Bridges between Logic and Algebra

Part 1: Intuitionistic Logic

George Metcalfe

Mathematical Institute
University of Bern

TACL 2019 Summer School, Île de Porquerolles, June 2019

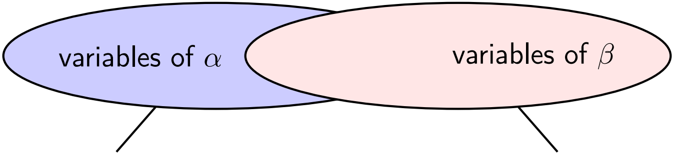
A Problem in Logic

Does some logic L admit **interpolation**?

$$\alpha(\overline{x}, \overline{y}) \qquad \vdash_{\mathsf{L}} \qquad \beta(\overline{y}, \overline{z})$$

A Problem in Logic

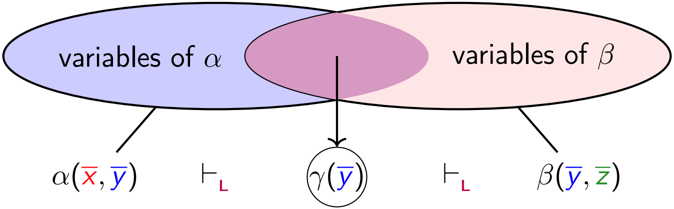
Does some logic L admit **interpolation**?



$$\alpha(\overline{x}, \overline{y}) \qquad \vdash_{\mathsf{L}} \qquad \beta(\overline{y}, \overline{z})$$

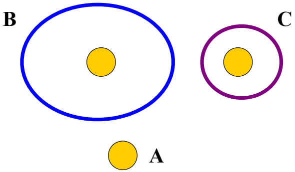
A Problem in Logic

Does some logic L admit **interpolation**?



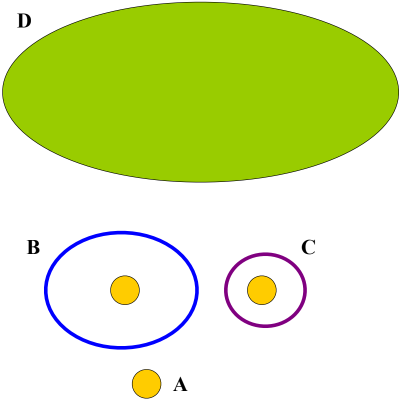
A Problem in Algebra

Does some class of algebras $\mathcal{K}$ have the **amalgamation property**?



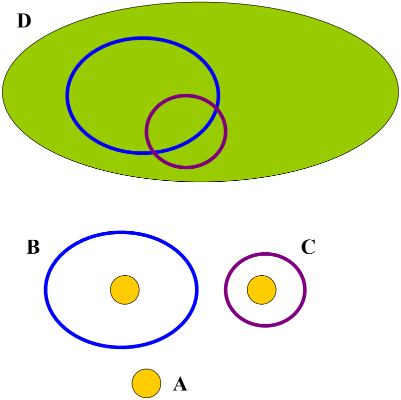
A Problem in Algebra

Does some class of algebras $\mathcal{K}$ have the **amalgamation property**?

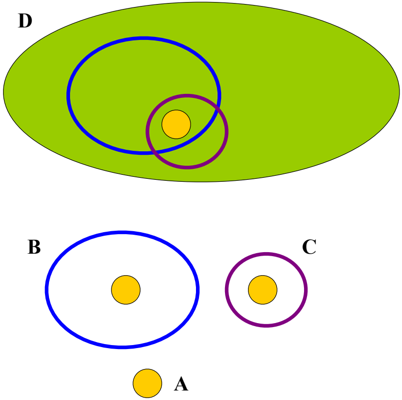


A Problem in Algebra

Does some class of algebras $\mathcal{K}$ have the **amalgamation property**?



Does some class of algebras $\mathcal{K}$ have the **amalgamation property**?



A Bridge Theorem

L admits **interpolation** $\iff$ $\mathcal{K}_{\mathrm{L}}$ has the **amalgamation property**

This Tutorial

How can we build and cross bridges between **logic** and **algebra**?

