Possibility Semantics

TACL 2019

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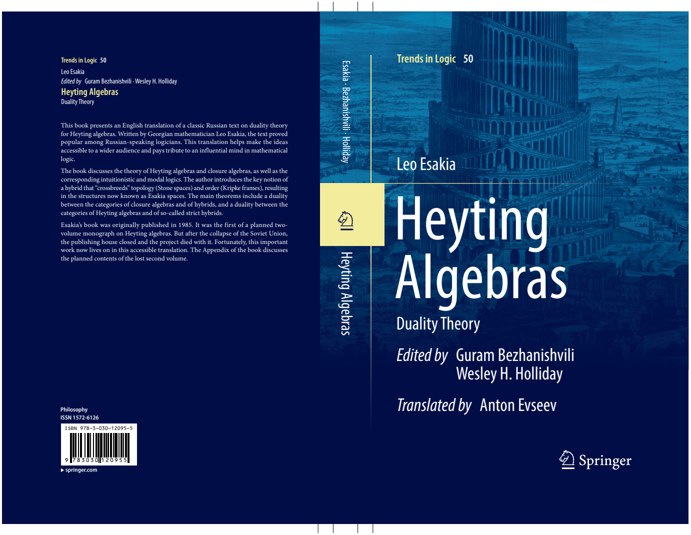
First, an advertisement: Esakia's *Heyting Algebras*



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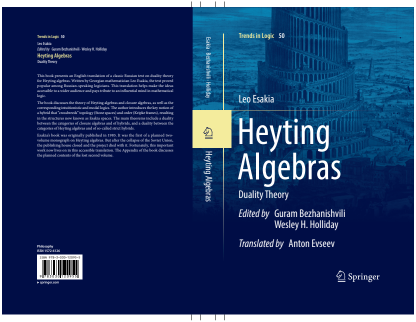

Thanks to Nick Bezhanishvili, David Gabelaia, Mai Gehrke, Mamuka Jibladze, and Hilary Priestley for their roles in this project!

Thanks to Heinrich Wansing and Christi Lue for publishing!

Outline

A survey based on my chapter in progress for the volume *Research Trends in Contemporary Logic*, College Publications, 2020.

1. The basic contrast: worlds vs. possibilities
2. From world incompleteness to possibility completeness
3. Relational incompleteness
4. Intuitionistic generalization
5. Open questions

Chronological staring point

The starting point of my work on this project was Lloyd Humberstone's 1981 paper

"From Worlds to Possibilities",

which gives possibility semantics for classical normal modal logics.

While Humberstone motivated the semantics with philosophical considerations, here I'll focus on mathematical ramifications.

The basic contrast (classical case)

Classical **possible world semantics** is based on the following BAs:

1. the powerset algebra of a set;
2. the algebra of clopen sets of a Stone space.

Classical **possibility semantics** is based on the following BAs:

$1'$. the regular open algebra of a poset;

$2'$. the algebra of compact regular open sets of a UV-space.

With 1 we represent only complete and atomic BAs, whereas with $1'$ we represent all complete BAs.

With 2 we need a nonconstructive choice principle to represent all BAs, whereas with $2'$ we represent all BAs choice free.

Regular open algebra of a space

Stone and Tarski observed that the **regular opens** of any topological space X, i.e., those opens such that $U = \mathsf{int}(\mathsf{cl}(U))$, form a complete BA with

$$\begin{array}{rcl} \neg U & = & \mathsf{int}(X \setminus U) \\ \bigwedge\{U_i \mid i \in I\} & = & \mathsf{int}(\bigcap\{U_i \mid i \in I\}) \\ \bigvee\{U_i \mid i \in I\} & = & \mathsf{int}(\mathsf{cl}(\bigcup\{U_i \mid i \in I\}). \end{array}$$

In fact, any complete BA arises (isomorphically) in this way from an Alexandroff space, i.e., as the regular opens in the downset/upset topology of a poset.

