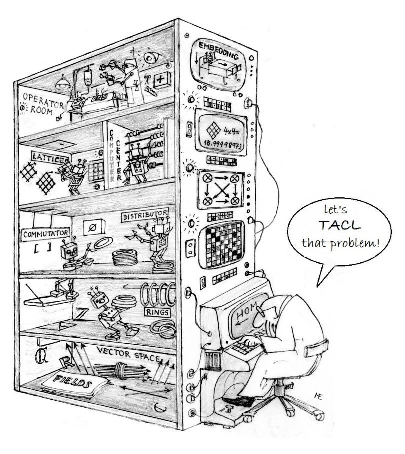
TALK: Topologie-Algebra-Logik-Kategorien



GENERALIZED CONTINUOUS CLOSURE SPACES: A TOPOLOGICAL APPROACH TO DOMAIN THEORY

TACL 2019

NICE

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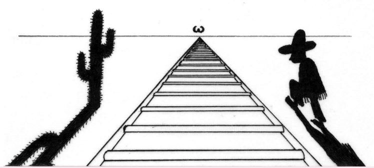
June 28, 2019

Part I

TOPOLOGY - ALGEBRA - CATEGORIES - LOGIC

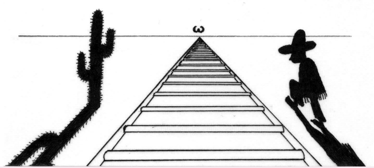
Topology Algebra Categories Logic

Natural numbers, finite sets, chains and directed sets



- ω is the first infinite ordinal (the set of all natural numbers).
- Ω is the first uncountable ordinal.
- A chain is a nonempty totally ordered set.
- An ω-chain is a chain that is finite or isomorphic to ω.
- If F is a finite subset of X, we write $F \subseteq_\omega X$.
- A set is directed if each finite subset has an upper bound.

Natural numbers, finite sets, chains and directed sets



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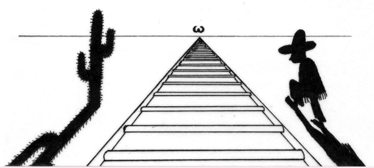

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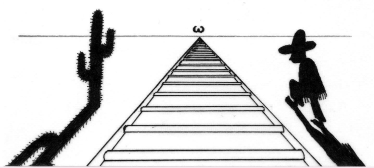

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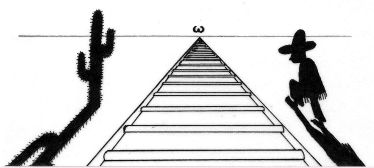

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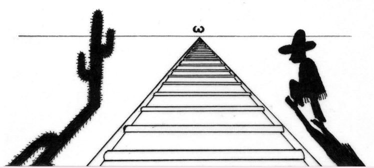

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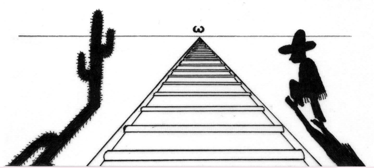

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Lower sets, upper sets, cuts and ω-ideals

Let X be a quasiordered set (qoset) with reflexive and transitive order relation $\leq$, and Y a subset of X. The

- principal ideal resp. principal filter (core) of $y \in X$ is $\downarrow y = \{x \in X : x \leq y\}$ resp. $\uparrow y = \{x \in X : x \geq y\}$,
- lower set (downset) resp. upper set (upset) generated by Y is $\downarrow Y = \bigcup\{\downarrow y : y \in Y\}$ resp. $\uparrow Y = \bigcup\{\uparrow y : y \in Y\}$,
- set of lower bounds resp. of set of upper bounds is $Y^{\downarrow} = \bigcap\{\downarrow y : y \in Y\}$ resp. $Y^{\uparrow} = \bigcap\{\uparrow y : y \in Y\}$,
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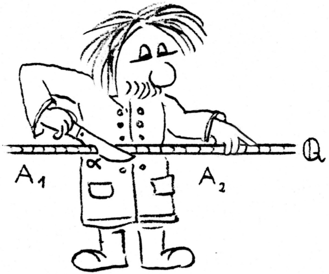
The ABC of order and topology



Dedekind creates a cut

ME, Einführung in die Ordnungstheorie, 1982

Standard extensions

A standard extension of a qoset X is a collection of lower sets that contains at least all principal ideals.

