PROGRAM FOR THE MASTER 2 MPA, 2022-2023

COORDINATED BY INDIRA CHATTERJI

Algebra-geometry bloc

Advanced geometry. Introduction to differential geometry (Jérémy Toulisse).

The goal of this course will be to give a general introduction to differential geometry (more specifically, differential topology and Riemannian geometry). We will cover some important aspects of differential geometry that are fundamental in many different areas of research, and this course will be intimately connected with the other courses of the Algebra-Geometry Bloc.

More specifically, we aim to cover:

- Theory of smooth manifolds: vector fields, differential forms, Frobenius theorem.
- Theory of vector bundles, Euclidean and Hermitian metrics, connections, curvature.
- Riemannian metrics, Levi-Civita connection, geoedesics, Riemann curvature tensor.
- Riemannian submanifolds, second fundamental form, Gauss, Ricci and Codazzi equations
- Comparison theorems in Riemannian geometry (if time permits).

The following references may be useful

(1) Paulin, *Groupes et Géométrie* (available of his webpage).
(2) Gallot, Hulin, Lafontaine. *Riemannian Geometry* - Springer 2004.
(3) Kobayashi, Nomizu. *Foundations of Differential Geometry Vol. I* - 1996.

Topics in algebra and geometry. Transcendental methods in complex geometry (Junyan Cao).

The goal of this course is to give an introduction to the transcendental aspect of complex geometry. We will start by recalling briefly

Date: November 2021.

the notions of connection, curvature, and Cherns classes on holomorphic vector bundles, which are closely related to Toulisse and Höring's courses. Then we will focus on the Hodge theory and its applications in complex geometry such as Hard lefschetz theorem, Kodaira vanishing theorems. In the last part, we will discuss its generalization in singular metric setting and its recent applications in complex geometry.

The following references may be useful

(1) Demailly, *Complex analytic and differential geometry* - available on his homepage.
(2) Demailly, *Analytic Methods in Algebraic Geometry* Higher Education Press, Surveys of Modern Mathematics, Vol. 1, 2010, - available on his homepage.
(3) Höring. *Kähler Geometry and Hodge theory* - available on his homepage.
(4) Huybrechts. *Complex Geometry* - Springer 2005.
(5) Voisin. *Théorie de Hodge et géométrie algébrique complexe.* Vol 10 of Cours Spécialisés, SMF, 2002.

Computational algebra or geometry. Introduction to topological data analysis (Indira Chatterji and Antoine Commaret).

Topological data analysis is an approach to understand large data sets through their shape. Data sets are often clouds of points in $\mathbf{R}^n$, and analyzing them is a challenge with many applications. Techniques from topology have been proven useful to analyze large data sets, reducing their complexity to allow computer to handle them. A related question is to reconstruct a geometrical object only knowing a finite set of points nearby, which is known as *geometric inference.* This course will study the basic tools from topology that are used in topological data analysis and geometric inference. More specifically, we shall study the following topics.

(1) Simplicial complexes and homology
(2) Simplicial complexes built from data sets
(3) Delaunay triangulations
(4) Distance functions, distance to measures
(5) Persistent homology

The lectures will be completely theoretical. The exercise sessions will have some hands-on experiments, as well as theoretical problems. We shall mainly follow the publicly available books below.

- J.-D. Boissonnat, F. Chazal, M. Yvinec. *Geometric and Topological Inference.* Cambridge Texts in Applied Mathematics, vol. 57, Cambridge University Press, 2018

- A. Hatcher. *Algebraic Topology.* Cambridge University Press 2002.

Advanced logic. Introduction to complex geometry (Andreas Höring)

The goal of this course is to give an introduction to complex geometry, that is the study of complex manifolds and more generally complex analytic spaces. Complex manifolds are special cases of differentiable manifolds, so the general techniques from Toulisse's course are useful for their study, but there are also tools which are specific for dealing with complex manifolds. In this course we will first see some examples, then discover various cohomological theories and how to use them for the classification of complex manifolds. This includes the cohomology defined via differential forms (de Rham and Dolbeault), as well as more general constructions like the Čech cohomology of sheaves. Time permitting I also want to talk about the intersection product for divisors on projective manifolds.

The following references may be useful

(1) Demailly, *Complex analytic and differential geometry* - available on his homepage.
(2) Fischer, *Complex analytic geometry* Springer 1976
(3) Höring. *Kähler Geometry and Hodge theory* - available on my homepage.
(4) Huybrechts. *Complex Geometry* - Springer 2005.

Topics in topology or dynamics. Lie groups and symmetric spaces (Vincent Pecastaing).

