
What's the structure of free algebras in $\mathcal{SA}$? Seeing how this works can elucidate the structure of Sugihara algebras generally.
Recall $\mathbf{F}_{\mathcal{SA}}(s) = \mathbf{F}_{\mathcal{SA}_{2n+1}}(s)$ if $n \geqslant s$. We can work at the finite level and use our duality for $\mathcal{SA}_{2n+1}$. Key properties:

- dual equivalence;
- $\mathbf{F}_{\mathcal{SA}_{2n+1}}(s) = \mathsf{E}(\underset{\sim}{\mathbf{M}}^s)$, where $\underset{\sim}{\mathbf{M}}$ is our alter ego for $\mathbf{M} := \mathbf{Z}_{2n+1}$;
- the duality is logarithmic.

[Not relevant here: strong properties of morphisms.]

Free Sugihara algebras: our objective

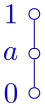
To describe the Birkhoff dual of the DLat reduct of $\mathbf{F}_{\mathcal{SA}_{2n+1}}(s)$. For this, we need to relate the natural duality at the finite level to Birkhoff duality.

General theory exists for this (Cabrer & Priestley (2015)). The process is easiest to understand by first passing to a multisorted duality. We motivate this by looking at the prototype example of Kleene algebras.

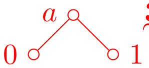
For our motivating example we could instead have used KlLat (Kleene lattices), in which any Sugihara algebra or monoid has a reduct, but KlAlg is better known and working with KlLat would involve minor distractions.

Kleene algebras: a good guy?

Davey & Werner (1983) found a strong duality for KlAlg $= \mathbb{ISP}(\mathbf{3})$ using a dualising object

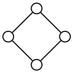


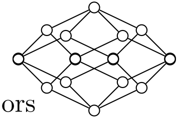
$\mathbf{3}$ with $\neg a = a$, $\neg 0 = 1$, $\neg 1 = 0$



$\underset{\sim}{\mathbf{3}}$ with order (as shown), and further structure (not shown)

Birkhoff duals of BDLat-reducts of free Kleene algebras:

1 generator 

2 generators 

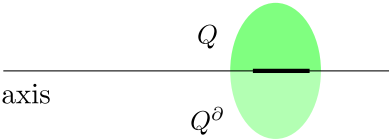
KlAlg not such a good guy?

- Lower pictures show duals of free algebras are not cartesian products for a restricted Priestley duality.
- The passage from the Davey–Werner duality to Birkhoff duals of reducts of free Kleene algebras is not transparent.

How a Kleene negation operates

Kleene algebras and lattices and Sugihara algebra/monoid quasivarieties have a Kleene negation $\neg$ on their generating chains. For finite algebras of this type, the lattice reducts have Birkhoff duals which are vertically symmetric, by the restricted Priestley duality for KlALg (Cornish & Fowler (1979)).

Any such dual is a sum got by vertical reflection of a finite poset Q: the posets Q and Q^∂ are stacked so that $x \geqslant x^\partial$, for each $x \in Q$ and corresponding $x^\partial \in Q^\partial$; here (indicated by thick line) $\min Q \cap \max Q^\partial \neq \emptyset$ may occur; no other overlapping does.



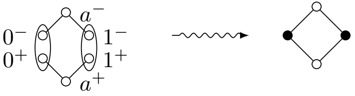
Think of Q as a tent pitched by a lake, with Q^∂ as its reflection.

Under Cornish–Fowler duality, the involution map $^\partial$ on the sum is used to capture $\neg$ on the associated Kleene algebra.

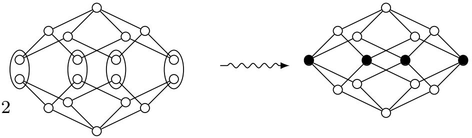
Kleene algebras: doubling up an alter ego

The diagrams show quotient maps to the Birkhoff duals of $\mathbf{F}_{\mathrm{KlAlg}}(1)$ and $\mathbf{F}_{\mathrm{KlAlg}}(2)$ from the pre-ordered set $\underset{\sim}{\mathfrak{M}}$ and from $\underset{\sim}{\mathfrak{M}}^2$ (the power is calculated 'by sorts').