Specifically, we have the following standard extensions:

$\mathcal{A} = \{Y : Y = \downarrow Y\}$,	the Alexandroff completion
$\mathcal{B} = \{Y : Y = \downarrow B,\ B\ \omega\text{-chain}\}$,	the ω-based ideal extension
$\mathcal{C} = \{Y : Y = \downarrow C,\ C \text{ chain}\}$,	the greatest chain-ideal extension
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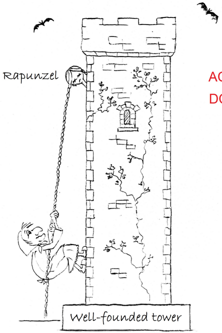
Chain conditions



ACC (Ascending Chain Condition) excludes ω-chains
DCC (Descending Chain Condition) excludes their duals

ME, Einführung in die Ordnungstheorie, 1982

Many results in due course are essentially simplified in presence of the ACC or DCC, but we are not discussing these variants.

Idempotent ideal relations

The square of a relation ϱ on X is $\varrho^2 = \{(x,z) : \exists\, y\, (x \;\varrho\; y \;\varrho\; z)\}$.

For $Y \subseteq X$, $\varrho Y = \{x : \exists\, y \in Y\, (x \;\varrho\; y)\}$, $Y\varrho = \{x : \exists\, y \in Y\, (y \;\varrho\; x)\}$.

The relation ϱ is

- transitive if $\varrho \supseteq \varrho^2$,
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- ideal if each $\varrho y = \{x : x \;\varrho\; y\}$ is an ideal in the quasiorder $\leq$ given by $x \leq y \Leftrightarrow \varrho x \subseteq \varrho y$.

The abstract bases of domain theory are the pairs (X, ϱ) where ϱ is an idempotent ideal relation on X.

The specialization order

Let X be a topological space with open set lattice (topology) $\mathcal{O}$.

- The specialization order of X is given by
$$x \leq y \;\Leftrightarrow\; x \in \overline{\{y\}} \;\Leftrightarrow\; \forall\, U \in \mathcal{O}\,(x \in U \Rightarrow y \in U).$$
All order-theoretical statements on spaces refer to $\leq$. The specialization qoset of X is the underlying set ordered by $\leq$.
- The (neighborhood) core of a point $x \in X$ is $\uparrow x = \{x : x \leq y\}$, the intersection of all neighborhoods of x.
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A-space if all cores (and intersections of open sets) are open,
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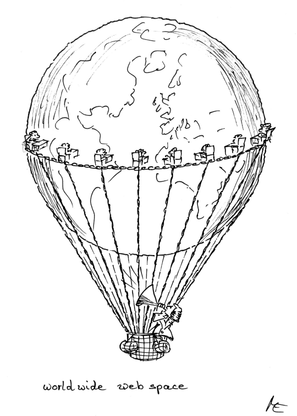
Worldwide web spaces

C-spaces are also known as locally supercompact spaces or Worldwide Web Spaces (ME 2010)



A fundamental link between order and topology

- **Q** is the category of qosets and isotone (order-preserving) maps.
- The interior relation ϱ of a topological space is given by $x \ \varrho \ y \Leftrightarrow y \in int \uparrow x$.

Theorem (ME 1991)

Sending topological spaces to their interior relations, one obtains functorial isomorphisms, concrete over **Q**, *between pairs of categories with the following objects:*

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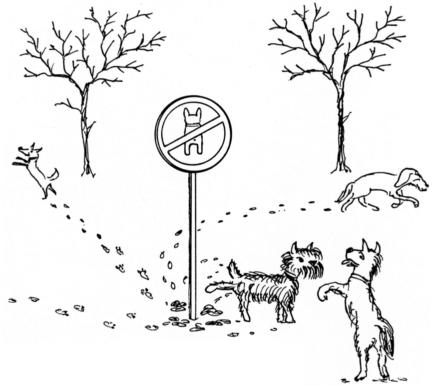
Approximating relations



Accumulation and approximation

ME, Lectures on Topology, 1984

Relational tools of domain theory

In its widest sense, domain theory may be regarded as the theory of approximation, both in the topological and in the relational sense.

As we shall see, there are close connections, in fact, categorical isomorphisms, between certain relational structures with suitable approximation properties, closure spaces, and convergence spaces. Instances of that phenomenon are provided by the ABC-Theorem.