Regular open algebra of a poset

In the case of upsets of a poset, the regular opens are the U s.th.

$$U = \{x \in X \mid \forall y \geq x \, \exists z \geq y : z \in U\},$$

which is equivalent to:

- persistence: if $x \in U$ and $x \leq y$, then $y \in U$, and
- refinability: if $x \notin U$, then $\exists y \geq x$: $y \in \neg U$.

The BA operations are given by:

$$\begin{aligned} \neg U &= \{x \in X \mid \forall y \geq x : y \notin U\} \\ \bigwedge\{U_i \mid i \in I\} &= \bigcap\{U_i \mid i \in I\} \\ \bigvee\{U_i \mid i \in I\} &= \{x \in X \mid \forall y \geq x \, \exists z \geq y : z \in \bigcup\{U_i \mid i \in I\}\}. \end{aligned}$$

The basic contrast (classical case)

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Upper Vietoris space of a Stone space

Let X be a Stone space.

Let $\mathsf{F}(X)$ be the set of all nonempty closed subsets of X.

The upper Vietoris topology has the basis

$$[U] = \{F \in \mathsf{F}(X) \mid F \subseteq U\},\ U \in \Omega(X).$$

The upper Vietoris space $UV(X)$ of X is $\mathsf{F}(X)$ with this topology.

UV spaces

"Choice-free Stone duality" with Nick Bezhanishvili
forthcoming in *The Journal of Symbolic Logic*.

Definition

A **UV-space** is a T_0 space X such that:

1. the family of compact regular open sets of X, $\mathsf{CRO}(X)$, is closed under $\cap$ and $\mathsf{int}(X \setminus \cdot)$;
2. if $x \nleqslant y$ in the specialization order $\leqslant$ of X, then there is $U \in \mathsf{CRO}(X)$ with $x \in U$ and $y \notin U$;
3. every proper filter in $\mathsf{CRO}(X)$ is $\mathsf{CRO}(x)$ for some $x \in X$.

Proposition

For any Stone space X, $UV(X)$ is a UV-space; and assuming AC, every UV-space is homeomorphic to $UV(X)$ for a Stone space X.

UV representation

For any UV-space X, $\mathsf{CRO}(X)$ is a BA under $\cap$ and $\mathsf{int}(X \setminus \cdot)$.

Theorem (Bezhanishvili and H.)

(ZF) Each BA $\mathbb{A}$ is isomorphic to $\mathsf{CRO}(X)$ for a UV-space X, namely the space of proper filters of $\mathbb{A}$ with the topology generated by the sets $\widehat{a} = \{F \in \mathsf{PropFilt}(\mathbb{A}) \mid a \in F\}$ for $a \in \mathbb{A}$.

Such filter spaces appear in previous work including that of van Benthem, Goldblatt, and Moshier and Jipsen.

UV duality

Theorem (Bezhanishvili and H.)

(ZF) The category of BAs with BA homomorphisms is dually equivalent to the category of UV-spaces with UV-maps (spectral maps that are p-morphisms w.r.t. the specialization order).

This is the "choice-free Stone duality" in the title of our JSL paper.

There we give examples of how to prove facts about BAs using this constructive—but still *spatial*—duality instead of Stone duality.

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The basic contrast (modal case)

General neighborhood world frames are triples $\mathfrak{F} = \langle W, \{N_i\}_{i \in I}, P \rangle$ where:

- W is a nonempty set;
- $N_i : W \to \wp(\wp(W))$;
- $P \subseteq \wp(W)$ yields a subalgebra of the BAE $\langle \wp(W), \{\Box_i\}_{i \in I} \rangle$ where

$$\Box_i U = \{w \in W \mid U \in N_i(w)\}.$$

If for each $w \in W$, $\bigcap N_i(w) \in N_i(w)$, we can define R_i on W by

$$wR_i v \text{ iff } v \in \bigcap N_i(w), \text{ so } \Box_i U = \{w \in W \mid R_i(w) \subseteq U\}.$$

General relational world frames: $\mathfrak{F} = \langle W, \{R_i\}_{i \in I}, P \rangle$.

