Simulation of Quantum Resources and the Degrees of Contextuality

S Abramsky, RS Barbosa, M Karvonen, S Mansfield

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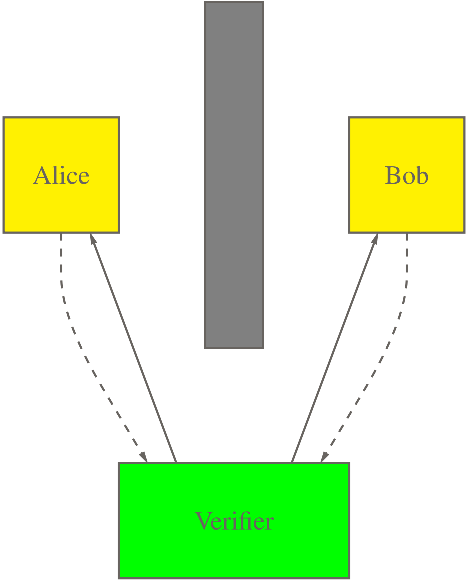
- The possibility of quantum advantage is intimately related to the non-classicality of quantum mechanics. And this non-classicality manifests itself in logical terms.
- This non-classical picture of the world lives "at the borders of paradox", as indicated by foundational results such as the EPR paradox, the Kochen-Specker paradox, the Hardy paradox, etc.
- In articulating the mathematical structure of these phenomena, we use tools from category theory, topology, algebra.

Alice-Bob games



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A table of conditional probabilities $p(a,b|x,y)$ defines a *probabilistic strategy* for this game. The *success probability* for this strategy is:

$$1/4[p(a=b|x=0,y=0)+p(a=b|x=0,y=1)+p(a=b|x=1,y=0)$$

$$+p(a \neq b|x=1,y=1)]$$

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Example: The Bell Model

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0	0	$1/2$	0	0	$1/2$
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The Bell table exceeds this bound. Since it is *quantum realizable* using an entangled pair of qubits, it shows that quantum resources yield a *quantum advantage* in an information-processing task.

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$$p_N \le \mathsf{Prob}(\bigvee_{i=1}^{N-1} \neg\phi_i) \le \sum_{i=1}^{N-1} \mathsf{Prob}(\neg\phi_i) = \sum_{i=1}^{N-1} (1 - p_i) = (N-1) - \sum_{i=1}^{N-1} p_i.$$

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Hence we obtain the inequality

$$\sum_{i=1}^{N} p_i \leq N - 1.$$

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All Bell inequalities arise this way.
Abramsky, Hardy, *Logical Bell inequalities*, Physical Review A 2012.

Science Fiction? – The News from Delft

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First Loophole-free Bell test, 2015

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NATURE | LETTER

日本語要約

Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres

B. Hensen, H. Bernien, A. E. Dréau, A. Reiserer, N. Kalb, M. S. Blok, J. Ruitenberg, R. F. L. Vermeulen, R. N. Schouten, C. Abellán, W. Amaya, V. Pruneri, M. W. Mitchell, M. Markham, D. J. Twitchen, D. Elkouss, S. Wehner, T. H. Taminiau & R. Hanson