Today

How can we do this for **intuitionistic logic** and **Heyting algebras**?

Intuitionistic Logic

Intuitionistic logic was introduced by Heyting in the 1930s to formalize certain principles used in Brouwer's constructive mathematics.

Intuitionistic Logic

Intuitionistic logic was introduced by Heyting in the 1930s to formalize certain principles used in Brouwer's constructive mathematics.

The Brouwer-Heyting-Kolmogorov interpretation presents the validity of formulas in intuitionistic logic in terms of the construction of proofs,

Intuitionistic Logic

Intuitionistic logic was introduced by Heyting in the 1930s to formalize certain principles used in Brouwer's constructive mathematics.

The Brouwer-Heyting-Kolmogorov interpretation presents the validity of formulas in intuitionistic logic in terms of the construction of proofs, e.g.,

"A proof of $\alpha \vee \beta$ is given via a proof of α or a proof of β."

Intuitionistic Logic

Intuitionistic logic was introduced by Heyting in the 1930s to formalize certain principles used in Brouwer's constructive mathematics.

The Brouwer-Heyting-Kolmogorov interpretation presents the validity of formulas in intuitionistic logic in terms of the construction of proofs, e.g.,

"A proof of $\alpha \vee \beta$ is given via a proof of α or a proof of β."

Intuitionistic logic may be presented **syntactically** via

- axiom systems, natural deduction, tableau or sequent calculi, etc.

Intuitionistic Logic

Intuitionistic logic was introduced by Heyting in the 1930s to formalize certain principles used in Brouwer's constructive mathematics.

The Brouwer-Heyting-Kolmogorov interpretation presents the validity of formulas in intuitionistic logic in terms of the construction of proofs, e.g.,

"A proof of $\alpha \vee \beta$ is given via a proof of α or a proof of β."

Intuitionistic logic may be presented **syntactically** via

- axiom systems, natural deduction, tableau or sequent calculi, etc.

or **semantically** via

- Heyting algebras, Kripke models, topological semantics, etc.

An Axiom System

Formulas $\alpha, \beta, \gamma \ldots$ are defined inductively for a propositional language with binary connectives $\wedge, \vee, \to$ and constants $\bot, \top$ over a countably infinite set of variables $x, y, z \ldots$, where $\alpha \leftrightarrow \beta := (\alpha \to \beta) \wedge (\beta \to \alpha)$.

An Axiom System

Formulas $\alpha, \beta, \gamma \ldots$ are defined inductively for a propositional language with binary connectives $\wedge, \vee, \to$ and constants $\bot, \top$ over a countably infinite set of variables $x, y, z \ldots$, where $\alpha \leftrightarrow \beta := (\alpha \to \beta) \wedge (\beta \to \alpha)$.

We write $T \vdash_{\mathrm{IL}} \alpha$ to denote that a formula α is **derivable** from a set of formulas T using

An Axiom System

Formulas $\alpha, \beta, \gamma \ldots$ are defined inductively for a propositional language with binary connectives $\wedge, \vee, \rightarrow$ and constants $\bot, \top$ over a countably infinite set of variables $x, y, z \ldots$, where $\alpha \leftrightarrow \beta := (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$.

We write $T \vdash_{\mathrm{IL}} \alpha$ to denote that a formula α is **derivable** from a set of formulas T using the axiom schema

$$\begin{array}{ll}
\alpha \rightarrow (\beta \rightarrow \alpha) & (\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)) \\
(\alpha \wedge \beta) \rightarrow \alpha & (\alpha \wedge \beta) \rightarrow \beta \\
\alpha \rightarrow (\alpha \vee \beta) & \beta \rightarrow (\alpha \vee \beta) \\
\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta)) & (\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow ((\alpha \vee \beta) \rightarrow \gamma)) \\
\alpha \rightarrow \top & \bot \rightarrow \alpha
\end{array}$$

together with the *modus ponens* rule: from α and $\alpha \rightarrow \beta$, infer β.