Full frames: $P = \wp(W)$. **Kripke frames**: full relational frames.

The basic contrast (modal case)

General neighborhood possibility frames are quadruples $\mathcal{F} = \langle S, \sqsubseteq, \{N_i\}_{i \in I}, P \rangle$ where:

- $\langle S, \sqsubseteq \rangle$ is a poset;
- $N_i : S \to \wp(\mathrm{RO}(S, \sqsubseteq))$ is such that for all $U \in \mathrm{RO}(S, \sqsubseteq)$,
$$\Box_i U = \{x \in S \mid U \in N_i(x)\} \in \mathrm{RO}(S, \sqsubseteq);$$
- $P \subseteq \mathrm{RO}(S, \sqsubseteq)$ is a subalgebra of BAE $\langle \mathrm{RO}(S, \sqsubseteq), \{\Box_i\}_{i \in I} \rangle$.

If for each $x \in S$, $\bigcap N_i(x) \in N_i(x)$, we can define R_i on S by

$$xR_i y \text{ iff } y \in \bigcap N_i(x), \text{ so } \Box_i U = \{x \in S \mid R_i(x) \subseteq U\}.$$

General relational possibility frames: $\mathcal{F} = \langle S, \sqsubseteq, \{R_i\}_{i \in I}, P \rangle$.

Full frames: $P = \mathrm{RO}(S, \sqsubseteq)$. **World frames**: $\sqsubseteq$ is identity.

The basic contrast (modal case)

General neighborhood possibility frames are quadruples $\mathcal{F} = \langle S, \sqsubseteq, \{N_i\}_{i \in I}, P\rangle$ where:

- $\langle S, \sqsubseteq\rangle$ is a poset;
- $N_i : S \to \wp(\mathrm{RO}(S, \sqsubseteq))$ is such that for all $U \in \mathrm{RO}(S, \sqsubseteq)$,

$$\Box_i U = \{x \in S \mid U \in N_i(x)\} \in \mathrm{RO}(S, \sqsubseteq);$$

- $P \subseteq \mathrm{RO}(S, \sqsubseteq)$ is a subalgebra of BAE $\langle \mathrm{RO}(S, \sqsubseteq), \{\Box_i\}_{i \in I}\rangle$.

If for each $x \in S$, $\bigcap N_i(x) \in N_i(x)$, we can define R_i on S by

$$xR_iy \text{ iff } y \in \bigcap N_i(x), \text{ so } \Box_i U = \{x \in S \mid R_i(x) \subseteq U\}.$$

General relational possibility frames: $\mathcal{F} = \langle S, \sqsubseteq, \{R_i\}_{i \in I}, P\rangle$.

Key point I'm skipping: first-order interaction between $\sqsubseteq$ and R_i.

The basic contrast (full relational frames)

Theorem (Thomason 1975)

The category of **complete and atomic** BAs with a completely multiplicative $\Box$ & complete BA homomorphisms preserving $\Box$ is dually equivalent to the category of Kripke frames & p-morphisms.

Theorem (H. 2015)

The category of **complete** BAs with a completely multiplicative $\Box$ & complete BA homomorphisms preserving $\Box$ is dually equivalent to a reflective subcategory of the category of full relational possibility frames & p-morphisms (for both $\sqsubseteq$ and R_i).

Analogues hold for BAEs, full neighborhood world frames (cf. Dǒsen 1989), and full neighborhood possibility frames.

The basic contrast (general relational frames)

Theorem (Goldblatt 1974, 2006)

(ZF + Ultrafilter Principle) The category of MAs with MA homomorphisms is dually equivalent to a reflective subcategory of the category of general relational world frames with modal maps—namely, the category of "descriptive" relational world frames with p-morphisms.

Theorem (H. 2015)

(ZF) The category of MAs with MA homomorphisms is dually equivalent to a reflective subcategory of the category of general relational possibility frames with possibility morphisms—namely, the category of "filter-descriptive" relational possibility frames with p-morphisms.

