Generic Models for Topological Evidence Logics

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1. Topological models for epistemic logics
2. Topological evidence logics
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Topological models for epistemic logics
• Let \((X, \tau)\) be a topological space, \(\text{Prop}\) a set of propositional variables and \(V : \text{Prop} \rightarrow \mathcal{P}(X)\) a valuation.

• Let us start with a language \(\mathcal{L}\) defined as follows:

\[
\phi ::= p | \phi \land \psi | \neg \phi | K\phi,
\]

with \(p \in \text{Prop}\).

### Interior semantics

- \(\|p\| = V(p)\);
- \(\|\phi \land \psi\| = \|\phi\| \cap \|\psi\|\);
- \(\|\neg \phi\| = X \backslash \|\phi\|\);
- \(\|K\phi\| = \text{Int} \|\phi\|\).
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The interior semantics

• Generalisation of Kripke semantics for preordered frames: preorders are Alexandroff topologies.
  From \((X, \leq)\) consider the topology \(\tau = \text{Up}(\leq)\).
  We have: \(x \in \text{Int} \|\phi\|\) iff \(y \in \|\phi\|\) for all \(y \geq x\).

• Evidential view of knowledge: knowing a proposition amounts to having a piece of evidence (i.e. an open set) than entails it.

• McKinsey & Tarski (1944) proved two important results to this respect:
  • The logic of topological spaces under this semantics is \(S_4\);
  • The logic of a single dense-in-itself metrisable space (such as \(\mathbb{R}\)) under this semantics is \(S_4\).

But:

• The interior semantics equates “knowing” to “having evidence”.

• Some attempts at introducing belief (Steinsvold, 2006; Baltag et al., 2013) equate true belief to knowledge or confine us to work with really weird spaces.
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Following this and building on evidence models (van Benthen & Pacuit, 2011) a new framework is introduced by Baltag, Bezhanishvili, Özgün & Smets (2016).

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$$(X, \tau, E_0, V)$$

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- \(K\): defeasible knowledge;
- \(B\): belief;
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### The dense interior semantics

- $x \in \|K\phi\|$ iff $x \in \text{Int } \|\phi\|$ and $\text{Int } \|\phi\|$ is dense;
- $x \in \|B\phi\|$ iff $\text{Int } \|\phi\|$ is dense;
- $x \in \|[\forall]\phi\|$ iff $\|\phi\| = X$;
- $x \in \|\Box\phi\|$ iff $x \in \text{Int } \|\phi\|$;
- $x \in \|\Box_0\phi\|$ iff $x \in e \subseteq \|\phi\|$ for some $e \in E_0$.

In this framework, knowing $P = \text{having an evidence for } P$ that can’t be defeated by any other evidence (i.e. a dense evidence).
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### The logic of some fragments

**The knowledge fragment** $\mathcal{L}_K$: $S4.2$.

**The knowledge-belief fragment** $\mathcal{L}_{KB}$: $S4.2$ axioms for $K$, $KD45$ axioms for $B$ plus:

- (PI) $B\phi \rightarrow KB\phi$;
- (NI) $\neg B\phi \rightarrow K\neg B\phi$;
- (KB) $K\phi \rightarrow B\phi$;
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- (FB) $B\phi \rightarrow BK\phi$.

(This is the logic outlined in Stalnaker, 2006.)

**The evidence fragment** $\mathcal{L}_{\forall\Box\Box}$: $S5$ for $[\forall]$, plus $S4$ for $\Box$, plus:

- ($4\Box_0$) $\Box_0\phi \rightarrow \Box_0\Box_0\phi$;
- (Universality) $[\forall]\phi \rightarrow \Box_0\phi$;
- (Factive evidence) $\Box_0\phi \rightarrow \Box\phi$;
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The logic of topological spaces under the interior semantics is $S_4$, and so is the logic of $\mathbb{R}$ under the interior semantics.

• This result tells us that $\mathbb{R}$, as a topological space, is generic enough to capture the logic of topological spaces.

• How to apply this idea to our framework? First, let us formalise:

A topological space $(X, \tau)$ is a **generic model** for a language $\mathcal{L}$ if the sound and complete $\mathcal{L}$-logic of topo-e-models is precisely the logic of the class

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Theorems

• Any dense-in-itself metrisable space \((X, \tau)\) is a generic model for the knowledge fragment \(\mathcal{L}_K\) and the knowledge-belief fragment \(\mathcal{L}_{KB}\).

**Examples:** \(\mathbb{R}, \mathbb{Q}, \mathbb{I}\), the Baire space, the Cantor space, the binary tree...

The \(\mathcal{L}_K\)-logic of any of these spaces is S4.2.

• Any dense-in-itself metrisable space \((X, \tau)\) which is idempotent (i.e. homeomorphic to \((X, \tau) \oplus (X, \tau)\)) is a generic model for the fragments \(\mathcal{L}_\forall\Box\), \(\mathcal{L}_\forall K\) and \(\mathcal{L}_\forall\Box\Box\Box\).