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Properties of auxiliary relations

Let X be a set, quasiordered by $\leq$, and $\mathcal{M}$ a collection of subsets of X, ordered by inclusion. A relation ϱ on X is

- separating if $x \leq y \Leftrightarrow \forall z \in X \, (\varrho z \subseteq {\downarrow} x \Rightarrow \varrho z \subseteq {\downarrow} y)$
- defining if $x \leq y \Leftrightarrow \varrho x \subseteq \varrho y$
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- ω-interpolating if $x \, \varrho \, z \Rightarrow \exists F \subset_\omega \varrho z \, (F^\uparrow \subseteq x\varrho)$
- an $\mathcal{M}$-relation if $x \mapsto \varrho x$ is an isotone map from X to $\mathcal{M}$ with $\varrho x \subseteq {\downarrow} x$ for all $x \in X$.

The $\mathcal{A}$-relations are the classical auxiliary relations, whereas our approximating $\mathcal{D}$-relations are the approximating auxiliary relations in the sense of Continuous Lattices and Domains.

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Some basic implications

Lemma

Let ϱ be an auxiliary relation on a qoset X.

(1) *If ϱ is approximating then it is separating and defining.*

(2) *If ϱ is interpolating then it is ω-interpolating.*

(3) *If ϱ is an ω-interpolating $\mathcal{D}$-relation then it is interpolating.*

(4) *If ϱ is ω-interpolating and defining then it is approximating.*

(5) *If X is a poset then ϱ is approximating iff $x = \bigvee \varrho x$ for all x.*

(6) *If X is a chain then ϱ is approximating iff it is interpolating and defining.*

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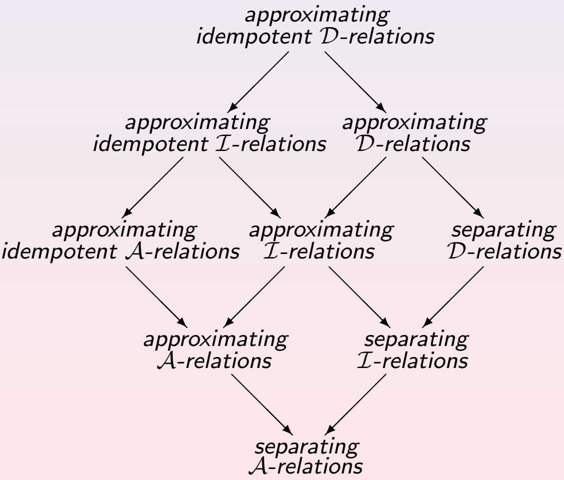
Implications between properties of auxiliary relations



Some examples

(1) The only reflexive auxiliary relation on a qoset X is the order relation.

(2) While the relation $<$ on $\mathbb{R}$, the reals, is an approximating and interpolating $\mathcal{D}$-relation, the relation $<$ on $\mathbb{Z}$, the integers, or on ω, is a separating and defining $\mathcal{D}$-relation for $\leq$ but neither approximating nor interpolating.

(3) The relation ϱ on $\mathbb{Z}$ with $x \, \varrho \, y \Leftrightarrow x \leq y$ and $(x < y$ if $y > 0)$ is a separating $\mathcal{D}$-relation but not defining, since $\varrho \, 0 = \varrho \, 1$.

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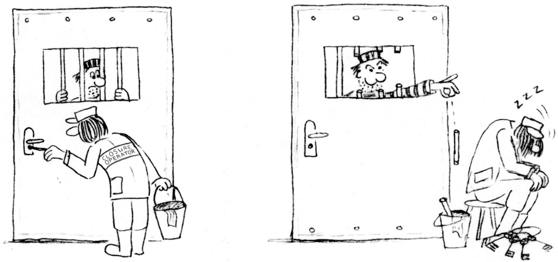
Closure and preclosure operations



Closure operator and clopen sets

ME, Lectures on Topology, 1984

Preclosure operations and operators

- A preclosure operation on a qoset X is an isotone and extensive map $p : X \to X$, that is, $x \leq y$ implies $x \leq px \leq py$.
- A closure operation is an idempotent preclosure operation.
- A (pre)closure operator on a set X is a (pre)closure operation on the power set lattice $\mathcal{P}X$. The pair (X, p) is then a (pre)closure space.
- Its specialization order is given by $x \leq_p y \Leftrightarrow p\{x\} \subseteq p\{y\}$.
- (X, p) is a lower preclosure space if $p{\downarrow}Y = {\downarrow}pY = pY$.
- (X, p) is a topped preclosure space if $p{\downarrow}x = {\downarrow}x$ for all $x \in X$.

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Every closure space is a topped lower preclosure space.

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Every closure space is a topped lower preclosure space.