Similarities

Despite these contrasts, we have:

- An analogue of the **Goldblatt-Thomason theorem** for full relational possibility frames and for general relational possibility frames (H. 2015);
- An analogue of the **Sahlqvist theorem** for full relational possibility frames (Yamamoto 2017).

W. H., "Possibility frames and forcing for modal logic"
UC Berkeley Working Paper in Logic & the Methodology of Science, 2015.

K. Yamamoto, "Results in modal correspondence theory for possibility semantics"
Journal of Logic and Computation, 2017.

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A warning sign about worlds

Theorem (Venema 2003)

There are normal unimodal logics that are not the logic of any class of atomic modal algebras and normal polymodal logics that are not even sound with respect to any atomic MAs.

A simple example from the wild

Let's consider a simple logic that cannot be characterized by atomic BAEs but can be by full possibility frames.

Fix a bimodal propositional language with modalities $\Box$ and Q.

No atomic BAE—hence no full nbhd frame—validates $\Box\top$ and

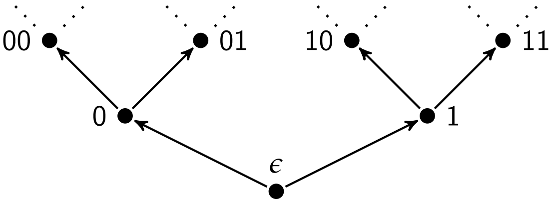
$$p \to \big(\Diamond(p \wedge Qp) \wedge \Diamond(p \wedge \neg Qp)\big). \qquad (\textsc{Split})$$

Let S5SE be the bimodal extension of S5 for $\Box$ with $\textsc{Split}$ and $\Box(\varphi \leftrightarrow \psi) \to (Q\varphi \leftrightarrow Q\psi)$.

Theorem (Ding and H.)

S5SE is the logic of a full neighborhood possibility frame.

Let $(S, \sqsubseteq)$ be the full infinite binary tree:



Interpret $\Box$ as the global modality. Define $N_Q(x)$ inductively by:

$$\begin{array}{rcl} N_Q(\epsilon) & = & \varnothing \\ N_Q(x0) & = & N_Q(x) \cup \{P \in \mathrm{RO}(S, \sqsubseteq) \mid x \in P, \mathrm{Parent}(x) \notin P\} \\ N_Q(x1) & = & N_Q(x). \end{array}$$

This frame validates $p \to (\Diamond(p \wedge Qp) \wedge \Diamond(p \wedge \neg Qp))$.

A simple example from the wild

$$p \to \big(\Diamond(p \wedge Qp) \wedge \Diamond(p \wedge \neg Qp)\big). \qquad (\textsc{Split})$$

Split arises in the wild with an arithmetical interpretation.

By restricting the induction axioms of PA, we obtain subtheories:

$$\mathsf{AR}_0 = I\Delta_0 + \mathsf{Exp} \qquad \mathsf{AR}_{n+1} = I\Sigma_{n+1}.$$

For an arithmetic sentence φ, let $\mathrm{Q}\varphi$ be the arithmetic sentence:

$$\forall n(\mathsf{Con}_n\varphi \to \mathsf{Con}_n(\varphi \wedge \mathsf{Con}_n\varphi)).$$

Theorem (Shavrukov and Visser 2014)

If φ is consistent in PA, then both $\varphi \wedge \mathrm{Q}\varphi$ and $\varphi \wedge \neg\mathrm{Q}\varphi$ are consistent in PA.

Normal examples

So far the examples of *normal* modal logics that cannot be characterized by Kripke frames but can be by relational possibility frames are artificially constructed rather than found in the wild.

Using an algebraic incompleteness result of Litak (2004), we have:

Theorem (H. 2015)

There are continuum many unimodal logics that are Kripke incomplete but full relational possibility frame complete.

Examples with propositional quantifiers

With more expressive languages, then world incompleteness + possibility completeness arises even more easily.

