Goodwillie calculus for Gamma-spaces

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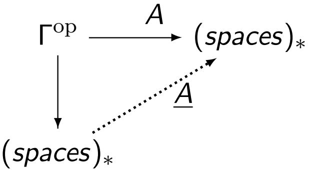
Definition (Γ-spaces)

$\mathrm{Ob}(\Gamma) = \{\underline{n} = \{*, 1, \ldots, n\}, n \geq 0\}$
$\Gamma^{\mathrm{op}}(\underline{m}, \underline{n}) = \mathrm{Sets}_*(\underline{m}, \underline{n})$
A Γ*-space* is a functor $A : \Gamma^{\mathrm{op}} \to (\mathit{spaces})_*$ sth. $A(\underline{0}) = *$.

space=simplicial set !

Each based space $(X, *)$ defines a functor $X^- : \Gamma \to (\mathit{spaces})_*$
Each Γ-space defines an endofunctor $\underline{A} : (\mathit{spaces})_* \to (\mathit{spaces})_*$

$$\underline{A}(X) = A \otimes_\Gamma X^-$$



Lemma (properties of the endofunctor $\underline{A}$)

- $\underline{A}$ preserves weak equivalences i.e. $\underline{A}$ is a *homotopy functor*
- $\underline{A}$ preserves connectivity
- $X \wedge \underline{A}(Y) \longrightarrow \underline{A}(X \wedge Y)$
- $\underline{A}$ takes (symmetric) spectra to (symmetric) spectra

Definition (Segal spectrification functor)

$\Phi : (\Gamma\text{-spaces}) \to (\mathit{spectra}) : A \mapsto \underline{A}(\text{sphere spectrum})$

Theorem (Segal '74)

Φ induces an equivalence of categories

$$\mathbf{Ho}(\text{very special } \Gamma\text{-spaces}) \simeq \mathbf{Ho}(\text{connective spectra})$$

Theorem (Bousfield-Friedlander '78)

There is a "stable" model structure on Γ-spaces such that

- the cofibrations are the Reedy cofibrations of Γ-spaces
- the weak equivalences are the *stable equivalences* of Γ-spaces
- the fibrant objects are the very special Γ-spaces

Moreover, $\Phi : (\Gamma\text{-spaces}) \to (\mathit{spectra})$ is a left Quillen functor inducing the aforementioned equivalence of homotopy categories.

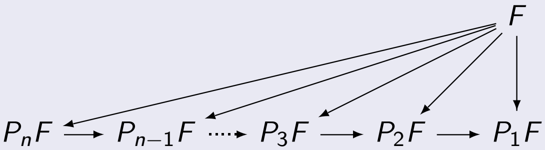
Remark

There is also a Reedy model structure on Γ-spaces with *pointwise weak equivalences* as weak equivalences and Reedy fibrations as fibrations because Γ^{op} is a generalised Reedy category (B-M '10).

The "stable" model structure is a left Bousfield localisation of the Reedy model structure.

Theorem (Goodwillie)

Each endofunctor $F : (spaces)_* \to (spaces)_*$ defines a tower of *n-excisive endofunctors* $P_nF : (spaces)_* \to (spaces)_*$ together with universal natural tranformations $p_n : F \to P_nF$



such that, if F is a connectivity-preserving homotopy functor, the *homotopy fiber* of $p_n : F \to P_nF$ multiplies connectivity by $n+1$.

Example (Freudenthal Suspension Theorem)

$\pi_*(X) \cong \pi_*((P_1 id)(X)) = \pi_*(\Omega^\infty \Sigma^\infty X)$ for $* < 2 \cdot \text{conn}(X)$

Definition (n-excisive)

An endofunctor is n-excisive if it takes *strongly h-cocartesian* $(n+1)$-cubes to h-cartesian $(n+1)$-cubes.
A homotopy functor is n-excisive iff it takes *cofibration* $(n+1)$*-cubes* to h-cartesian $(n+1)$-cubes.

Example (Linear endofunctor)

A homotopy functor F is 1-excisive (or *linear*) iff F takes any cone cofibration square to an h-cartesian square

$$\begin{array}{ccc} X & \longrightarrow & CX \\ \downarrow & & \downarrow \\ CX & \longrightarrow & \Sigma X \end{array} \qquad \begin{array}{ccc} F(X) & \longrightarrow & F(CX) \sim * \\ \downarrow & & \downarrow \\ F(CX) \sim * & \longrightarrow & F(\Sigma X) \end{array}$$

i.e. iff the canonical map $F(X) \to \Omega F \Sigma(X)$ is a weak equivalence.

For based spaces $(X_i)_{1\leq i\leq n+1}$, the wedge $(n+1)$-cube $\Xi_{X_1,\ldots,X_{n+1}}$ takes on $S\subset\{1,\ldots,n+1\}$ the value $\bigvee_{j\notin S}X_j$ with obvious arrows.

Example (Ξ_{X_1,X_2})

$$\begin{array}{ccc} X_1\vee X_2 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & * \end{array} \qquad \begin{array}{ccc} F(X_1\vee X_2) & \longrightarrow & F(X_1) \\ \downarrow & & \downarrow \\ F(X_2) & \longrightarrow & * \end{array}$$

F takes wedge squares to h-cartesian squares iff F takes (up to homotopy) coproducts to products.
For $F=\underline{A}$ and $X_1=\underline{m}$ and $X_2=\underline{n}$ this is what Segal calls a *special* Γ-space A.