Preclosure operations and operators

- A preclosure operation on a qoset X is an isotone and extensive map $p : X \to X$, that is, $x \leq y$ implies $x \leq px \leq py$.
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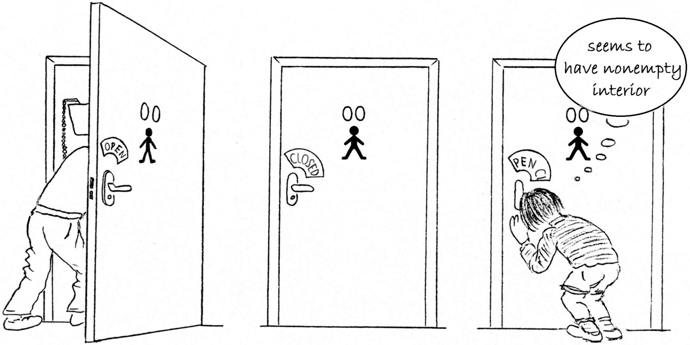

ME, Lectures on Analysis, 1981

Preinterior relations and precore spaces

- The preinterior operator of a preclosure space (X, p) is given by $i\,Y = X \setminus p(X \setminus Y)$.
- The preinterior relation of a preclosure space (X, p) is given by $x \ll_p y \Leftrightarrow y \in i{\uparrow}x \Leftrightarrow y \notin p(X \setminus {\uparrow}x)$.
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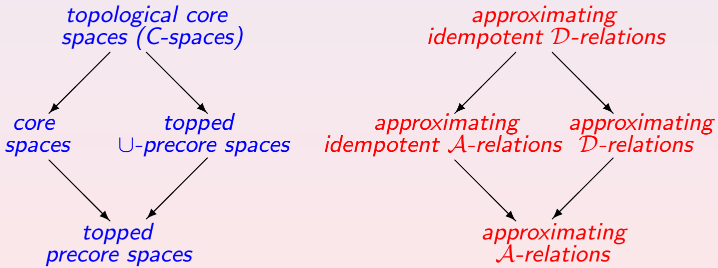

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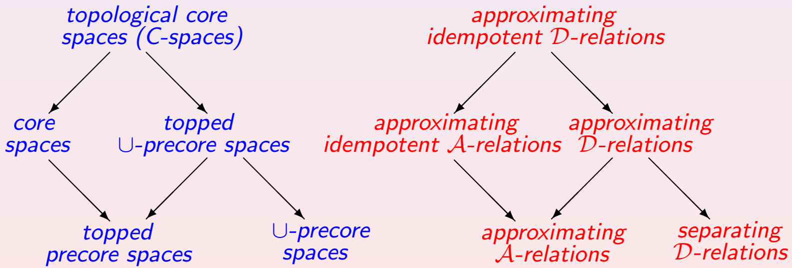

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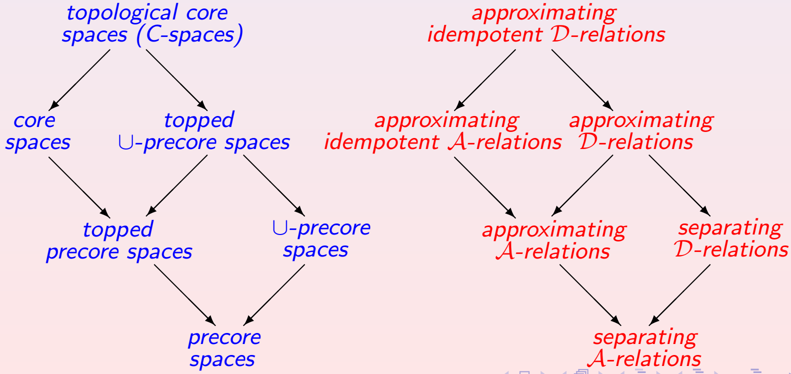

Convergence and distributivity

There are diverse surprising connections between properties of natural convergence structures on complete lattices and certain infinitary distributive laws.

A typical instance is provided by the continuous lattices, which may be characterized equationally (with infinitary operations) but also by the property that their lower lim-inf convergence, alias Scott convergence, is topological.

These observations extend to quasiordered sets and to closure spaces equipped with their specialization order.

A famous application of continuous lattice theory to logic is Scott's discovery that continuous lattices provide models for the λ-calculus. Of course, other distributive laws are of importance in logic as well.

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$\diamond$ holds for all $\mathcal{Y} \subseteq \mathcal{A}$ iff X is completely distributive.

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Completely distributive lattices are also called supercontinuous.
The frames are just the complete Heyting algebras.