Consequence

It is easy to check that $\vdash_{\text{IL}}$ is a **finitary structural consequence relation**;

Consequence

It is easy to check that $\vdash_{\mathsf{IL}}$ is a **finitary structural consequence relation**; that is, for any set of formulas $T \cup T' \cup \{\alpha\}$,

(i) if $\alpha \in T$, then $T \vdash_{\mathsf{IL}} \alpha$ *(reflexivity)*;

Consequence

It is easy to check that $\vdash_{\mathsf{IL}}$ is a **finitary structural consequence relation**; that is, for any set of formulas $T \cup T' \cup \{\alpha\}$,

(i) if $\alpha \in T$, then $T \vdash_{\mathsf{IL}} \alpha$ *(reflexivity)*;

(ii) if $T \vdash_{\mathsf{IL}} \alpha$ and $T \subseteq T'$, then $T' \vdash_{\mathsf{IL}} \alpha$ *(monotonicity)*;

Consequence

It is easy to check that $\vdash_{\text{IL}}$ is a **finitary structural consequence relation**; that is, for any set of formulas $T \cup T' \cup \{\alpha\}$,

(i) if $\alpha \in T$, then $T \vdash_{\text{IL}} \alpha$ *(reflexivity)*;

(ii) if $T \vdash_{\text{IL}} \alpha$ and $T \subseteq T'$, then $T' \vdash_{\text{IL}} \alpha$ *(monotonicity)*;

(iii) if $T \vdash_{\text{IL}} \alpha$ and $T' \vdash_{\text{IL}} \beta$ for every $\beta \in T$, then $T' \vdash_{\text{IL}} \alpha$ *(transitivity)*;

Consequence

It is easy to check that $\vdash_{\mathrm{IL}}$ is a **finitary structural consequence relation**; that is, for any set of formulas $T \cup T' \cup \{\alpha\}$,

(i) if $\alpha \in T$, then $T \vdash_{\mathrm{IL}} \alpha$ *(reflexivity)*;

(ii) if $T \vdash_{\mathrm{IL}} \alpha$ and $T \subseteq T'$, then $T' \vdash_{\mathrm{IL}} \alpha$ *(monotonicity)*;

(iii) if $T \vdash_{\mathrm{IL}} \alpha$ and $T' \vdash_{\mathrm{IL}} \beta$ for every $\beta \in T$, then $T' \vdash_{\mathrm{IL}} \alpha$ *(transitivity)*;

(iv) if $T \vdash_{\mathrm{IL}} \alpha$, then $\sigma[T] \vdash_{\mathrm{IL}} \sigma(\alpha)$ for any substitution σ *(structurality)*;

Consequence

It is easy to check that $\vdash_{\mathsf{IL}}$ is a **finitary structural consequence relation**; that is, for any set of formulas $T \cup T' \cup \{\alpha\}$,

(i) if $\alpha \in T$, then $T \vdash_{\mathsf{IL}} \alpha$ *(reflexivity)*;

(ii) if $T \vdash_{\mathsf{IL}} \alpha$ and $T \subseteq T'$, then $T' \vdash_{\mathsf{IL}} \alpha$ *(monotonicity)*;

(iii) if $T \vdash_{\mathsf{IL}} \alpha$ and $T' \vdash_{\mathsf{IL}} \beta$ for every $\beta \in T$, then $T' \vdash_{\mathsf{IL}} \alpha$ *(transitivity)*;

(iv) if $T \vdash_{\mathsf{IL}} \alpha$, then $\sigma[T] \vdash_{\mathsf{IL}} \sigma(\alpha)$ for any substitution σ *(structurality)*;

(v) if $T \vdash_{\mathsf{IL}} \alpha$, then $T' \vdash_{\mathsf{IL}} \alpha$ for some finite $T' \subseteq T$ *(finitarity)*.

A Deduction Theorem

Theorem

For any set of formulas $T \cup \{\alpha, \beta\}$,

$$T \vdash_{\mathsf{IL}} \alpha \to \beta \iff T \cup \{\alpha\} \vdash_{\mathsf{IL}} \beta.$$

A Deduction Theorem

Theorem

For any set of formulas $T \cup \{\alpha, \beta\}$,

$$T \vdash_{\mathsf{IL}} \alpha \to \beta \iff T \cup \{\alpha\} \vdash_{\mathsf{IL}} \beta.$$

Proof.