Consider modal logic with propositional quantification: $\forall p\varphi$, $\exists p\varphi$.

In a complete MA, we can interpret $\forall$ and $\exists$ with meets and joins:

$$v(\forall p\varphi) = \bigwedge\{v'(\varphi) \mid v' \text{ a valuation differing from } v \text{ at most at } p\}.$$

$$v(\exists p\varphi) = \bigvee\{v'(\varphi) \mid v' \text{ a valuation differing from } v \text{ at most at } p\}.$$

Examples with propositional quantifiers

For any normal modal logic L, let $\mathsf{L}\Pi$ be the propositionally quantified extension with the following axioms and rule:

- $\forall$-distribution: $\forall p(\varphi \to \psi) \to (\forall p\varphi \to \forall p\psi)$.
- $\forall$-instantiation: $\forall p\varphi \to \varphi^p_\psi$ where ψ is free for p in φ;
- Vacuous-$\forall$: $\varphi \to \forall p\varphi$ where p is not free in φ.
- $\forall$-generalization: if φ is a theorem, so is $\forall p\varphi$.

Theorem (H. 2019)

$\mathsf{S5}\Pi$ is the logic of all complete (simple) S5 algebras and hence of all full possibility frames with $\Box$ as universal modality.

"A note on algebraic semantics for S5 with propositional quantifiers,"
Notre Dame Journal of Formal Logic, 2019

Examples with propositional quantifiers

For any normal modal logic L, let LΠ be the propositionally quantified extension with the following axioms and rule:

- $\forall$-distribution: $\forall p(\varphi \to \psi) \to (\forall p\varphi \to \forall p\psi)$.
- $\forall$-instantiation: $\forall p\varphi \to \varphi^p_\psi$ where ψ is free for p in φ;
- Vacuous-$\forall$: $\varphi \to \forall p\varphi$ where p is not free in φ.
- $\forall$-generalization: if φ is a theorem, so is $\forall p\varphi$.

Theorem (H. 2019)

S5Π is the logic of all complete (simple) S5 algebras and hence of all full possibility frames with $\Box$ as universal modality.

Yet S5Π is Kripke incomplete. For if we restrict to atomic cBAs, we obtain additional validities not derivable in S5Π, such as:

$$\exists q(q \wedge \forall p(\Box(q \to p) \vee \Box(q \to \neg p)).$$

Examples with propositional quantifiers

Below S5, so far Yifeng Ding (UC Berkeley) has proved:

Theorem (Ding 2019)

1. $\mathsf{KD45}\Pi + \forall p\Box\varphi \to \Box\forall p\Box\varphi$ is the logic of all complete KD45 algebras and hence of all full KD45 nhbd possibility frames.
2. $\mathsf{KD45}\Pi + \forall p\Box\varphi \to \Box\forall p\varphi$ is the logic of all complete KD45 algebras with completely multiplicative $\Box$ and hence of all full KD45 relational possibility frames.

Yet both are Kripke incomplete for the same reason as before.

Examples with propositional quantifiers

What about still weaker logics?

Theorem (Fine 1970)

Let $\mathcal{C}$ be the class of Kripke frames for one of the following logics: K, T, K4, S4, S4.2, B. Then the propositionally quantified logic of $\mathcal{C}$ is not recursively axiomatizable.

Does the situation improve if we move from atomic and complete MAs to complete MAs, i.e., to full possibility frames?

1. The basic contrast: worlds vs. possibilities ✓
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Relational incompleteness

The duals of Kripke frames are complete and atomic MAs with completely multiplicative $\Box$ operations.

Theorem (Venema 2003)

There are normal unimodal logics that are not the logic of any class of atomic modal algebras and normal polymodal logics that are not even sound with respect to any atomic MAs.

Theorem (Litak 2004)

There are continuum-many modal logics that are not the logic of any class of complete MAs.

Relational incompleteness

The natural next question, raised in Litak's dissertation (2005) and by Venema in the Handbook of Modal Logic (2006):

- do such incompleteness or unsoundness results also apply to completely multiplicative MAs?