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Complete distributivity of chains

Theorem (Raney 1953)

Complete sublattices of products of complete chains are completely distributive. The converse holds under a mild choice principle.

ME, Einführung in die Ordnungstheorie, 1982

Superalgebraic lattices and homomorphic representations

- A complete lattice is superalgebraic if each of its elements is a join of supercompact elements x (satisfying $x \in \downarrow Y$ for all Y with $x \leq \bigvee Y$).

Theorem

(1) *The superalgebraic lattice are exactly the isomorphic copies of Alexandroff topologies.*

(2) *The supercontinuous lattices are exactly the images of superalgebraic lattices under maps preserving arbitrary joins and meets.*

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Stacks, filters and grills

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- A stack on X is an upper set in $\mathcal{P}X$.
- A filter on X is a proper dual ideal in $\mathcal{P}X$.
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Lemma

(1) *$\mathcal{Y}$ is a stack iff $\mathcal{Y}^{\sharp\sharp} = \mathcal{Y}$.*

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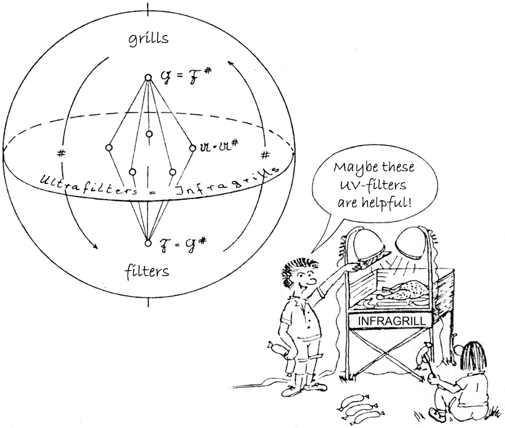
The involution ♯ on the superalgebraic stack lattice



ME, Lectures on Topology, 1984

Convergence spaces

- A (filter) convergence relation on a set X is a relation C between filters on X and elements of X such that $\mathcal{F} \; C \; x$ implies $\mathcal{G} \; C \; x$ for all filters $\mathcal{G} \supseteq \mathcal{F}$ and $\dot{x} \; C \; x$ for the ultrafilters $\dot{x} = \{Y \subseteq X : x \in Y\}$.
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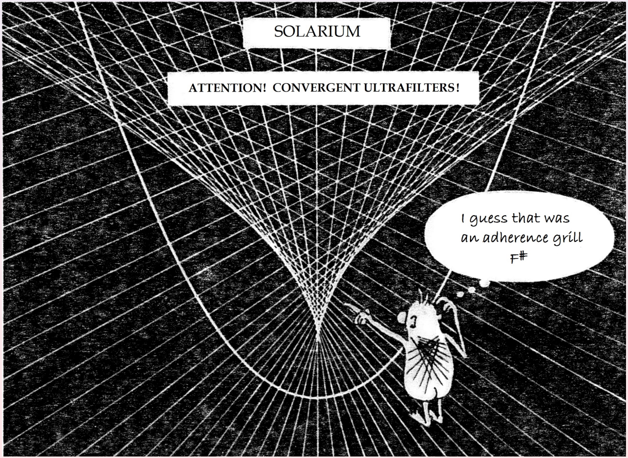
Convergence and adherence



Core generated and core based convergence spaces

A convergence-theoretical generalization of the notion of C-spaces (alias topological core spaces) is provided by the following definition, in which cores refer to the induced topology.

- A convergence relation or space is core generated, resp. core based, if each filter converging to x contains a filter that has a subbase, resp. base, consisting of cores and converges to x.

Lemma

The core generated, resp. core based, pretopological spaces are those in which every preneighborhood filter $\mathcal{V}_C x$ has a subbase, resp. base, of cores. Hence, the C-spaces are, up to a categorical isomorphism, just the core based topological (convergence) spaces.

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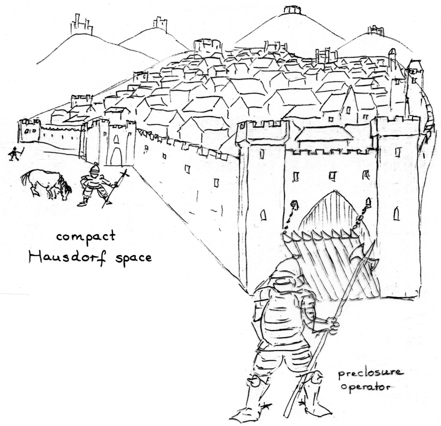
Closure, Limits and Separation



ME, History of comps, Middle Age, 2000

$\cup$-precore spaces = core generated pretopological spaces

Since the pioneering work of Hausdorff, who was the first to present a common theory of set theory, order theory and topology, one knows that, in modern language, the category of Čech closure spaces is concretely isomorphic to the category of pretopological spaces, by passing from convergence spaces (X, C) to the preclosure spaces (X, ch_C), where $x \in ch_C Y \Leftrightarrow Y \in \mathcal{V}_C x^\sharp$.