$\underset{\sim}{\mathcal{M}} = \underset{\sim}{\mathbf{3}}{}_{+}^{-}$



$\underset{\sim}{\mathcal{M}}^2 = (\underset{\sim}{\mathbf{3}}{}_{+}^{-})^2$



Not shown: the 'vertical reflection' involutions which capture Kleene negation dually.

This suggests we'd like a version of natural duality theory which uses multisorted alter egos, like $\underset{\sim}{\mathfrak{M}}$ above.

Multisorted natural dualities

Theory exists (Davey & Priestley (1987)). It was devised to find a nice duality, piggyback fashion, for Kleene algebras.

Consider $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\boldsymbol{\mathfrak{M}})$, where $\boldsymbol{\mathfrak{M}} = \{\mathbf{M}_1, \mathbf{M}_2\}$, and $\mathbf{M}_1, \mathbf{M}_2$ are finite algebras, possibly equal, with disjoint(ified) universes. [Finitely many sorts handled likewise.]
An alter ego $\underset{\sim}{\mathfrak{M}}$ is $M_1 \dot\cup M_2$ equipped with

- structure on each individual sort;
- structure linking pairs of sorts.

Then the natural duality framework extends, *mutatis mutandis*.
Critical point: Powers on the dual side are formed 'by sorts'. So

$$\mathsf{D}(\mathbf{F}_{\boldsymbol{\mathcal{A}}}(s)) = \underset{\sim}{\mathfrak{M}}^s, \text{ with universe } M_1^s \dot\cup M_2^s.$$

Bonus: we can encompass finitely generated varieties not expressible as $\mathbb{ISP}(\mathbf{M})$, whenever Jónsson's Lemma is available.

From a multisorted alter ego to a Birkhoff dual

Theory is drawn from Cabrer & Priestley (2015).

For $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$, where $\mathbf{M}$ is finite and has reduct in DBLAT or DLat, there is a multisorted alter ego $\underset{\sim}{\mathfrak{M}}$ yielding a strong duality, with $\mathsf{D}(\mathbf{F}_{\boldsymbol{\mathcal{A}}}(s)) = \underset{\sim}{\mathfrak{M}}^s$.
The structure $\underset{\sim}{\mathfrak{M}}$ induces a pre-order $\preccurlyeq$ on the universe of $\underset{\sim}{\mathfrak{M}}^s$.
Form the quotient by $\preccurlyeq \cap \succcurlyeq$.

Bingo!

The resulting poset is the Birkhoff dual of the reduct of $\mathbf{F}_{\boldsymbol{\mathcal{A}}}(s)$.

Postscript: the single-sorted and multisorted dualities for KlAlg compared. The structure on the Davey–Werner alter ego $\underset{\sim}{\mathbf{3}}$, which we avoided describing in full, encodes the information needed to create the 2-sorted alter ego $\underset{\sim}{\mathbf{3}}{}^{-}_{+}$ with its pre-order $\preccurlyeq$. This requires work. Conclusion: the two dualities are essentially the same, but the 2-sorted alter ego is easier to understand.

Multisorted duality for $\mathcal{SA}_{2n+1}$

Treat $\mathcal{SA}_{2n+1}$ as $\mathbb{ISP}(\mathbf{P}^-, \mathbf{P}^+)$ where $\mathbf{P}^-, \mathbf{P}^+$ are disjointified copies of $\mathbf{Z}_{2n+1}$.

In the alter ego:

- The structure on each sort is provided by the partial endomorphisms of $\mathbf{Z}_{2n+1}$ (a (nice!) generating set for these suffices).
- Linking structure between the sorts is provided by natural isomorphisms between them.

Shall omit the official statement.

[Note: the best way to handle the even case is to obtain a 3-sorted duality for $\mathbb{HSP}(\mathbf{Z}_{2n}) = \mathbb{ISP}(\mathbf{Z}_{2n}, \mathbf{Z}_{2n-1})$.]

Free Sugihara algebras: a taster

Given the multisorted alter ego $\underset{\sim}{\mathfrak{M}}$ for $\mathcal{SA}_{2n+1}$, we want to obtain the Birkhoff dual of (the reduct of) $\mathbf{F}_{\mathcal{SA}_{2n+1}}(s)$.