**Examples:** all of the above except \(\mathbb{R}\) and the binary tree.

So:

• Whatever is true in any of the logics defined earlier is true in any topo-e-model whose topological space is \(\mathbb{Q}\), and conversely,

• Whatever is not provable in these logics has a countermodel based on \(\mathbb{Q}\).
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S4.2 is complete with respect to $\mathbb{R}$ as a topo-e-model.

We use this (from Bezhanishvili $\times 2$, Lucero-Bryan & van Mill, 2018):

- Completeness wrt finite rooted S4.2 Kripke frames (rooted preorder $B \cup$ final cluster $A$).
- Partition lemma: for each $n \geq 1$, $\mathbb{R}$ can be partitioned in $\{U_1, ..., U_n, G\}$, where $G$ is a dense-in-itself set with dense complement and each $U_i$ is open.
- There exists an open, continuous and surjective map $f : G \to B$.

We can extend $f$ to a surjective map $\tilde{f} : \mathbb{R} \to B \cup A$ such that:

- The preimage of an upset is a dense open set (dense-continuous);
- The image of a dense open set is an upset (dense-open).

If something can be refuted in $B \cup A$, we can construct a valuation using $\tilde{f}$ that refutes it in $\mathbb{R}$. Completeness follows.
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\[ f \text{ open and cont.} \]

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Two-agent topo-e-models
Let us make this framework multi-agent.

We will consider two epistemic agents (1 and 2) each of them having different sets of evidence on a common space $X$ ($\tau_1$ and $\tau_2$).

How do we account for defeasibility and infallible knowledge?

A first naive approach would be to just use density as in the single agent case:

$$x \in \|K_i\phi\| \text{ iff } \exists U \in \tau_i, \text{ dense such that } x \in U \subseteq \|\phi\|.$$  

Two issues with this approach:

- Same set of worlds is compatible with both agents’ information.
- The logic of this semantics contains theorems like

$$\hat{K}_1K_1p \to K_2\hat{K}_1K_1p.$$  

If agent 1 doesn’t know that she doesn’t know $p$, then agent 2 knows this fact. We don’t want this!
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If agent 1 doesn’t know that she doesn’t know $p$, then agent 2 knows this fact. We don’t want this!
We want to account for defeasibility of knowledge (through some notion of density) that gives us a reasonable logic and doesn’t present this issues while being defined on a common space $X$.

To solve this, we make explicit which is the subset of $X$ which is compatible with each agent’s information at a world.

We make this through the use of partitions $\Pi_1$ and $\Pi_2$.

A topological partitional model is a tuple

$$(X, \tau_1, \tau_2, \Pi_1, \Pi_2, V)$$

where $X$ is a set, $V$ is a valuation, each $\tau_i$ is a topology and each $\Pi_i \subseteq \tau_i$ is an open partition of $X$. 

**Topological-partitional models**
• We want to account for defeasibility of knowledge (through some notion of density) that gives us a reasonable logic and doesn’t present this issues while being defined on a common space $X$.

• To solve this, we make explicit which is the subset of $X$ which is compatible with each agent’s information at a world.

• We make this through the use of partitions $\Pi_1$ and $\Pi_2$.

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For $U \subseteq X$ and $\pi \in \Pi_i$ we say that $U$ is $\pi$-locally dense whenever $U \cap \pi$ is dense in the subspace topology

$$\tau_i|_{\pi} = \{ V \cap \pi : V \in \tau_i \}.$$ 

Semantics for knowledge in topological-partitional models

For $i = 1, 2$ and $x \in X$, let $\pi \in \Pi_i$ be the unique cell with $x \in \pi$. We have:

$$x \in \|K_i \phi\| \iff \exists U \in \tau_i, \pi \text{-locally dense such that } x \in U \subseteq \|\phi\|.$$ 

The logic of knowledge in this setting is $S_{4.2K_1} + S_{4.2K_2}$. 
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$$x_{11} \models K_1 p \land \neg K_2 p$$
Theorem

The logic of the knowledge-only fragment of topological parititional models is the fusion logic $S_{4.2K_1} + S_{4.2K_1}$.

The infinite branching quaternary tree is an example of a generic model for this fragment.

- Each topology $\tau_i$ is given by the set of $R_i$-upsets.
- The open partitions are given by the equivalence relation: $x \sim_i y$ iff there exists a $z$ ($zR_ix$ and $zR_iy$).
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• Another generic model is found on the rational plane $\mathbb{Q} \times \mathbb{Q}$.

• The horizontal topology $\tau_H$ is originated by the family of sets

\[ \{ U \times \{x\} : U \text{ open}, x \in \mathbb{Q} \}. \]

Similarly, the vertical topology $\tau_V$ is originated by the sets $\{x\} \times U$.

• We prove that there exists an open partition on $\mathbb{Q} \times \mathbb{Q}$ making it into a generic model for the knowledge fragment.
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• Strong completeness?
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• Dynamic two-agent topological-partitional models.
I never know how to end talks. Please clap now.
Alexandru Baltag, Nick Bezhanishvili, Aybüke Özgün, and Sonja Smets.  
**The topology of belief, belief revision and defeasible knowledge.**  

Alexandru Baltag, Nick Bezhanishvili, Aybüke Özgün, and Sonja Smets.  
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