For the case of core based spaces, this amounts to the following

Lemma

The concrete isomorphism between pretopological spaces and Čech closure spaces induces an isomorphism between the category of core based pretopological spaces and that of $\cup$-precore spaces, and in particular, an isomorphism between the category of core based topological spaces and that of C-spaces.

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Theorem (ME 2019)

Sending pretopological spaces (X, C) to $(X, \leq_C, \ll_C)$, one obtains isomorphisms, concrete over $\mathbf{Q}$*, between these pairs of categories:*

core based topological spaces	*qosets with approximating interpolating $\mathcal{D}$-relations*

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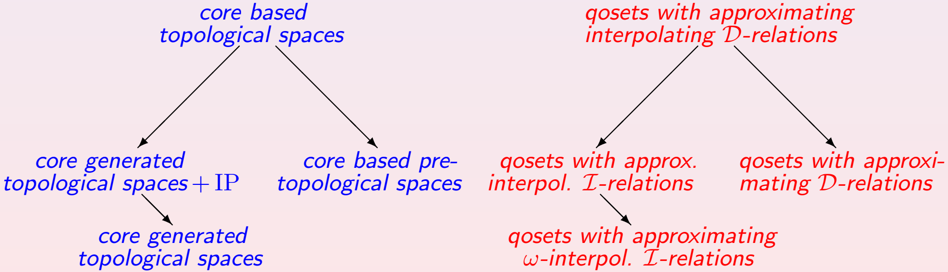

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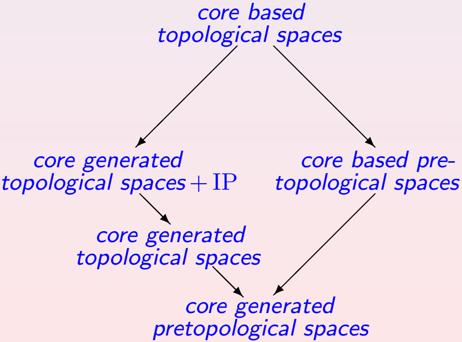
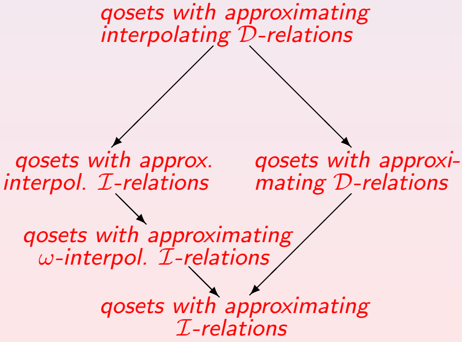
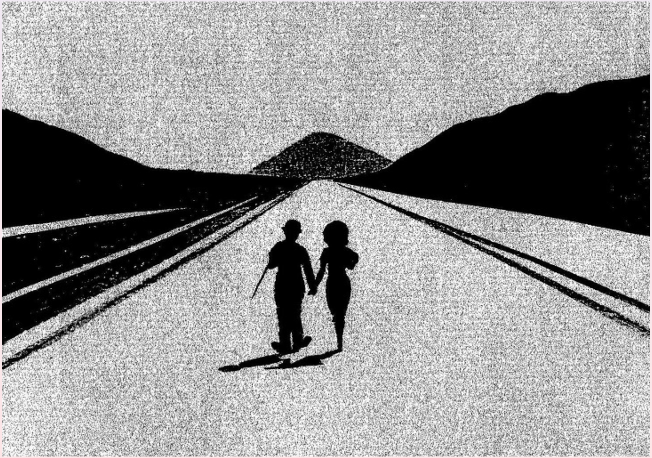




Part II

Generalized lim-inf convergence and continuity

The continuous long way to infinity



$\mathcal{M}$-precontinuity

Characterizations of continuous lattices

Theorem (Gierz et al. 1979, ME 1981)

Each of the following conditions characterizes continuous lattices:

- *the way-below sets $\Downarrow x$ are ideals with join (supremum) x*
- *the Scott preclosure operator $c_{\mathcal{D}}$ preserves meets of lower sets*
- *the cut operator preserves intersections of ideals*
- *$c_{\mathcal{D}} : \mathcal{A} \to \mathcal{A}$ is right adjoint to $w_{\mathcal{D}} : \mathcal{A} \to \mathcal{A}, Y \mapsto \Downarrow Y$*
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By definition, a filter $\mathcal{F}$ Scott converges to x iff there is a directed set D with $x \leq \bigvee D$ and $\uparrow z \in \mathcal{F}$ for all $z \in D$. Frequently, Scott convergence is expressed in terms of nets instead of filters, using eventual lower bounds. Extensions of the previous equivalences to posets have already been established in the last century (ME 1989).