Possibility semantics led to the answer to this question by providing a new perspective on complete multiplicativity of $\Box$...

Complete multiplicativity says $\Box$ distributes over the meet of **any set of elements** with a meet: $\Box \bigwedge\{a_i \mid i \in I\} = \bigwedge\{\Box a_i \mid i \in I\}$.

Surprisingly, this ostensibly second-order condition is equivalent to a first-order condition discovered for the purposes of turning completely multiplicative MAs into relational possibility frames.

Theorem (H. and Litak 2015)

The operation $\Box$ in an MA is completely multiplicative iff:

$$\text{if } x \not\leq \Box\neg y\text{, then } \exists \text{ nonzero } y' \leq y \text{ such that } xRy',$$

where xRy' means that $\forall$ nonzero $y'' \leq y'$: $x \not\leq \Box\neg y''$.

All of this could be stated in terms of complete additivity of $\Diamond$.

H. Andréka, Z. Gyenis, and I. Németi generalized the first-orderness result to arbitrary posets with completely additive operators.

Relational incompleteness

The first-order reformulation of complete multiplicativity from possibility semantics led to a solution to the problem about incompleteness with respect to complete multiplicative MAs.

Theorem (H. and Litak 2015)

There are continuum-many modal logics that are not the logic of any class of MAs with completely multiplicative $\Box$.

Theorem (H. and Litak 2015)

The bimodal provability logic GLB is not the logic of any class of MAs with completely multiplicative box operations.

"Complete Additivity and Modal Incompleteness" with Tadeusz Litak
forthcoming in *The Review of Symbolic Logic*.

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The basic contrast (intuitionistic case)

Classical world semantics vs. possibility semantics generalize from:

1. the powerset algebra of a set, vs.

1′. the regular open algebra of a poset.

In the intuitionistic case, the distinction becomes:

0. the algebra of upsets of a poset, vs.

0′. the algebra of fixpoints of a nucleus on the upsets of a poset.

The classical setting is then a special case, where for world semantics we take the poset to be discrete and for possibility semantics we take the nucleus to be double negation.

With 0 we can represent only the completely join-prime generated complete HAs, whereas with $0'$ we can represent all complete HAs.

Nuclei

Regular that a regular open set is a fixpoint of the operation $\mathrm{int}(\mathrm{cl}(\cdot))$ on the open sets of a space. Thinking in terms of the cHA of open sets, this is the operation $\neg\neg$ of double negation.

The operation $\neg\neg$ is an example of a *nucleus* on an HA.

A **nucleus** on an HA H is a function $j : H \to H$ satisfying:

1. $a \leq ja$ (inflationarity);
2. $jja \leq ja$ (idempotence);
3. $j(a \wedge b) = ja \wedge jb$ (multiplicativity).

The HA of fixpoints of a nucleus

For any HA H and nucleus j on H, let $H_j = \{a \in H \mid ja = a\}$.

Then H_j is an HA where for $a, b \in H_j$:

- $a \wedge_j b = a \wedge b$;
- $a \rightarrow_j b = a \rightarrow b$;
- $a \vee_j b = j(a \vee b)$;
- $0_j = j0$.

If H is a complete, so is H_j, where $\bigwedge_j S = \bigwedge S$ and $\bigvee_j S = j(\bigvee S)$.

In the case $j = \neg\neg$, we have that H_j is a BA.

Representing cHAs as fixpoints of a nucleus on upsets

Dragalin showed that every cHA can be represented using a triple $(S, \leq, j)$ where $(S, \leq)$ is a poset and j is a nucleus on $\mathrm{Up}(S, \leq)$.

Theorem (Dragalin 1981)

Every cHA is isomorphic to the algebra of fixpoints of a nucleus on the upsets of a poset.

But we would like to replace the operation j with something more concrete. . .

Intuitionistic possibility frames

A (normal) **FM-frame** is a triple $(S, \leq_1, \leq_2)$ where $\leq_1$ and $\leq_2$ are preorders on S such that $\leq_2$ is a subrelation of $\leq_1$.