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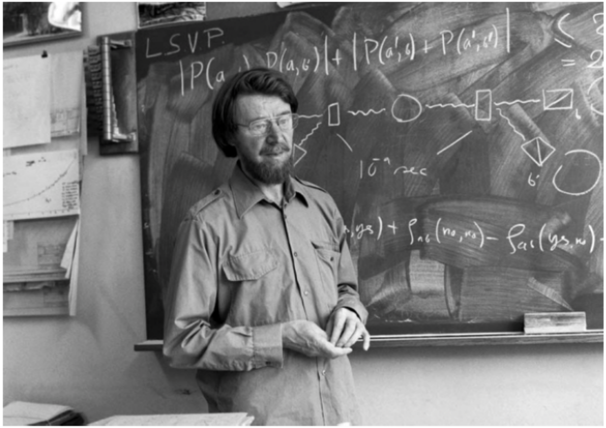
More than 50 years ago[1], John Bell proved that no theory of nature that obeys locality and realism[2] can reproduce all the predictions of quantum theory: in any local-realist theory, the correlations between outcomes of measurements on distant particles satisfy an inequality that can be violated if the particles are entangled. Numerous Bell inequality tests have been reported[3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]; however, all experiments reported so far required additional assumptions to obtain a contradiction with local realism, resulting in 'loopholes'[13, 14, 15, 16]. Here we report a Bell experiment that is free of any such additional assumption and thus directly tests the principles underlying Bell's inequality. We use an event-ready scheme[17, 18, 19] that enables the generation of robust entanglement between distant electron spins (estimated state fidelity of 0.92 ± 0.03). Efficient spin read-out avoids the fair-sampling assumption (detection loophole[14, 15]), while the use of fast random-basis selection and spin read-out combined with a spatial separation of 1.3 kilometres ensure the required locality conditions[13]. We performed 245 trials that tested the CHSH–Bell inequality[20] $S \leq 2$ and found $S = 2.42 \pm 0.20$ (where S quantifies the correlation between measurement outcomes). A null-hypothesis test yields a probability of at most $P = 0.039$ that a local-realist model for space-like separated sites could produce data with a violation at least as large as we observe, even when allowing for memory[16, 21] in the devices. Our data hence imply statistically significant rejection of the local-realist null hypothesis. This conclusion may be further consolidated in future experiments; for instance, reaching a value of $P = 0.001$ would require approximately 700 trials for an observed $S = 2.4$. With improvements, our experiment could be used for testing less-conventional theories, and for implementing device-independent quantum-secure communication[22] and randomness certification[23, 24].

Quantum ‘spookiness’ passes toughest test yet

Experiment plugs loopholes in previous demonstrations of 'action at a distance', against Einstein's objections — and could make data encryption safer.

Zeeya Merali

27 August 2015



CERN

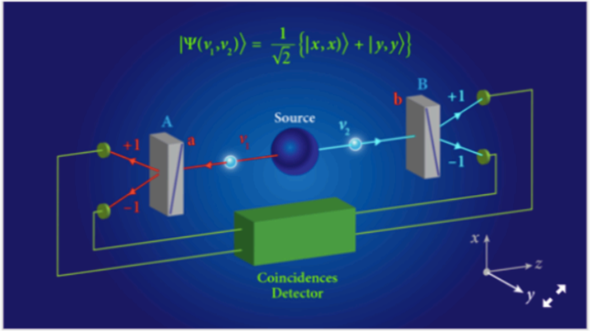
APS physics | Physics

Viewpoint: Closing the Door on Einstein and Bohr's Quantum Debate

Alain Aspect, Laboratoire Charles Fabry, Institut d'Optique Graduate School, CNRS, Université Paris-Saclay, Palaiseau, France

December 16, 2015 • *Physics* 8, 123

By closing two loopholes at once, three experimental tests of Bell's inequalities remove the last doubts that we should renounce local realism. They also open the door to new quantum information technologies.



APS/Alan Stonebraker

Figure 1: An apparatus for performing a Bell test. A source emits a pair of entangled photons ν_1 and ν_2. Their polarizations are analyzed by polarizers A and B (grey blocks), which are aligned, respectively

Timeline

1932	von Neumann's Mathematical Foundations of Quantum Mechanics
1935	EPR Paradox, the Einstein-Bohr debate
1964	Bell's Theorem
1982	First experimental test of EPR and Bell inequalities (Aspect, Grangier, Roger, Dalibard)
1984	Bennett-Brassard quantum key distribution protocol
1985	Deutch Quantum Computing paper
1993	Quantum teleportation (Bennett, Brassard, Crépeau, Jozsa, Peres, Wooters)
1994	Shor's algorithm
2015	First loophole-free Bell tests (Delft, NIST, Vienna)

Formalising empirical data*

*SA, Brandenburger, *New Journal of Physics*, 2011.

in\out	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
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A **measurement scenario X** $= \langle X, \Sigma, O \rangle$:

- X – a finite set of measurements
- Σ – a simplicial complex on X
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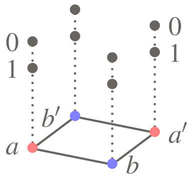

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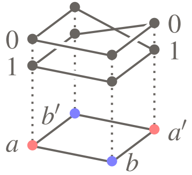
(i.e. marginals are well-defined)



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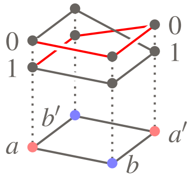

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We call such a d a *global section*.