The main purpose of the second part of this talk is to present a much more general theory that allows to establish the same equivalences (or some of them) for arbitrary closure spaces, and so for qosets equipped with their cut closure operator.

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$\mathcal{M}$-convergence and $\mathcal{M}$-topology I

X always denotes a closure or topological space with specialization order $\leq$ and closure operator c, or a qoset with cut operator c.

- Given a set $\mathcal{M} \subseteq \mathcal{P}X$, put $\mathcal{M}^{\wedge} = \{\downarrow Z : Z \in \mathcal{M}\}$.
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$\mathcal{M}$-convergence and $\mathcal{M}$-topology II

We generally assume that $\mathcal{M}$ is supporting. (If not, one may pass from $\mathcal{M}$ to $\mathcal{M} \cup \mathcal{E}$). A few immediate consequences:

Lemma

- *$\mathcal{M}$-convergence is in fact a convergence relation.*
- *The end filter of a net ξ $\mathcal{M}$-converges to x iff x lies in the closure of a set in $\mathcal{M}$ formed by eventual lower bounds of ξ.*
- $\sigma_{\mathcal{M}} \subseteq \tau_{\mathcal{M}}$. *Equality holds if all sets in $\mathcal{M}$ are directed.*
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$\mathcal{M}$-convergence and $\mathcal{M}$-topology II

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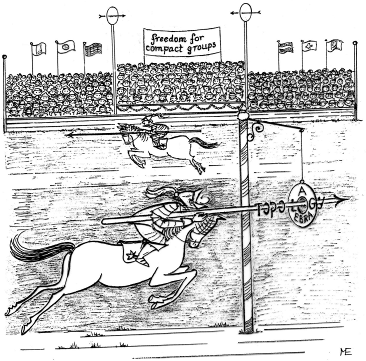
Continuity meets convergence, algebra meets topology



ME, Category Theory at Work, 1991

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Variants of continuity

We call a qoset or space X

- $\mathcal{M}$-precontinuous if (A) and (C)
- $\mathcal{M}$-d-precontinuous if (A), (C) and (D)
- $\mathcal{M}$-continuous if (A), (C) and (E)
- strongly $\mathcal{M}$-continuous if (A), (B) and (C)
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are fulfilled for all $x \in X$.

Theorem (ME 1991)

A qoset or space X with closure operator c is $\mathcal{M}$-precontinuous iff $c_{\mathcal{M}} : \mathcal{A} \to \mathcal{A}, Y \mapsto \bigcup\{cZ : Z \in \mathcal{M}, Z \subseteq {\downarrow} Y\}$ preserves meets iff $c_{\mathcal{M}}$ is right adjoint to the $\mathcal{M}$-below operator $\Downarrow$ restricted to $\mathcal{A}$.

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For a supporting set $\mathcal{M}$ of ω-ideals, a qoset or space X is

- *$\mathcal{M}$-precontinuous iff $\mathcal{M}$-convergence is pretopological,*
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An important example

- The Tychonoff plank is the product $T = (\Omega + 1) \times (\omega + 1)$.
- The deleted Tychonoff plank is $T \setminus \{(\Omega, \omega)\}$.
- The Tychonoff carpet is obtained from T by deleting the relations between the points (m, ω).

