Bridges between Logic and Algebra

Part 3: Interpolation and Amalgamation

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Equational Consequence Recalled

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(2) $\Delta \subseteq \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma)$.

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Proof hint. Consider $\Delta(\overline{y}) := \{\varepsilon(\overline{y}) \mid \Sigma(\overline{x}, \overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y})\}$.

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Note that the pair $\langle i^*, i^{-1} \rangle$ is an **adjunction**, i.e.,

$$i^*(\Theta) \subseteq \Psi \iff \Theta \subseteq i^{-1}(\Psi).$$

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The following are equivalent:

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(2) The following diagram commutes (where i, j, k, l are inclusion maps):

$$\begin{array}{ccc}
\mathrm{Con}\,\mathbf{F}(\overline{x}, \overline{y}) & \xrightarrow{\;i^{-1}\;} & \mathrm{Con}\,\mathbf{F}(\overline{y}) \\
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That is, for any $\Theta \in \mathrm{Con}\,\mathbf{F}(\overline{x}, \overline{y})$,

$$\mathrm{Cg}_{\mathbf{F}(\overline{x}, \overline{y}, \overline{z})}(\Theta) \cap F(\overline{y}, \overline{z})^2 = \mathrm{Cg}_{\mathbf{F}(\overline{y}, \overline{z})}(\Theta \cap F(\overline{y})^2).$$

But now. . .

What does deductive interpolation mean **algebraically**?

The Amalgamation Property

A variety $\mathcal{V}$ has the **amalgamation property** if for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{V}$ and embeddings $i \colon \mathbf{A} \to \mathbf{B}$ and $j \colon \mathbf{A} \to \mathbf{C}$,

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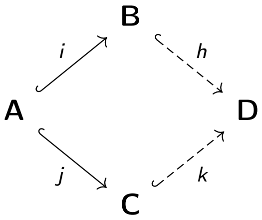

A Key Lemma

Lemma (Pigozzi 1972)

$\mathcal{V}$ has the amalgamation property if and only if for any $\Theta \in \mathrm{Con}\,\mathbf{F}(\overline{x}, \overline{y})$, $\Psi \in \mathrm{Con}\,\mathbf{F}(\overline{y}, \overline{z})$ *satisfying*

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there exists $\Phi \in \mathrm{Con}\,\mathbf{F}(\overline{x}, \overline{y}, \overline{z})$ satisfying

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This property of congruences of free algebras can be reformulated in terms of consequence as the so-called **Robinson property**.

Proof Sketch ($\Rightarrow$)

Suppose that $\mathcal{V}$ has the amalgamation property and $\Theta \in \mathrm{Con}\,\mathbf{F}(\overline{x}, \overline{y})$, $\Psi \in \mathrm{Con}\,\mathbf{F}(\overline{y}, \overline{z})$ satisfy $\Phi_0 := \Theta \cap F(\overline{y})^2 = \Psi \cap F(\overline{y})^2$.

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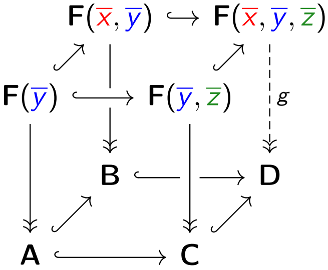
$$\mathbf{A} = \mathbf{F}(\overline{y})/\Phi_0, \quad \mathbf{B} = \mathbf{F}(\overline{x},\overline{y})/\Theta, \quad \text{and} \quad \mathbf{C} = \mathbf{F}(\overline{y},\overline{z})/\Psi,$$



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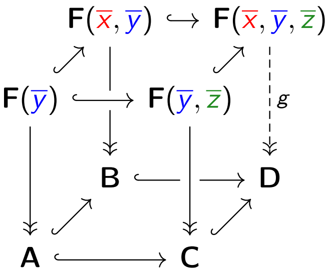

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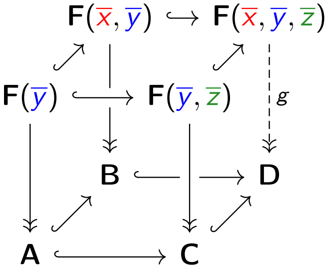
yielding an amalgam $\mathbf{D}$ and a surjective homomorphism $g\colon \mathbf{F}(\overline{x},\overline{y},\overline{z}) \to \mathbf{D}$



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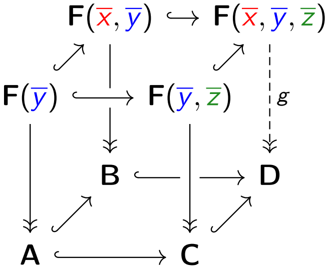

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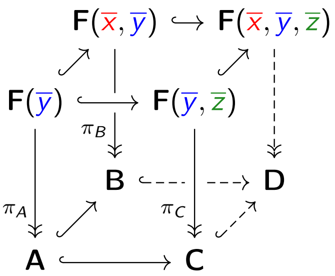

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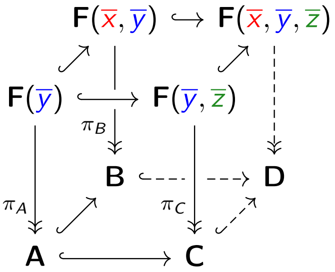

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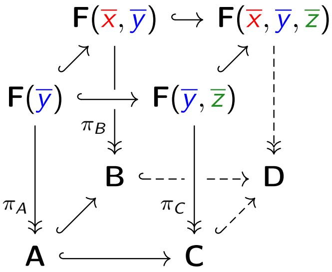

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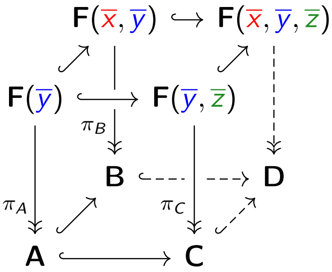

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$$\top \approx \bigwedge \{\gamma \leftrightarrow \delta \mid \gamma \approx \delta \in \Pi\} \to (\alpha \leftrightarrow \beta).$$

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Theorem (Bacsich, Czelakowski, Pigozzi, Ono, . . .)