Proposition (Fairtlough and Mendler 1997)

For any such $(S, \leq_1, \leq_2)$, the operation $\Box_1 \Diamond_2$ given by

$$\Box_1 \Diamond_2 U = \{x \in S \mid \forall y \geq_1 x \, \exists z \geq_2 y : z \in U\}$$

is a nucleus on $\mathrm{Up}(S, \leq_1)$.

This approach is related to **Urquhart**'s representation of lattices using doubly-ordered sets.

Intuitionistic possibility frames

Recall that a Kripke frame (poset) $(S, \leq)$ can represent only the very special *completely join-prime generated* complete Heyting algebras via their algebras $\mathsf{Up}(S, \leq)$ of upsets.

By contrast, FM-frames $(S, \leq_1, \leq_2)$ can be used to represent *all* complete Heyting algebras.

Theorem (Bezhanishvili and H. 2016, Massas 2016)

Every complete Heyting algebra is isomorphic to the algebra of $\Box_1\Diamond_2$-fixpoints of an FM-frame.

G. Bezhanishvili & W.H., "Locales, nuclei, and Dragalin frames," AiML 2016.

G. Massas, *Possibility Spaces, Q-Completions and Rasiowa-Sikorski Lemmas for Non-Classical Logics*, ILLC MSc Thesis, 2016.

Intermediate incompleteness

Theorem (Shehtman 1977)

There are Kripke incomplete intermediate logics

Theorem (Litak 2002)

There are continuum-many Kripke incomplete intermediate logics.

Theorem (Shehtman 1980, 2005)

There are Kripke incomplete but *topologically complete* intermediate logics.

Thus, there are intermediate logics incomplete w.r.t. Kripke frames but complete w.r.t. intuitionistic possibility frames (FM-frames).

Kuznetsov's problem: is every intermediate logic topologically complete? Or cHA complete, i.e., possibility frame complete?

A semantic hierarchy for intuitionistic logic

The nuclei-based perspective is applied to other semantics in the "semantic hierarchy" for intuitionistic logic investigated in:

"A semantic hierarchy for intuitionistic logic"
with Guram Bezhanishvili, *Indagationes Mathematicae*, 2019
special issue on "L.E.J. Brouwer after fifty years."

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So far...

So far the work on possibility semantics has led to some new results concerning (in)completeness of **modal and intermediate logics**.

More recently, it has led to new semantics for **inquisitive logic** and an answer to a question raised in the inquisitive logic community: what should **inquisitive intuitionistic** logic be?

"Algebraic and topological semantics for inquisitive logic via choice-free duality? with N. Bezhanishvili and G. Grilletti, Proceedings of WoLLIC 2019.

"Inquisitive intuitionistic logic" with G. Bezhanishvili, Manuscript.

Open Questions

But many questions remain:

1. What are other examples of modal logics incomplete w.r.t. atomic BAEs or MAs?
2. Which classes of complete MAs or HAs have recursively axiomatizable propositionally quantified modal logics?
3. Can completeness w.r.t. MAs with completely additive operators be characterized by conservativity of some rule?
4. **Kuznetsov's problem**: is every intermediate logic topologically complete?
5. Variant of Kuznetsov's problem: is every intermediate logic cHA complete?

Thank you!

G. Bezhanishvili and W. H. Holliday, "A semantic hierarchy for intuitionistic logic," *Indagationes Mathematicae*, 2019.

N. Bezhanishvili and W. H. Holliday, "Choice-free Stone duality," *The Journal of Symbolic Logic*, forthcoming.

W. H. Holliday, "Algebraic semantics for S5 with propositional quantifiers," *Notre Dame Journal of Formal Logic*, 2019.

W. H. Holliday, "Possibility frames and forcing for modal logic," UC Berkeley Working Paper, 2015.

W. H. Holliday and T. Litak, "Complete additivity and modal incompleteness," *The Review of Symbolic Logic*, forthcoming.