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An empirical model $\{e_C\}_{C\in\Sigma}$ on a measurement scenario (X, Σ, O) is *non-contextual* if there is a distribution d on $\prod_{x\in X} O_x$ such that, for all $\sigma \in \Sigma$:

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The import of Bell's theorem and similar results is that there are empirical models arising from quantum mechanics which are contextual.

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The compatibility/no-signalling condition is just naturality.

There is also topology at work here. We can use *Čech cohomology* of our (pre)sheaf to define invariants to capture contextuality.

- Abramsky, Barbosa, Mansfield, *The cohomology of non-locality and contextuality*, QPL 2011.
- Abramsky, Barbosa, Kishida, Lal, Mansfield, *Contextuality, Cohomology and Paradox*, CSL 2015.

Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
ab	✓	✓	✓	✓
ab'	×	✓	✓	✓
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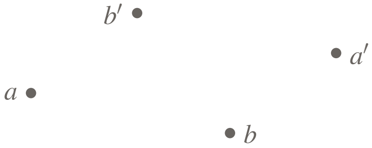
$a'b'$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$



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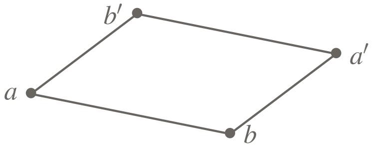

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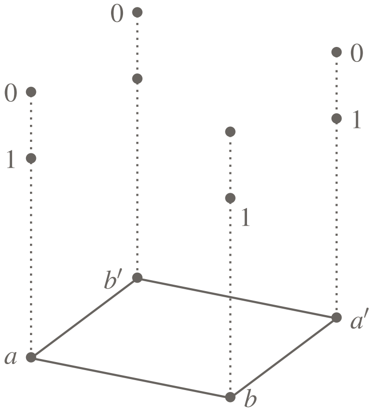

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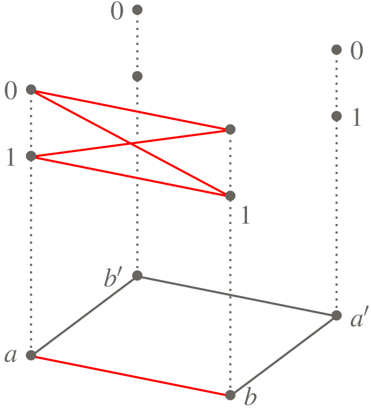

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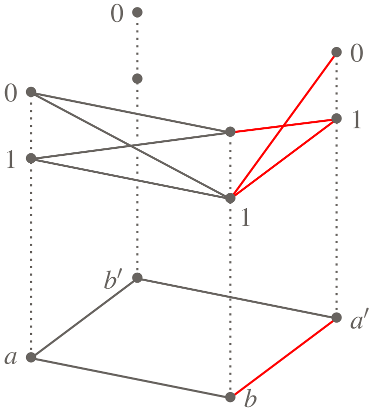

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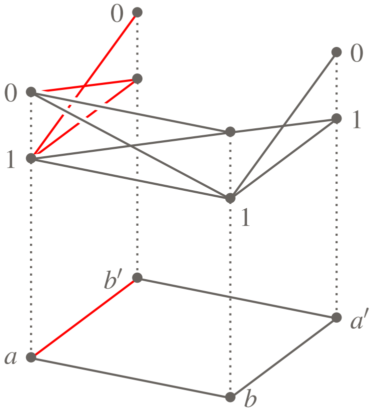

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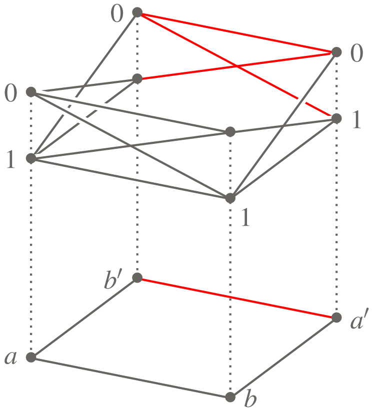

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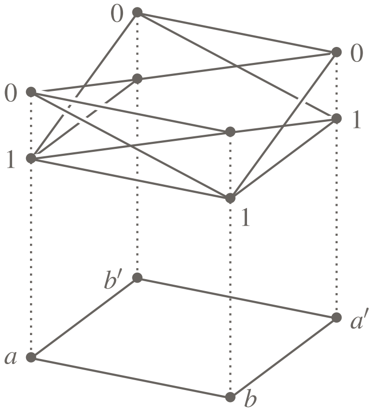