Key point: $\underset{\sim}{\mathfrak{M}}$ encodes exactly the information we need to do this.

The structure of $\underset{\sim}{\mathfrak{M}}$ lifts pointwise to $(P^-)^s \,\dot\cup\, (P^+)^s$ to give $\underset{\sim}{\mathfrak{M}}^s = \mathsf{D}(\mathbf{F}_{\mathcal{SA}_{2n+1}}(s))$. This encodes a pre-order $\preccurlyeq$ on $(P^-)^s \,\dot\cup\, (P^+)^s$. The Birkhoff dual we seek, $(Y, \sqsubseteq)$ say, is the quotient by $\preccurlyeq \cap \succcurlyeq$ of $(P^-)^s \,\dot\cup\, (P^+)^s$.

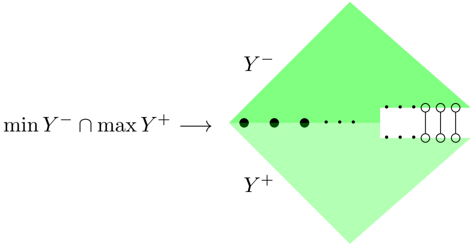
Tasks: describe $\preccurlyeq$, and then $\sqsubseteq$,

- when restricted to $(P^-)^s$ and $(P^+)^s$, so obtaining posets $(Y^-, \sqsubseteq)$ and $(Y^+, \sqsubseteq)$;
- when acting on pairs of tuples drawn from different sorts, to see how Y^- and Y^+ glue together.

Free Sugihara algebras, a taster

The ordered set $(Y, \sqsubseteq)$ is a sum by vertical reflection of Y^-, the upper layer. The lower layer Y^+ is dually order-isomorphic to Y^-.

So how do the layers glue together in the middle? There is a canonical labelling of the s-tuples in Y^-. Let $\mathbf{a} \in \min Y^-$ and $\mathbf{a}^\partial$ be its 'mate' in $\max Y^+$. Then $\mathbf{a}$ and $\mathbf{a}^\partial$ get glued together iff $\mathbf{a}$ has no zero coordinates, and otherwise they form a 2-element chain in the sum. No other glueing occurs.



A stylised picture of the Birkhoff dual of a free Sugihara algebra, to indicate how the layers are related.

Free Sugihara algebras, a taster

What about the internal structure of the layers?

The upper layer

- is a tree, growing downwards;
- the canonical labelling of the elements and the partial endomorphisms in $\mathfrak{M}$ together tell us
  - the upper-cover relation (given by the partial endomorphism g in $\mathfrak{M}$) and
  - the way the tree branches and how multiple copies of isomorphic subtrees arise.

Crucially, the tree can be built recursively.

We have corresponding claims for the lower layer, which is an upward-growing tree. The entire Birkhoff dual can then be built recursively.

All this is good news for performing a structural and combinatorial analysis of the free algebras.

Aside: other classes in the Sugihara family

We have focused on finitely generated Sugihara algebra quasivarieties in the odd case. Only localised adaptations are needed to handle other classes—we do not need to start afresh,

For odd Sugihara monoids we get essentially the same pictures but now without glueing between upper and lower layers.

For the even case (algebras and monoids) we use 3-sorted dualities. The third sort contributes an antichain lying between the two layers we had in the odd cases.

END OF 2019 SNAPSHOT

And there's a lot more I could have said …

Outline: snapshot from 50 years ago

- Putting the T back into TACL
- How it all began . . .

Marshall Stone's legacy

A cardinal principle of modern mathematical research may be stated as a maxim: "One must always topologize."

Marshall Stone, 1938)

Was Stone right?

Stone's duality for Boolean algebras:
Dual category STONE of Stone spaces (a.k.a. Boolean spaces)—purely topological.

Full marks to Stone!

Stone's 1937 duality for distributive lattices

Stone's duality for BDLat uses a purely topological dual category, SPEC, of spectral spaces.
His paper lay virtually dormant for nearly 40 years. Why?

Unloved?