The goal of this course is to give an introduction to Riemannian symmetric spaces, an important class of homogeneous Riemannian manifolds which generalizes notably hyperbolic spaces. Most concepts seen in the course of J. Toulisse will be important. Starting from a differential-geometric definition, we will see that all the geometry of a symmetric space is encoded into a pair of Lie groups. We will discuss the interplay between these two viewpoints, and if time permits we will introduce important dynamical systems attached to these geometries. The content will be:

- Lie groups, Lie algebras, Homogeneous spaces
- General definition and properties of Riemannian symmetric spaces
- De Rham decomposition, Non-compact type
- Classification of semi-simple Lie algebras
- Maximal flats, Weyl chambers, and visual boundary

References

- A.W. Knapp, *Lie Groups Beyond an Introduction*, Progress in Mathematics, Vol. 140, Birkhäuser, Basel, 2002.
- S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, vol. 80, Academic Press, 1979
- F. Paulin, *Groupes et géométries, Version préliminaire*, Lecture notes.

Analysis bloc

Numerical methods and deterministic PDEs. Numerical Approximation of Hyperbolic Systems of Conservation Laws (Afeintou SANGAM).

The aim of this course unit is at familiarising students with base methods of numerical computation and numerical simulation of Hyperbolic Systems of Conservation Laws, the most famous example of which is gas dynamics, that is studied during this course. Numerical illustrations on computers are proposed for the practical implementation of studied algorithms. It will be also shown how these methods are integrated in the recent upgrades of major industrial computer codes and, research software as RealFluids (Universität Zürich) and PlaTo (INRIA and Université Côte d'Azur). Finally, it will be a timely opportunity to report how Mathematics and real applications interweave.

Contents

- Introduction to theoretical analysis of hyperbolic scalar equations and systems of equations.
- Introduction to numerical approximation.
- Numerical schemes for hyperbolic scalar equations.
- Numerical schemes for hyperbolic 1-D systems of equations.
- Introduction to numerical schemes for hyperbolic 2-D scalar equations and systems of equations.

Selected bibliography

- Edwige Godlewski, Pierre-Arnaud Raviart, *Numerical Approximation of Hyperbolic System of Conservation Laws*, Second Edition, Applied Mathematics Sciences, **118**, Springer, New York (2021).
- Hervé Guillard, Rémi Abgrall, *Modélisation Numérique des Fluides Compressibles*, Series in Applied Mathematics, **5**, Gauthier-Villars, Paris, North-Holland, Amsterdam (2001).
- Eleuterio F. Toro, *Riemann Solvers and Numerical Methods in Fluid Dynamics, a Practical Introduction*, 3rd Edition, Springer, Heidelberg, 2009.
- Bernardo Cockburn, Chi-Wang Shu, Claes Johnson, Eitan Tadmor, Advanced Numerical Approximation of Nonlinear Hyperbolic Equations, Editor Alfio Quarteroni, Lecture Notes in Mathematics book series, **1697**, Springer, Berlin, Heidelberg (1998).

Analysis of PDEs. The Cauchy problem in collisionless kinetic theory (N. Besse).

The aim of this lecture is to present the state of art of the Cauchy problem for the collisionless kinetic equations such as the Vlasov–Poisson and Vlasov–Maxwell systems. Collisionless kinetic equations, which are Hamiltonian systems, appear among others in plasma physics and astrophysics. In plasma physics, these models describe accurately the wave-particle interaction which plays a crucial role in turbulent plasmas such as magnetic fusion plasmas (ITER project). In astrophysics theses models allow to describe the large scale structure of the universe such as clusters of galaxies and the dark matter. Here, we present the theory of weak solutions and classical regular solutions. Existence theory of classical solutions is based on natural a priori estimates like among others the conservation of energy, on the theory of characteristics and on the control of the velocity support of the distribution function. Uniqueness follows from regularity properties of classical solutions. Existence theory of weak solutions relies on a priori functional estimates and on compactness results such as standard compact embeddings in Sobolev spaces for the Vlasov–Poisson equations, or averaging lemmas for the Vlasov–Maxwell system. Uniqueness of weak solutions is a more tricky task and sometimes it is still an open issue.
Contents:

1) Introduction: theory of characteristics; formal properties of the Vlasov–Poisson and Vlasov–Maxwell systems; conservation laws; basic a priori estimates.
2) Weak and classical solutions for the Vlasov–Poisson system.
3) Weak and classical solutions for the Vlasov–Maxwell system.

Mathematical models for physics. Mathematical modelling for cell mechanics (R. Allena).

Cell mechanics plays a fundamental role in several mechanobiological phenomena such as bone remodelling, cancer , embryogenesis and immune response. During the last decades, biologists have started exploring the role of mechanics and more specifically of forces, stress and strains exerted and undergone by the cells on their environment. During this course we will review the basic principles of continuum mechanics in order to be able to address mathematical modelling for several biological processes such as adhesion, migration, invasion and cell-cell interactions. Various mathematical approaches based on ODEs, PDEs, diffusion-reaction equations or others will be presented in order to highlight how phenomena happening at different scales can be modelled and coupled to account for multiscale aspects. Specific focus will be on single and collective cell migration as well as nucleus mechanics.