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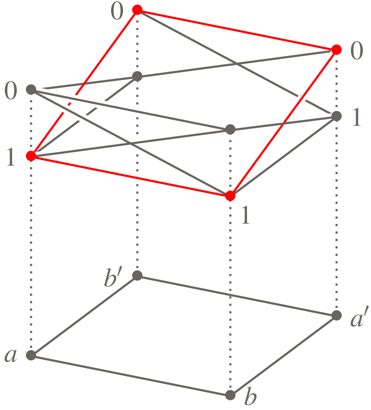

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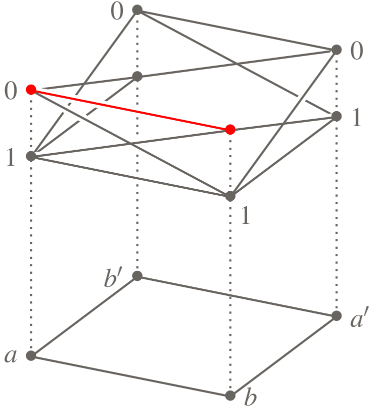

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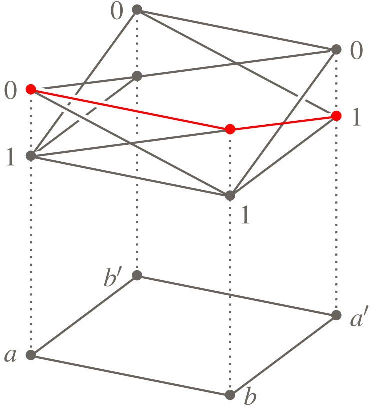

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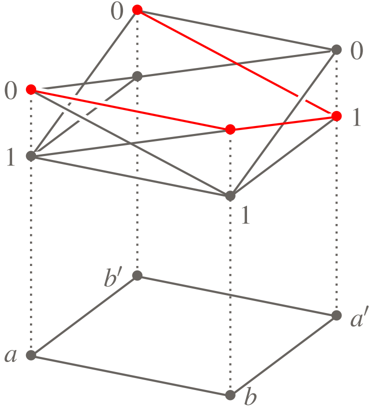

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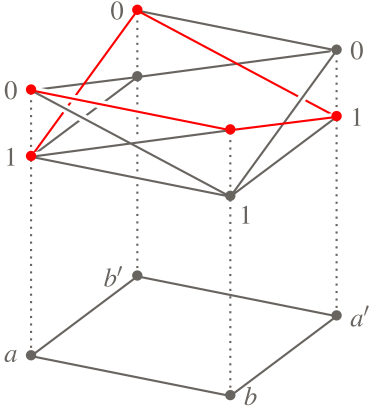

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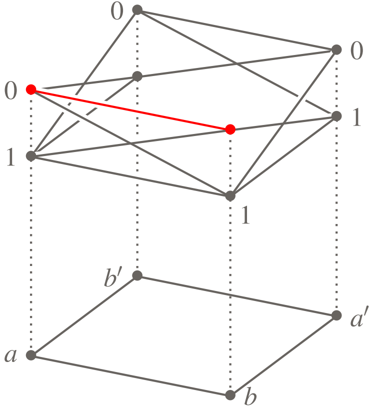

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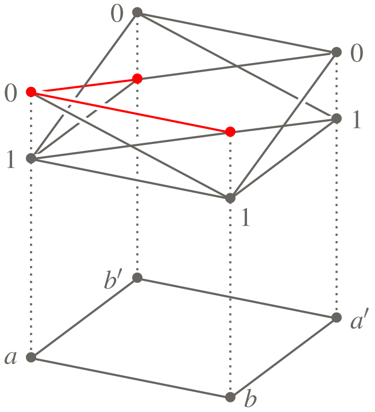

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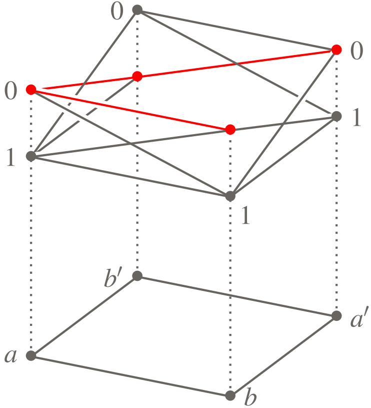