($\Rightarrow$) Suppose that $T \vdash_{\mathsf{IL}} \alpha \to \beta$.

A Deduction Theorem

Theorem

For any set of formulas $T \cup \{\alpha, \beta\}$,

$$T \vdash_{\mathsf{IL}} \alpha \to \beta \iff T \cup \{\alpha\} \vdash_{\mathsf{IL}} \beta.$$

Proof.

($\Rightarrow$) Suppose that $T \vdash_{\mathsf{IL}} \alpha \to \beta$. By monotonicity, $T \cup \{\alpha\} \vdash_{\mathsf{IL}} \alpha \to \beta$

A Deduction Theorem

Theorem

For any set of formulas $T \cup \{\alpha, \beta\}$,

$$T \vdash_{\mathsf{IL}} \alpha \to \beta \iff T \cup \{\alpha\} \vdash_{\mathsf{IL}} \beta.$$

Proof.

($\Rightarrow$) Suppose that $T \vdash_{\mathsf{IL}} \alpha \to \beta$. By monotonicity, $T \cup \{\alpha\} \vdash_{\mathsf{IL}} \alpha \to \beta$ and, by reflexivity, $T \cup \{\alpha\} \vdash_{\mathsf{IL}} \alpha$.

A Deduction Theorem

Theorem

For any set of formulas $T \cup \{\alpha, \beta\}$,

$$T \vdash_{\mathrm{IL}} \alpha \to \beta \iff T \cup \{\alpha\} \vdash_{\mathrm{IL}} \beta.$$

Proof.

($\Rightarrow$) Suppose that $T \vdash_{\mathrm{IL}} \alpha \to \beta$. By monotonicity, $T \cup \{\alpha\} \vdash_{\mathrm{IL}} \alpha \to \beta$ and, by reflexivity, $T \cup \{\alpha\} \vdash_{\mathrm{IL}} \alpha$. So, by modus ponens, $T \cup \{\alpha\} \vdash_{\mathrm{IL}} \beta$.

A Deduction Theorem

Theorem

For any set of formulas $T \cup \{\alpha, \beta\}$,

$$T \vdash_{\mathsf{IL}} \alpha \to \beta \iff T \cup \{\alpha\} \vdash_{\mathsf{IL}} \beta.$$

Proof.

($\Rightarrow$) Suppose that $T \vdash_{\mathsf{IL}} \alpha \to \beta$. By monotonicity, $T \cup \{\alpha\} \vdash_{\mathsf{IL}} \alpha \to \beta$ and, by reflexivity, $T \cup \{\alpha\} \vdash_{\mathsf{IL}} \alpha$. So, by modus ponens, $T \cup \{\alpha\} \vdash_{\mathsf{IL}} \beta$.

($\Leftarrow$) By induction on the length of a derivation of β from $T \cup \{\alpha\}$ in IL.

A Deduction Theorem

Theorem

For any set of formulas $T \cup \{\alpha, \beta\}$,

$$T \vdash_{\mathsf{IL}} \alpha \to \beta \iff T \cup \{\alpha\} \vdash_{\mathsf{IL}} \beta.$$

Proof.

($\Rightarrow$) Suppose that $T \vdash_{\mathsf{IL}} \alpha \to \beta$. By monotonicity, $T \cup \{\alpha\} \vdash_{\mathsf{IL}} \alpha \to \beta$ and, by reflexivity, $T \cup \{\alpha\} \vdash_{\mathsf{IL}} \alpha$. So, by modus ponens, $T \cup \{\alpha\} \vdash_{\mathsf{IL}} \beta$.

($\Leftarrow$) By induction on the length of a derivation of β from $T \cup \{\alpha\}$ in IL. For $\beta = \alpha$, note that $T \vdash_{\mathsf{IL}} \alpha \to \alpha$.

A Deduction Theorem

Theorem

For any set of formulas $T \cup \{\alpha, \beta\}$,

$$T \vdash_{\mathrm{IL}} \alpha \to \beta \iff T \cup \{\alpha\} \vdash_{\mathrm{IL}} \beta.$$

Proof.