- Part I
  Fundamental concepts in continuum mechanics.
  Solid mechanics.
- Part II
  Introduction on cell mechanical properties.
  Extra, inter and intracellular mechanisms involved in cell motility.
- Part III
  Different models of single and collective cell migration: physical, mathematical and computational approaches.
- Part IV
  Nucleus mechanics.

Topics in PDEs. Elliptic PDEs and regularity (I. Moyano).

This course is an introduction to the theory of linear second order elliptic partial differential equations, which play a fundamental role in many areas of mathematics including Non-linear PDEs, Calculus of Variations or Riemannian Geometry.

In this course we aim at providing a rigorous treatment of both classical and weak solutions to linear elliptic equations (Poisson problem, divergence operators, etc). The question of existence and uniqueness of solutions to the Dirichlet problem on several geometric setting (balls, an more general domains) and its regularity in terms of the data will be central in the course. Some specific topics in this field are: harmonic functions, maximum principles, Schauder estimates, Perron's method and De Giorgi–Nash–Moser estimates.

Prerequisites (helpful but not essential): Some courses of M1 level that may be helpful but not essential are "Analyse de Fourier et distributions", "Equations aux dérivées partielles et différences finies" ou "Introduction aux équations aux dérivées partielles," "Introduction à l'analyse fonctionnelle."

Some references: Gilbarg & Trudinger, Evans, Qin & Lin.

Advanced PDEs. Nonlinear Schrödinger equations (S. Rota Nodari).

This course is an introduction to nonlinear Schrödinger equations (NLS). This kind of equations is relevant from a physical point of view, in particular because of their applications to nonlinear optics. Moreover, nonlinear Schrödinger equations also arise in quantum field theory, and more precisely in the Hartree-Fock theory. From a mathematical point of view, the NLS equation can be seen as a good model of dispersive equation which is technically simpler than other dispersive equations as the wave or the Korteweg-de Vries (KdV) equations.

The aim of this course is to discuss local and global existence results, conservation laws, and the existence and the qualitative properties of solitary wave solutions. This will involve a wide variety of branches of mathematics: functional analysis (Lebesgue and Sobolev spaces), harmonic analysis (Strichartz estimates) and variational methods.

Probability and statistics bloc

Stochastic Calculus and Applications. (R. Catellier).

This course is devoted to the introduction of the basic concepts of continuous time stochastic processes which are used in many fields : physics, finance, biology, medicine, filtering theory, decision theory. It will consist of a presentation of Brownian motion, Itô integral, stochastic differential equations and Girsanov theorem. Several applications from the aforementioned topcis will be given.

Probabilistic Numerical Methods. (S. Rubenthaler).

Probabilistic numerical methods are widely used in machine learning algorithms as well as in mathematical finance for pricing financial derivatives and computing strategies. The course will present the basic methods used for simulating random variables and implementing the Monte-Carlo methods. Simulation in Python of stochastic processes used in mathematical finance, such as Brownian motion and solutions to stochastic differential equations, will be discussed as well.

Advanced Stochastics. (F. Delarue).

The first part of the course provides the basic knowledge in stochastic control, control for diffusion processes, dynamic programming principle, dynamic programming equation, Hamilton Jacobi Bellman equation, control for counting processes.

As for the second part, one path among the following two ones has to be chosen:

- A first path addresses the theory of mean-field models. Applications to collective optimization in finance, or self-organisation and phase transition in neuroscience will be considered.
- A second path addresses stochastic optimization. Stochastic gradient descent (Robbins-Monro, 1951) is the workhorse of many statistical and probabilistic procedure. In particular, it is widely used in machine learning for training artificial neural networks, support vector machines. This course is intended to provide a mathematical foundation to this algorithm and variants of it, along with a numerical intuition of its behavior on practical examples.

Statistical Learning Methods. (D. Garreau).

Statistical Learning is a successful framework for computers to learn and perform complicated tasks. In recent years, human-like performance has been achieved in some applications such as object detection in images. In this course, we will introduce a selected subset of methods ranging from linear methods to neural networks. Each time, we will demonstrate their use in practice (practical sessions in Python), and present some theoretical insights when possible.

Advanced Statistics. (E. Di Bernardino).

This course focuses on three pillars of modern statistical inference: parameter estimation, hypothesis testing, and model selection. Its aim is to provide a good understanding of the current methods via a thorough treatment of the existing theoretical guarantees. A particular emphasis will be placed on the estimation, with a focus on the asymptotic setting. After revisiting the method of moments and maximum likelihood estimation, we will turn to U-statistics and look at some applications.