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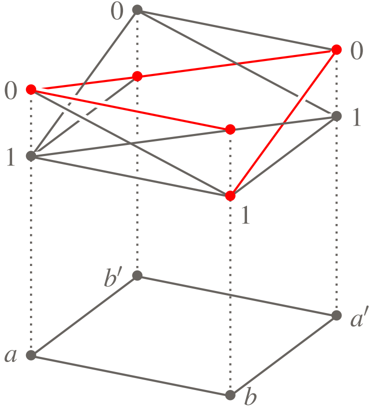

Bundle Pictures

Strong Contextuality

- E.g. the PR box:

	00	01	10	11
ab	✓	×	×	✓
ab'	✓	×	×	✓
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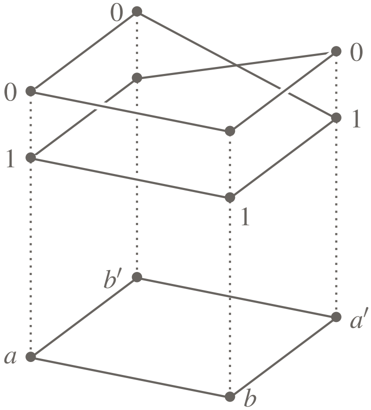


Contextuality and quantum advantage

- Measurement-based quantum computation (MBQC)
  - Raussendorf, *Physical Review A*, 2018.
  - Abramsky, Barbosa, SM, *Physical Review Letters*, 2018.

$$\overbrace{1-\bar{p}_S}^{\text{error}} \geq \underbrace{[1-\mathsf{CF}(e)]}_{\text{classicality}} \overbrace{\nu(f)}^{\text{hardness}}$$

quantifiable relationship!

- Magic state distillation
  - Howard, Wallman, Veitch, Emerson, *Nature*, 2014.
- Shallow circuits
  - Bravyi, Gossett, Kœnig, *Science*, 2018.

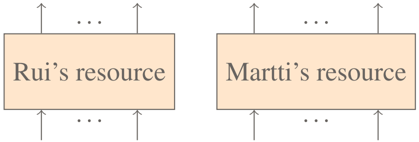
Contextuality analysis using empirical models, logical Bell inequalities, contextual fraction:

- Aasnæss, *Forthcoming*, 2019.

Contextuality as a resource

Comparing contextual behaviours

- When can we say that one resource is more powerful than another?
- Can one resource simulate the usefulness of another?



Example

Barrett, Pironio, *PRL*, 2005.

- PR boxes simulate all 2-outcome bipartite boxes
- A tripartite quantum box that cannot be simulated from PR boxes

Structure of resources

Two views

1. **Resource theories:** An algebraic theory of *free operations* which do not use any of the resource in question, *i.e.* under which contextuality is non-increasing (Physics approach).

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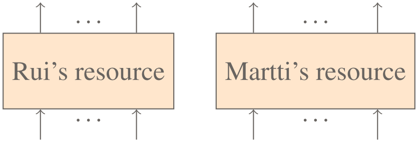
   A *category* of resources and simulations (CS approach*).
   *Cf. (in)computability, degrees of unsolvability, complexity classes

   - Karvonen, *QPL*, 2018.
     Abramsky, Barbosa, Karvonen, Mansfield, *LiCS*, 2019.

Towards morphisms

- We have defined mathematical objects (empirical models)
- What are the morphisms?
  1. Given $e : \mathbf{X}$ and $d : \mathbf{Y}$, a morphism $d \to e$ is a way of transforming d to e using *free operations*
  2. Alternatively: a morphism $d \to e$ is a way of *simulating* e using d



Free operations

From Abramsky, Barbosa, SM, *Contextual fraction*, PRL 2017.