- Stone comes across as unimpressed by his own results.
- 'akertGian-Carlo Rota (1972):
  The theory of distributive lattices is richer than that of Boolean algebras; nevertheless it has had an abnormal development. Stone's representation closely imitated his representation for Boolean algebras and turned out to be too contrived. (I have yet to find a person who can state the entire theorem from memory.)
  Second, a strange prejudice circulated among mathematicians that distributive lattices are just Boolean algebras' 'weak sisters'.

Stone's 1937 duality

Ahead of its time?

- T_0 spaces not in vogue in the 1930's.
  Dana Scott, who kick-started domain theory in his *Continuous lattices* paper (1971), makes a case for them, contrasting this with disparaging terminology (eg *feebly semi-separated*) and implicit mild contempt in topologists' textbooks in which T_0 grudgingly appeared as a source of exercises.

Or was Stone's paper simply too hard to access and so unread?

- Published in Czechoslovak journal, Časopis Pěst. Mat. Fys.

Fresh perspectives, 1968–1972

Two very different PhD theses:–

- Compact totally order-disconnected spaces make their first appearance.
  (Michael Canfell, *Type 1 Semialgebras of Continuous Functions* (University of Edinburgh, 1968)).

- Spectral spaces get a make-over with the introduction of the patch topology construction and use of specialisation order, in the context of prime spectra of commutative rings (Melvin Hochster (Princeton University, 1968)).

  Ground-breaking work. Nothing on distributive lattices.

Fresh perspectives, 1968–1972

Canfell mentions compact totally order-disconnected spaces on 2 pages, for a density result (via Stone–Weierstrass Theorem).

- Nachbin's *Topology and Order* book did feature, and
- the topologies $\mathcal{U}$ (open up-sets), $\mathcal{L}$ (open down-sets) did appear, fleetingly.

But:

- No lattices. No categories.

$$\text{BA} \longleftrightarrow \text{STONE}$$

$$?? \dashleftarrow \text{Canfell's spaces}$$

?? turned out to be bounded distributive lattices (Priestley, Bull. London Math. Soc. (1970)).

How did this come about?

Serendipity!

Functional analysis met set theory and logic.

John L. Bell (of Bell & Slomson, *Models and Ultraproducts* (1969)) developed at an early age his interest in set theory, including choice principles, and logic. He indoctrinated his undergraduate tutor, David Edwards.
David was later my graduate supervisor and acted as external examiner of Canfell's thesis. He suggested I look at the thesis, mentioning Stone duality ...

[David and John are both still active researchers, and both maintain the interests in the topics they discussed in John's freewheeling 1960's Oxford tutorials.]

A prototype for natural dualities

Our results in the 2019 snapshot rely our being able to draw on the rich theory of natural dualities.
For this, Priestley duality provided a trailblazer example, hinting at the form a 'natural' duality, based on a dualising object, should take.
The alter ego for Stone duality for BA is a structure which is just a set (with discrete topology). Too special to reveal a general pattern.

Enter another key player . . .

An airletter from Australia, March 1971

Dear Dr. Priestly,

I have just received a preprint of T.P. Speed's paper. "Profinite Posets", in which he refers to two of your papers, namely, "Representations of distributive lattices by means of ordered stone spaces" and "Ordered topological spaces and the representation of distributive lattices".

I am greatly interested in this topic and if possible I would like to purchase a copy of your thesis, but if that is not possible copies of these papers would be appreciated.

Yours sincerely,

Sender: a Mr B. Davey, who later learned to spell my name correctly.

Priestley duality in its infancy

Special cases, and categorical features and constructions.

- We capture
  - Birkhoff duality, for the finite case (when the topology is discrete);
  - Stone duality for BA (when the order is discrete);
  - minimal Boolean extension (forget the order).
- PRIES profinite (Speed (1972), as mentioned by Davey).

All the above have precise categorical formulations. One later result deserves a mention alongside them.

- Canonical extension of L in BDLAT (all up-sets of L's-dual space). Paved the way for canonical extensions for DLO's and DLE's (Gehrke & Jónsson (1995, 2004)).

Another route to Priestley duality (nearly)

André Joyal (two abstracts in Notices AMS (1971), submitted Dec. 1970) covered in a couple of paragraphs:

- patch topology on a spectral space;
- profiniteness;
- the adjunction giving the minimal Boolean extension;
- the dual equivalence between BDLat and a category, call it $\mathfrak{Z}$, of Stone spaces equipped with a closed order relation (it's generated profinitely).

