Nearness Posets

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Nice, France, June 17th 2019
Topology, Algebra and Categories in Logic
Classical Uniformities

- Familiar concepts from metric spaces, e.g. uniform continuity, completion etc., can be generalised to uniform spaces.
- Following Tukey (1940), a uniformity is just a family of ‘uniform’ covers on a set $X$ satisfying certain conditions.
- These covers determine a canonical topology, specifically where the ‘stars’ of $x \in X$ form a neighbourhood base at $x$.

Theorem
A topological space $X$ is uniformisable iff $X$ is completely regular.

- To extend to regular and even $T_1$ spaces, Morita (1951) weakened the star-refinement axiom for the covers.
- Katětov (1963) and Herrlich (1974) independently came up with equivalent versions of Morita’s generalised uniformities.
- These are now usually called nearness spaces, i.e. a nearness is again just a special family of ‘uniform’ covers of a set $X$. 
Nearness Frames

- More recently, people have considered point-free nearnesses.
- First, take a frame $\mathbb{L}$, i.e. a complete lattice which we think of as representing the open sets of a topological space.
- Again, a nearness is a family of covers, which are now subsets $C \subseteq \mathbb{L}$ each with $\bigvee C = 1$, again subject to some conditions.
- However, we can already see a key difference between this approach and the classical pointy notion of a nearness – Now the nearness is placed on top of a pre-existing topological structure, rather than defining it like in the classical case.

Question
What if we instead replace the lattice structure with covers?
- So we would instead start with just a set $S$ together with some distinguished family of subsets $\Theta \subseteq \mathcal{P}(S)$, nothing more.
- Here we could even consider $S$ to represent a more general basis or even just a subbasis of open sets of some space.
Recovering Spaces from Covers

- First question – can we recover a space $X$ from such a weak abstract covering structure? Yes, as long as $X$ is $T_1$.
- To see this, take a subbasis $S$ of a $T_1$ space $X$ with covers

$$\Theta = \{ C \subseteq S : X \subseteq \bigcup S \}.$$

- For each $x \in X$, consider its subbasic neighbourhoods

$$N_x = \{ s \in S : x \in s \}.$$

- As each $C \in \Theta$ covers $X$, each $N_x$ is $\Theta$-Cauchy, i.e.

$$C \in \Theta \quad \Rightarrow \quad N_x \cap C \neq \emptyset. \quad (\Theta\text{-Cauchy})$$

- As $X$ is $T_1$, each $N_x$ is minimal $\Theta$-Cauchy, i.e.

$$s \in N_x \quad \Rightarrow \quad \exists C \in \Theta \ (N_x \cap C = \{ s \}).$$

- Moreover, there are no other minimal $\Theta$-Cauchy subsets:

- Say $M \subseteq S$ does not contain any $N_x$.
- Then $X \subseteq \bigcup S \setminus M \in \Theta$ so $M$ is not $\Theta$-Cauchy.
The Spectrum

Definition
Given $\Theta \subseteq \mathcal{P}(S)$, the spectrum is the space

$$\hat{\Theta} = \{ N \subseteq S : N \text{ is minimal } \Theta\text{-Cauchy} \}$$

with the topology generated by the sets $\hat{\Theta}_s = \{ N \in \hat{\Theta} : s \in N \}$.

So what we just proved is the following.

Proposition
If $S$ is a subbasis of a $T_1$ space $X$ and $\Theta = \{ C \subseteq S : X \subseteq \bigcup S \}$ is the family of all $S$-covers of $X$ then $\hat{\Theta}$ is homeomorphic to $X$.

Conversely, say we start with abstract $\Theta \subseteq \mathcal{P}(S)$.

By minimality, $\hat{\Theta}$ is a $T_1$ space.

By Cauchyness, each $C \in \Theta$ yields a cover $(\hat{\Theta}_c)_{c \in C}$ of $\hat{\Theta}$.

However, there could be many other covers, e.g. if $C \in \Theta$ then any $D \supseteq C$ also covers $\hat{\Theta}$, even when $D \notin \Theta$.

Also, we could have $\hat{\Theta}_s = \hat{\Theta}_t$ even when $s \neq t$. 
The Canonical Order

- Any $\Theta \subseteq \mathcal{P}(S)$ defines a preorder on $S$ by

\[ s \leq_\Theta t \iff \Theta^s \subseteq \Theta^t, \]

where $\Theta^s = \{ D \subseteq S : \{s\} \cup D \in \Theta \}$.