Definition (n-special)

An endofunctor F is *n-special* if $F(\Xi_{X_1,\dots,X_{n+1}})$ is h-cartesian $\forall X_i$.
A Γ-space A is *n-special* if $\underline{A}(\Xi_{\underline{m}_1,\dots,\underline{m}_{n+1}})$ is h-cartesian $\forall \underline{m}_i$.
A Γ-space A is *very n-special* if A is *n-special* and *fully fibrant*.

Fully fibrant $=$ Reedy fibrant $+$ horn matching maps are surjective.
In particular, the bisimplicial set $\underline{A}(X)$ fulfills π_*-Kan condition for every based simplicial set $(X,*)$.

Remark (very 1-special=very special)

A Γ-space A is 1-special iff $A(\underline{m+n}) \xrightarrow{\sim} A(\underline{m}) \times A(\underline{n})$ $\forall m,n$.
A Γ-space A is very 1-special iff A is 1-special and "group-like".

Remark

A Γ-space is n-special iff it has the RLP with respect to a certain finite set of cofibrations into representable Γ-sets.

Lemma

An n-excisive homotopy functor is n-special.

Consider the cofibration $(n+1)$-cube generated by
$X_1 \vee \cdots \vee X_i \vee \cdots \vee X_{n+1} \to X_1 \vee \cdots \vee CX_i \vee \cdots \vee X_{n+1}$

Lemma (A vs $\underline{A}$)

For a fully fibrant Γ-space A, A is n-special iff $\underline{A}$ is n-special.

Use π_*-Kan and $\underline{A}$(wedge-cube) degree-wise h-cartesian.

Proposition (n-special vs n-excisive)

For a fully fibrant Γ-space A, $\underline{A}$ is n-special iff $\underline{A}$ is n-excisive.

Use relationship between homotopy total fibers of retraction cube and associated section cube.

Definition (*n*-equivalence)

A map of Γ-spaces $A \to B$ is an *n-equivalence* if $(P_n\underline{A})(X) \to (P_n\underline{B})(X)$ is a weak equivalence for any $(X, *)$.

A pointwise weak equivalence is an n-equivalence for all $n \geq 1$. Stable equivalences of Γ-spaces are precisely 1-equivalences.

Theorem (n-excisive model structure, B-B '12)

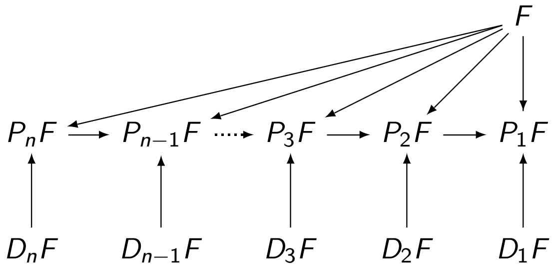
For each $n \geq 1$, there is a model structure on Γ-spaces such that

- the cofibrations are the Reedy cofibrations of Γ-spaces
- the weak equivalences are the n-equivalences of Γ-spaces
- the fibrant objects are the very n-special Γ-spaces

For $n = 1$ this is precisely the stable model structure of Bousfield-Friedlander.

Corollary

For each Γ-space A, the n-th approximation $p_n : \underline{A} \to P_n\underline{A}$ is induced by a fibrant replacement in the n-excisive model structure.



D_nF is n-excisive and its $(n-1)$-th approximation is trivial.

Theorem (Goodwillie)

$(D_nF)(X) \simeq \Omega^\infty(\partial_n F \wedge X^{\wedge n})_{h\Sigma_n}$ for a Σ_n-spectrum $\partial_n F$.

Conjecture (*n*-homogeneous model structure)

For each $n \geq 1$, there is a model structure on $(n-1)$-reduced Γ-spaces transferred from the *n*-excisive model structure such that

$$\mathbf{Ho}((n-1)\text{-reduced } \Gamma\text{-spaces}) \simeq \mathbf{Ho}((-n)\text{-connected } \Sigma_n\text{-spectra})$$

Corollary

For each Γ-space A, the derivatives $\partial_n \underline{A}$ can be deduced from this Quillen equivalence.

Γ and Δ share many formal properties. In particular, there is the notion of *representable* Γ-set $\Gamma^n = \Gamma(-, \underline{n})$, of its *boundary* $\partial\Gamma^n$, and hence of the *quotients* $\Gamma^n/\partial\Gamma^n$, the Γ-*spheres*.

Problem

What are the endofunctors associated to the Γ-spheres ?

- $\underline{\Gamma^n}(X) = X^n$
- $\exists$ outer boundary $\partial_{out}\Gamma^n \subset \partial\Gamma^n$ such that $\underline{\partial_{out}\Gamma^n}(X) = \{x \in X^n | x_i = * \text{ for at least one } i\}$
- $\underline{\Gamma^n/\partial_{out}\Gamma^n}(X) = X^{\wedge n}$
- $\partial\Gamma^n/\partial_{out}\Gamma^n \hookrightarrow \Gamma^n/\partial_{out}\Gamma^n$ has as quotient the Γ-sphere

Proposition

There is a functor $\phi_n : \Pi_n \longrightarrow$ (Γ-sets) whose value at a partition with k blocks is $\Gamma^k/\partial_{out}\Gamma^k$ such that

$$\varinjlim_{\Pi_n \setminus \{\text{finest partition}\}} \phi_n = \partial\Gamma^n/\partial_{out}\Gamma^n.$$

Conjecture

The derivatives of the identity can be deduced from this decomposition of the Γ-spheres. More precisely, $\mathrm{Hom}_\Gamma(\Gamma^n/\partial\Gamma^n, A)$ is stably equivalent to the n-th derivative of $\underline{A}$.

Thank you !