IV Duality & Logic

0) Recap

- $MA \equiv^{op} TKS$
- $(-)^{+} : KS \to MA$
- $(-)_{\circ} : MA \to KS$

Define

$Cm(K) := \{ \$^{+} \mid \$ \text{ in } K \}$

$Cst(C) := \{ A_{\circ} \mid A \text{ in } C \}$

$Str(C) := \{ \$ \mid \$^{+} \text{ in } C \}.$

1) (Basic) Modal Logic – Semantics

- Language

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond\varphi$$

$(p \in \mathrm{PropVar})$

- Algebraic logic: ~~formulas~~ formulas as terms
  ~ formulas as equations

Here: focus on inequalities $\varphi \leq \psi$.

- Semantics: interpret formulas in complex algebras.

Given (S, R) consider assignments / valuations

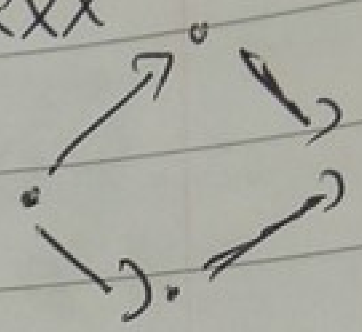
$$V : \mathrm{PropVar} \to \mathcal{P}(S).$$

- Validity: $\mathfrak{S}^+ \models \varphi \leq \psi$ $\varphi \leq \psi$ valid in $\mathfrak{S} / \mathfrak{S}^+$

$K \models \varphi \leq \psi$ if $\mathfrak{S} \models \varphi \leq \psi$ for all $\mathfrak{S}$ in K.

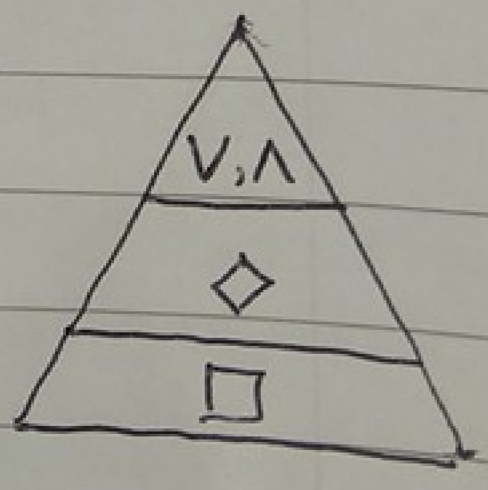
class of structures.

2) Correspondence

- Examples
  - $\mathfrak{F}^+ \models p \leq \Diamond p$ iff $\mathfrak{F} \models Rxx$
  - $\mathfrak{F}^+ \models \Diamond\Box p \leq \Box\Diamond p$ iff $\mathfrak{F} \models$ 
  - $\mathfrak{F}^+ \models \Box(p \to \Box p) \leq \Box p$ iff R is transitive & noetherian.

- General
  - modal inequalities corr. to second-order prop's
  - first-order correspondence is undecidable!

Def. Sahlqvist inequality

$\varphi \leq \psi$ where φ: 

ψ: positive

Thm Algorithmic correspondence for Sahlqvist

Pf Ackermann's Lemma: if β positive in p then $\exists p\,(p \leq \alpha \wedge \beta(p)) \equiv \beta[\alpha/p]$

3) Completeness

logic as deductive systems.

3a) Basic case: $K \sim MA$.

- $\vdash_K \varphi \leq \psi$ iff $MA \models \varphi \leq \psi$.
- Lindenbaum–Tarski alg. of K = free modal algebra.

Thm (Soundness & Completeness for K):

$$\vdash_K \varphi \leq \psi \iff KS \models \varphi \leq \psi$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

Pf $\quad MA \models \varphi \leq \psi \qquad Cm(KS) \models \varphi \leq \psi.$

$$\mathbb{A} \hookrightarrow (\mathbb{A}_\circ)^+$$

3b) Completeness with ~~frame~~ relational conditions

Example: transitive structures.

- correspondence: $\Diamond\Diamond p \leq \Diamond p \sim$ trans.
- logic K4: Add $4^{\nearrow}$ to K.
- Alg. Cpl: $\vdash_{K4} \varphi \leq \psi \Leftrightarrow MA_4 \models \varphi \leq \psi$.

Thm (Soundness & Completeness for K4).

$$\vdash_{K4} \varphi \leq \psi \Leftrightarrow \text{TraKS} \models \varphi \leq \psi.$$

$$\Updownarrow$$

Pf $MA_4 \models \varphi \leq \psi \Leftrightarrow Cm(\text{TraKS}) \models \varphi \leq \psi.$

MA_4 is generated by $Cm(\text{TraKS})$

Thm There is an order-~~preserving~~ reversing isomorphism between the lattices of

- normal modal logics (axiom. ext's of K)
- varieties of modal algebras.

Def A variety of MAs is complete if it is generated by its complex members.

Note: not all $\left\{ \begin{array}{l} \text{varieties of MAs} \\ \text{normal modal log's} \end{array} \right\}$ are complete!

4) Canonicity.

▷ Def.

- $\mathbb{A}^\delta := (\mathbb{A}_\bullet)^+$ is the canonical extension of $\mathbb{A}$
- An ineq $\varphi \leq \psi$ is canonical if $\mathbb{A} \models \varphi \leq \psi \Rightarrow \mathbb{A}^\delta \models \varphi \leq \psi$.
- A class of MAs is canonical if it is closed under taking can ext's.

▷ Thm. If V is canonical then it is complete.

▷ Ex. K4: V_4 is canonical.

▷ Thm. Sahlqvist inequalities are canonical.

▷ Cor. Let $\varphi \leq \psi$ be a Sq ineq, α its FO corr. Then $K + \varphi \leq \psi$ is complete for the Kripke structures satisfying α.

▷ Thm (Fine). If $K \subseteq KS$ is elementary then $HSP(\mathrm{cm}K)$ is canonical.

6. Interpolation.

- **Def.** A modal logic Λ has Craig Interpolation if

  if $\vdash_\Lambda \varphi \leq \psi$ then there is a θ such that
  - $L_\theta = L_\varphi \cap L_\psi$
  - $\vdash_\Lambda \varphi \leq \theta$ and $\vdash_\Lambda \theta \leq \psi$.

- Algebraic counterpart: superamalgamation

$$\begin{tikzcd} & \mathbb{C} & \\ \mathbb{B}_0 \arrow[ur, dashed, "f_0"] & & \mathbb{B}_1 \arrow[ul, dashed, "f_1"'] \\ & \mathbb{A} \arrow[ul, tail, "e_0"] \arrow[ur, tail, "e_1"'] & \end{tikzcd}$$

  - $f_0 \circ e_0 = f_1 \circ e_1$
  - $f_0 b_0 \leq f_1 b_1 \Rightarrow \exists a \in A$
    $f_0 b_0 \leq f_0 e_0 a \leq f_1 e_1$.
  - $f_1 b_1 \leq f_0 b_0 \Rightarrow \ldots$

- **Thm** (Maksimova) Λ has CIP $\Leftrightarrow$ V_Λ has SUPAP.

- **Thm** Let Γ be a set of inequalities, s.t.
  - every $\varphi \leq \psi$ is canonical
  - every $\varphi \leq \psi$ corresponds to a univ. Horn sentence

  Then $K.\Gamma$ has Craig Interpolation.