- **Zero model** z: unique empirical model on the empty measurement scenario

$$\langle \emptyset, \Delta_0 = \{\emptyset\}, () \rangle \ .$$

- **Singleton model** u: unique empirical model on the 1-outcome 1-measurement scenario

$$\langle \mathbf{1} = \{\star\}, \Delta_1 = \{\emptyset, \mathbf{1}\}, (O_\star = \mathbf{1}) \rangle \ .$$

- **Probabilistic mixing**: Given empirical models e and d in $\mathbf{X}$ and $\lambda \in [0,1]$, the model $e +_\lambda d : \mathbf{X}$ is given by the mixture $\lambda e + (1-\lambda) d$

Free operations ctd

- **Tensor**: Let $e : \langle X, \Sigma, O \rangle$ and $d : \langle Y, \Theta, P \rangle$. Then

$$e \otimes d : \langle X \sqcup Y, \Sigma * \Theta, (O_x)_{x \in X} \cup (P_y)_{y \in Y} \rangle$$

  where $\Sigma * \Theta := \{\sigma \cup \theta | \sigma \in \Sigma, \theta \in \Theta\}$. *Runs e and d independently and in parallel*

- **Coarse-graining**: Given $e : \langle X, \Sigma, O \rangle$ and a family of functions $h = (h_x \colon O_x \longrightarrow O'_x)_{x \in X}$, get a coarse-grained model

$$e/h : \langle X, \Sigma, O' \rangle$$

- **Measurement translation**: Given $e : \langle X, \Sigma, O \rangle$ and a simplicial map $f \colon \Sigma' \longrightarrow \Sigma$, the model $f^* e : \langle X', \Sigma', O \rangle$ is defined by pulling e back along the map f

New free operation

- **Conditioning on a measurement**: Given $e : \langle X, \Sigma, O\rangle$, $x \in X$ and a family of measurements $(y_o)_{o\in O_x}$ with $y_o \in \mathsf{Vert}(\mathrm{lk}_x\Sigma)$. Consider a new measurement $x?(y_o)_{o\in O_x}$, abbreviated $x?y$. Get

$$e[x?y] : \langle X \cup \{x?y\}, \Sigma[x?y], O[x?y \mapsto O_{x?y}]\rangle$$

that results from adding $x?y$ to e.

The link

If Σ is a simplicial complex and a $\sigma \in \Sigma$ is a face, the link of σ in Σ is the subcomplex of Σ whose faces are

$$\mathrm{lk}_\sigma\Sigma := \{\tau \in \Sigma \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Sigma\} \ .$$

Summary of operations

The operations generate terms

$$\mathsf{Terms} \ni t := a \in \mathsf{Var} \mid \mathsf{z} \mid \mathsf{u} \mid f^* t \mid t/h$$
$$\mid t +_\lambda t \mid t \otimes t \mid t[x?y]$$

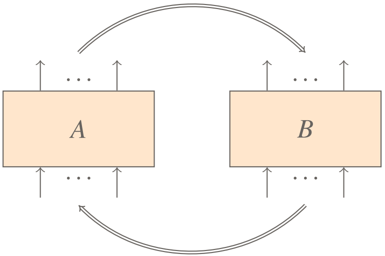
typed by measurement scenarios.

Proposition

A term without variables always represents a noncontextual empirical model. Conversely, every noncontextual empirical model can be represented by a term without variables.

- Can e be built from d using free operations?
- Formally: is there a typed term $\xi : \mathbf{Y} \vdash t : \mathbf{X}$ such that $t[d/\xi] = e$?

Basic simulations



To simulate B using A:

- map inputs of B (measurements) to inputs of A
- run A
- map outputs of A (measurement outcomes) back to outputs of B

Formally

A morphism of scenarios $(\pi, h) : \langle X, \Sigma, O\rangle \to \langle Y, \Theta, P\rangle$ is given by:

- A simplicial map $\pi : \Theta \to \Sigma$.
- For each $y \in Y$, a map $h_y : O_{\pi(y)} \to P_y$.

Basic simulations

A morphism of scenarios induces a natural action on empirical models:

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If e is an empirical model on (X, Σ, O), then $(\pi, h)^* e$ is an empirical model on (Y, Δ, P), given by:

$$(\pi, h)^*(e)_C = \mathscr{D}(\gamma)(e_{\pi(C)})$$

the push-forward of the probability measure $e_{\pi(C)}$ along the map

$$\gamma : \prod_{x \in \pi(C)} O_x \to \prod_{y \in C} P_y$$

given by $\gamma(s)_y = h_y\left(s_{\pi(y)}\right)$.