The following are equivalent:

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(3) $\mathcal{V}$ *has the* **congruence extension property***: for any subalgebra* $\mathbf{B}$ *of* $\mathbf{A} \in \mathcal{V}$ *and* $\Theta \in \mathrm{Con}\, \mathbf{B}$*, there exists* $\Phi \in \mathrm{Con}\, \mathbf{A}$ *with* $\Theta = \Phi \cap B^2$*.*

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and symmetrically, $\Phi \cap F(\overline{x}, \overline{y})^2 = \Theta \cap F(\overline{z})^2$. □

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- interpolation has been proved for many intermediate and modal logics by establishing the amalgamation property (often using dualities) for corresponding varieties of Heyting and modal algebras;
- the amalgamation property has been established for many varieties of residuated lattices by proving interpolation (often using proof theory) for corresponding substructural logics.

Further Relationships...

$$
\begin{array}{ccccccc}
\text{CEP} + \text{FAP} & & & & & & \\
\Updownarrow & & & & & & \\
\text{TIP} & \Longrightarrow & \text{AP} & \Longrightarrow & \text{WAP} & \Longrightarrow & \text{FAP} \\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow \\
\text{MIP} & \Longrightarrow & \text{RP} & \Longrightarrow & \text{CDIP} & \Longrightarrow & \text{DIP} \\
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But Now...

Can we describe **uniform interpolation** algebraically?

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$\mathcal{V}$ admits **deductive interpolation** if for any set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a set of equations $\Delta(\overline{y})$ such that

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$\mathcal{V}$ admits **right uniform deductive interpolation** if for any *finite* set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a *finite* set of equations $\Delta(\overline{y})$ such that

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But what does the extra ingredient in (ii) mean *algebraically*?

Finitely Generated Congruences

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Recall that the inclusion map $i\colon \mathbf{F}(\overline{y}) \to \mathbf{F}(\overline{x}, \overline{y})$ "lifts" to the maps

$$i^*\colon\ \mathrm{Con}\,\mathbf{F}(\overline{y}) \to \mathrm{Con}\,\mathbf{F}(\overline{x}, \overline{y}); \qquad \Theta \mapsto \mathrm{Cg}_{\mathbf{F}(\overline{x},\overline{y})}(i[\Theta])$$

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The **compact lifting of** i restricts i^* to $\mathrm{KCon}\,\mathbf{F}(\overline{y}) \to \mathrm{KCon}\,\mathbf{F}(\overline{x}, \overline{y})$; it has a right adjoint if i^{-1} restricts to $\mathrm{KCon}\,\mathbf{F}(\overline{x}, \overline{y}) \to \mathrm{KCon}\,\mathbf{F}(\overline{y})$.

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Useful Lemma

If $\mathbf{A} \in \mathcal{V}$ is finitely presented and isomorphic to $\mathbf{F}(\overline{y})/\Psi$ for some finite set $\overline{y}$ and congruence Ψ on $\mathbf{F}(\overline{y})$, then Ψ is finitely generated.

Coherence

Theorem (Kowalski and Metcalfe 2019)

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Note. Every **locally finite** variety is coherent.

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Another Bridge Theorem

Theorem (Kowalski and Metcalfe 2019)

A variety with the congruence extension property admits right uniform deductive interpolation if and only if it has the amalgamation property and is coherent.

Left Uniform Deductive Interpolation

$\mathcal{V}$ has **left uniform deductive interpolation** if for any finite set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a finite set of equations $\Delta(\overline{y})$ such that

$$\Pi(\overline{y}, \overline{z}) \models_{\mathcal{V}} \Sigma(\overline{x}, \overline{y}) \quad \Longleftrightarrow \quad \Pi(\overline{y}, \overline{z}) \models_{\mathcal{V}} \Delta(\overline{y}).$$

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Moreover, if $\mathcal{V}$ is locally finite, these are equivalent to

(3) *$\mathcal{V}$ has deductive interpolation, is congruence distributive, and for finite sets $\overline{x}$, $\overline{y}$, the compact lifting of $\mathbf{F}(\overline{y}) \hookrightarrow \mathbf{F}(\overline{x}, \overline{y})$ preserves meets.*

An Example

An **implicative semilattice** is an algebraic structure $\langle A, \wedge, \to \rangle$ satisfying (i) $\langle A, \wedge \rangle$ is a semilattice; (ii) $a \wedge b \leq c \iff a \leq b \to c$ for all $a, b, c \in A$.

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since such a Δ would give a definition of $y_1 \vee y_2$ for implicative semilattices.

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Note. The condition that $\mathrm{KCon}\,\mathbf{F}(\omega)$ is a Brouwerian join-semilattice is equivalent to the property of equationally definable principal congruences.

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Theorem (Wheeler 1976)

The theory of $\mathcal{V}$ has a model completion if and only if $\mathcal{V}$ is coherent, admits the amalgamation property, and has the conservative congruence extension property for its finitely presented members.

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