What's missing: an explicit description of the dual category.

Ordered Stone spaces versus Priestley spaces

Joyal did not claim that there is a dual equivalence between BDLAT and the category of all ordered Stone spaces, *viz.* Stone spaces equipped with a topologically-closed partial order.

He was right.

A famous example (Stralka (1980))
Take the Cantor chain with its usual topology. It is well known as being the dual space of a countable atomless BA.
Impose a partial order for which covering pairs in the Cantor chain are retained as 2-element chains, and the new order is otherwise discrete. The resulting space has a closed order relation but is not a Priestley space.

Priestley duality versus Stone's 1937 duality

At the object level:

Priestley space $(X, \leqslant, \mathcal{T})$ $\longrightarrow$ spectral space $(X, \mathcal{U})$

X with specialisation order & patch topology $\longleftarrow$ (X, τ)

For morphisms. the correspondence dates back to Nerode (1959)—a solitary follow-up to Stone's paper during its years in the wilderness. Nerode builds the dual space of the minimal Boolean extension of $L \in$ BDLAT. Implicitly and with hindsight, he came close to deriving Priestley duality.

More details: Bezhanishvili $\times 2$, Gabeleia, Kurz (2010).

Priestley duality versus Stone's 1937 duality

In fact, as is well known, PRIES and SPEC are isomorphic categories, not just equivalent categories (Cornish (1975), Fleisher (2000)). Same thing, two guises.
So why bother with both PRIES and SPEC?
Each has its merits.

- SPEC has a new book (Dickmann, Tressl, Schwartz (March 2019)), facing towards ring theory and algebraic geometry. 652 pages, NO pictures.
- In the context of BDLat and DLat (and of classes of DLE's), Priestley duality
  - embraces Birkhoff duality: pictorial features are a big plus;
  - links well to natural dualities.

Here's a challenge: describe the duals in SPEC of free Sugihara algebras/monoids.

Esakia Duality

Heyting algebras provide complete algebraic semantics for IPC (Intuitionistic Propositional Calculus). They have a special place in the history of duality theory.

Heyting algebras are dually equivalent to the category of Esakia spaces, a non-full subcategory of PRIES. This landmark duality was developed by Leo Esakia and presented in his 1985 monograph in Russian,

Heyting Algebras: Duality Theory.

Springer is publishing in 2019 an English version of Esakia's monograph, edited by Guram Bezhanishvili & Wesley Holliday, based on a English translation produced long ago by Anton Evseev, then an Oxford undergraduate.

It is very pleasing that Leo Esakia's pioneering contribution will shortly be accessible to a wider readership.

References (Snapshot 2019)

For admissibility algebras for R-mingle via natural dualities

- L.M. Cabrer, B. Freisberg, G. Metcalfe and H.A. Priestley, Checking admissibility using natural dualities. ACM Trans. Comput. Logic **20**, no.1, Art. 2 (2019) (preprint available at `www.arXiv:1801.02046v2`)
- L.M. Cabrer and H.A. Priestley, Sugihara algebras: admissibility algebras via the test spaces method (J. Pure Appl. Algebra (to appear); preprint available at `www/arxiv.org/abs/1809.07816v2`)

For free algebras in Sugihara algebras/monoids

- L.M. Cabrer and H.A. Priestley, Sugihara algebras and Sugihara monoids: multisorted dualities (submitted; preprint available at `www/arxiv.org/abs/1901.09533v2`)
- L.M. Cabrer and H.A. Priestley, Sugihara algebras and Sugihara monoids: free algebras (manuscript)

References (natural duality background)

Recommended as introductory reading on duality theory for those who wish to peek inside the black box.

- B.A. Davey, An invitation to natural dualities in general and Priestley duality in particular. Course at TACL Summer School 2017. Videos and slides for the four lectures available at `http://logica.dipmat.unisa.it/tacl/page/2/`
- B.A. Davey, Lonely Planet Guide to Natural Dualities. Available at `https://www.researchgate.net/publication/281803079_Lonely_Planet_Guide_to_the_Theory_of_Natural_Dualities`