- The Tychonoff plank is completely ($\mathcal{C}$-)distributive and ($\mathcal{D}$-)continuous but not $\mathcal{C}$-precontinuous.
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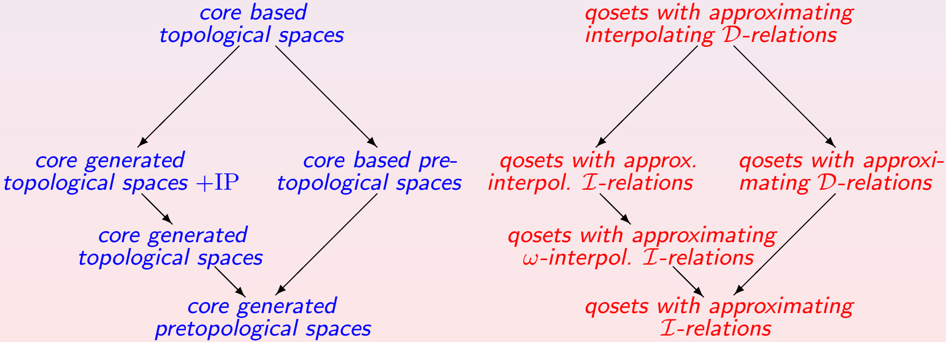
Tychonoff plank and Tychonoff carpet

Auxiliary relations and generalized continuous qosets

Theorem (ME 2019)

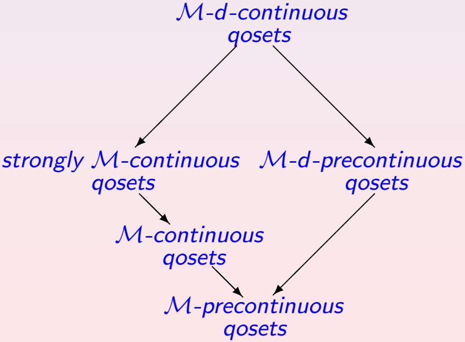
Sending (X, C) to $(X, \leq_C, \ll_C)$, one obtains isomorphisms, concrete over $\mathbf{Q}$*, between these pairs of categories:*

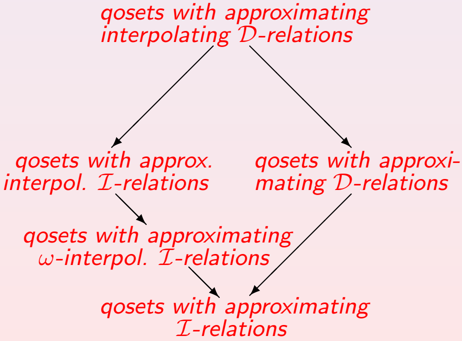


Auxiliary relations and generalized continuous qosets

Theorem (ME 2019)

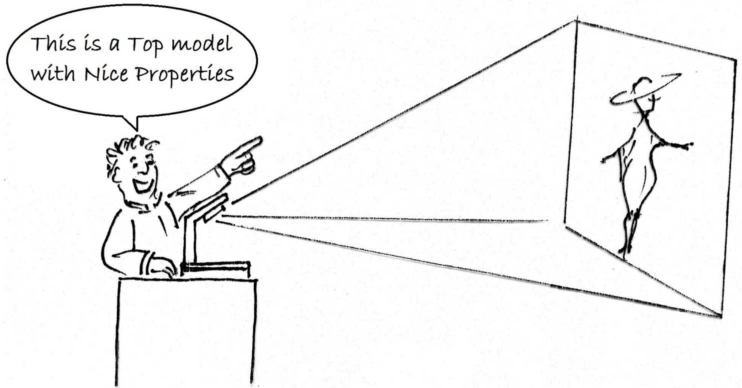
Sending $\mathcal{M}$-precontinuous qosets to their $\mathcal{M}$-below relations, one obtains surjections between the following objects:





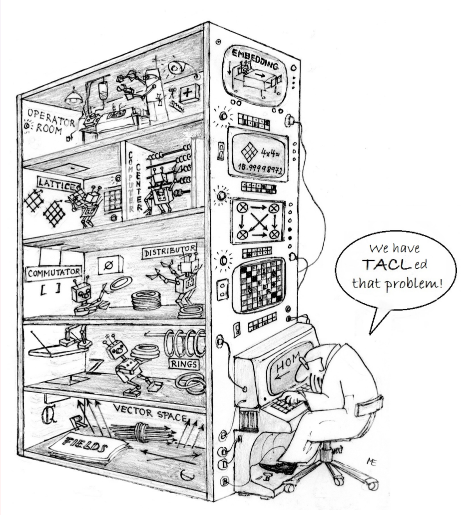
A Top Model with Nice Properties

IT WAS NICE TO BE AT NICE



ME, Flat models of topological spaces, 2005

Conclusion


EMBEDDING
OPERATOR ROOM
4x4=
16.99998972
LATTICE
COMPUTER CENTER
DISTRIBUTOR
COMMUTATOR
[ ]
Ø
RINGS
HOM
VECTOR SPACE
FIELDS
We have TACLed that problem!

Conclusion and Thanks