- If $S$ is a concrete subbasis of some $T_1$ space $X$ and $\Theta$ is the family of all covers then $\leq_\Theta$ coincides with containment, i.e.

\[ s \leq_\Theta t \iff s \subseteq t. \]

- For abstract $S$ and $\Theta$, we do at least have

\[ s \leq_\Theta t \Rightarrow \hat{\Theta}_s \subseteq \hat{\Theta}_t. \]

- So if we hope to represent $S$ faithfully as a subbasis on the spectrum then, at the very least, $\leq_\Theta$ should be a partial order.

- In this case, let us call $(S, \leq_\Theta, \Theta)$ a nearness poset.
Finitary Nearness Posets

Call $\Theta$ finitary if every $C \in \Theta$ is finite and, for all finite $F \subseteq S$,

$$F \supseteq C \in \Theta \Rightarrow F \in \Theta.$$  

We can now reformulate a classical result due to Wallman.

Theorem (Wallman 1938)

If $(S, \leq, \Theta)$ is a finitary nearness poset then $\hat{\Theta}$ is compact, 

$$s \leq t \iff \hat{\Theta}_s \subseteq \hat{\Theta}_t$$

and

$$\Theta = \{F \subseteq S : F \text{ is finite and } \hat{\Theta} \subseteq \bigcup_{s \in F} \hat{\Theta}_s\}.$$  

Conversely, if $S$ is a subbasis of compact $T_1$ $X$ and $\Theta$ is the family of all finite covers, $(S, \subseteq, \Theta)$ is a finitary nearness poset.

So we have a kind of duality

Finitary Nearness Posets $\leftrightarrow$ Compact $T_1$ Spaces.
Extensions

- It is then natural to investigate potential Wallman-type dualities for non-compact nearness spaces, e.g.

Star-Finitary Nearness Posets $\leftrightarrow$ Locally Compact $T_1$ Spaces.

- Using the Arhangel'skii-Stone metrisation theorem, we also have an analog for completely metrisable spaces, via regular $\Theta$ with a countable filter base (w.r.t. refinement).

- For the details see ArXiv:1902.07948 ‘Nearness Posets’.

- Aside: compact metric spaces are supercompact, i.e. they have a subbasis s.t. every cover has a 2-element subcover.

- Thus these correspond to 2-ary nearness posets.

- So all compact metric spaces arise as the spectrum of a countable graph, which could be worth exploring further.
Graded Posets

- Graded/ranked posets have a natural nearness structure coming from the rank levels, i.e. taking these as a base for $\Theta$.
- Many natural examples of arise in this way.
- E.g. the standard basis of the Cantor space $\{0, 1\}^\mathbb{N}$ coming from finite initial sequences yields the complete binary tree:
Similarly, the arc/interval $[0, 1]$, with the dyadic basis

$$\left\{ \left( \frac{k-1}{2^n}, \frac{k+1}{2^n} \right) : k, n \in \mathbb{N} \text{ and } 1 < k < 2^n - 1 \right\}$$

and $$\left\{ [0, \frac{1}{2^n}) : n \in \mathbb{N} \right\} \cup \left\{ (1 - \frac{1}{2^n}, 1] : n \in \mathbb{N} \right\} \cup \left\{ [0, 1] \right\}$$ yields
Graded Posets $\leftrightarrow$ Compact $T_1$ Spaces

Theorem
Every second countable compact $T_1$ space is the spectrum of a countable graded poset with finite levels.

- Analogous to the fact compact Hausdorff spaces are all inverse limits of simplicial complexes (Freudenthal 1937).
- But this is not merely of theoretical interest - it suggests we could actually construct interesting spaces by first constructing an appropriate graded poset, e.g. by recursively defining the levels and the relations between them.
- E.g. for the pseudoarc, we could consider the category of finite paths/linear graphs, where the morphisms are relations between them that preserve and reflect the graph structure.
- This category has the amalgamation property and hence a Fraïssé sequence, which we combine to form a graded poset.
- The spectrum of this poset is precisely the pseudoarc.
- To obtain the Lelek fan, replace paths with rooted trees.