Basic simulations

A morphism of scenarios induces a natural action on empirical models:

If e is an empirical model on (X, Σ, O), then $(\pi, h)^* e$ is an empirical model on (Y, Δ, P), given by:

$$(\pi, h)^*(e)_C = \mathscr{D}(\gamma)(e_{\pi(C)})$$

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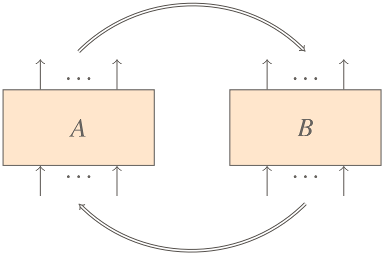
This gives a category **Emp**, with objects $e : (X, \Sigma, O)$, and morphisms $(\pi, h) : e \to e'$ such that $(\pi, h)^*(e) = e'$.

General simulations

Idea: allow *adaptive* use of A



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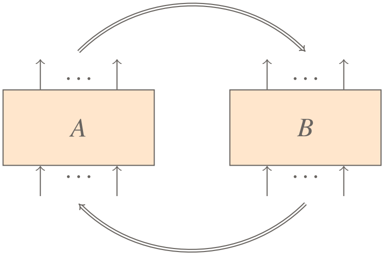

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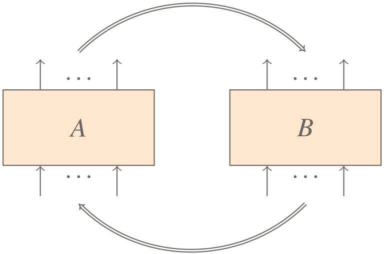

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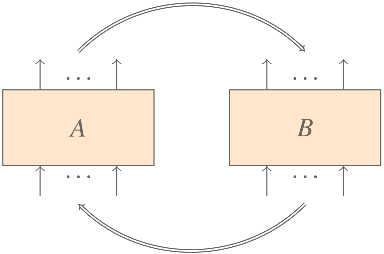

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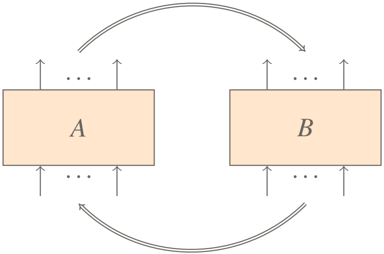

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Formally, we define the **measurement protocol completion** $\mathsf{MP}(X)$ of $\mathbf{X}$ recursively by

$$\mathsf{MP}(\mathbf{X}) ::= \emptyset \mid (x, f)$$

where $x \in X$ and $f\colon O_x \to \mathsf{MP}(\mathrm{lk}_x \Sigma)$

The MP construction

Given a scenario $\mathbf{X} = (X, \Sigma, O)$ we build a new scenario $\mathsf{MP}(\mathbf{X})$, where:

- measurements are the measurement protocols on $\mathbf{X}$
- measurement protocols are compatible if they can be combined consistently
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Roughly: comultiplication $\mathsf{MP}(\mathbf{X}) \to \mathsf{MP}^2(\mathbf{X})$ by "flattening", unit $\mathsf{MP}(\mathbf{X}) \to \mathbf{X}$, and $\mathsf{MP}(\mathbf{X} \otimes \mathbf{Y}) \to \mathsf{MP}(\mathbf{X}) \otimes \mathsf{MP}(\mathbf{Y})$.

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Given empirical models e and d, a *simulation* of e by d is a map

$$d \otimes c \to e$$

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We denote the existence of a general simulation by $d \rightsquigarrow e$.

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Theorem [Viewpoints agree]

Let $e : \mathbf{X}$ and $d : \mathbf{Y}$ be empirical models. Then $d \rightsquigarrow e$ if and only if there is a typed term $a : \mathbf{Y} \vdash t : \mathbf{X}$ such that $t[d/a] \simeq e$.

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Roughly: We develop the equational theory of free operations, and use this to obtain normal forms. These provide a means of decomposing morphisms into operations.

Theorem [Generalised no-cloning]

$e \rightsquigarrow e \otimes e$ if and only if e is noncontextual.

Roughly: Use the monotonicity properties of the *contextual fraction* under free operations

Degrees of Contextuality

The relation $d \rightsquigarrow e$ is a preorder on empirical models. The induced equivalence classes are the *degrees of contextuality*. They are partially ordered by the existence of simulations between representatives.

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More generally, we can ask for conditions on scenarios (X,Σ,O) and (Y,Δ,P) such that every empirical model over (Y,Δ,P) can be simulated by some empirical model over (X,Σ,O).

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- Graded versions of simulability: e.g. by adaptivity width or depth, available classical randomness, numbers of copies of resource, approximate simulations, . . .