($\Rightarrow$) Suppose that $T \vdash_{\mathrm{IL}} \alpha \to \beta$. By monotonicity, $T \cup \{\alpha\} \vdash_{\mathrm{IL}} \alpha \to \beta$ and, by reflexivity, $T \cup \{\alpha\} \vdash_{\mathrm{IL}} \alpha$. So, by modus ponens, $T \cup \{\alpha\} \vdash_{\mathrm{IL}} \beta$.

($\Leftarrow$) By induction on the length of a derivation of β from $T \cup \{\alpha\}$ in IL. For $\beta = \alpha$, note that $T \vdash_{\mathrm{IL}} \alpha \to \alpha$. If β is in T or an axiom, then $T \vdash_{\mathrm{IL}} \beta$

A Deduction Theorem

Theorem

For any set of formulas $T \cup \{\alpha, \beta\}$,

$$T \vdash_{\text{IL}} \alpha \to \beta \iff T \cup \{\alpha\} \vdash_{\text{IL}} \beta.$$

Proof.

($\Rightarrow$) Suppose that $T \vdash_{\text{IL}} \alpha \to \beta$. By monotonicity, $T \cup \{\alpha\} \vdash_{\text{IL}} \alpha \to \beta$ and, by reflexivity, $T \cup \{\alpha\} \vdash_{\text{IL}} \alpha$. So, by modus ponens, $T \cup \{\alpha\} \vdash_{\text{IL}} \beta$.

($\Leftarrow$) By induction on the length of a derivation of β from $T \cup \{\alpha\}$ in IL. For $\beta = \alpha$, note that $T \vdash_{\text{IL}} \alpha \to \alpha$. If β is in T or an axiom, then $T \vdash_{\text{IL}} \beta$ and, since $T \vdash_{\text{IL}} \beta \to (\alpha \to \beta)$, also $T \vdash_{\text{IL}} \alpha \to \beta$.

A Deduction Theorem

Theorem

For any set of formulas $T \cup \{\alpha, \beta\}$,

$$T \vdash_{\mathsf{IL}} \alpha \to \beta \iff T \cup \{\alpha\} \vdash_{\mathsf{IL}} \beta.$$

Proof.

($\Rightarrow$) Suppose that $T \vdash_{\mathsf{IL}} \alpha \to \beta$. By monotonicity, $T \cup \{\alpha\} \vdash_{\mathsf{IL}} \alpha \to \beta$ and, by reflexivity, $T \cup \{\alpha\} \vdash_{\mathsf{IL}} \alpha$. So, by modus ponens, $T \cup \{\alpha\} \vdash_{\mathsf{IL}} \beta$.

($\Leftarrow$) By induction on the length of a derivation of β from $T \cup \{\alpha\}$ in IL. For $\beta = \alpha$, note that $T \vdash_{\mathsf{IL}} \alpha \to \alpha$. If β is in T or an axiom, then $T \vdash_{\mathsf{IL}} \beta$ and, since $T \vdash_{\mathsf{IL}} \beta \to (\alpha \to \beta)$, also $T \vdash_{\mathsf{IL}} \alpha \to \beta$. For the induction step, suppose that $T \cup \{\alpha\} \vdash_{\mathsf{IL}} \gamma$ and $T \cup \{\alpha\} \vdash_{\mathsf{IL}} \gamma \to \beta$.

A Deduction Theorem

Theorem

For any set of formulas $T \cup \{\alpha, \beta\}$,

$$T \vdash_{\mathrm{IL}} \alpha \to \beta \iff T \cup \{\alpha\} \vdash_{\mathrm{IL}} \beta.$$

Proof.

($\Rightarrow$) Suppose that $T \vdash_{\mathrm{IL}} \alpha \to \beta$. By monotonicity, $T \cup \{\alpha\} \vdash_{\mathrm{IL}} \alpha \to \beta$ and, by reflexivity, $T \cup \{\alpha\} \vdash_{\mathrm{IL}} \alpha$. So, by modus ponens, $T \cup \{\alpha\} \vdash_{\mathrm{IL}} \beta$.

($\Leftarrow$) By induction on the length of a derivation of β from $T \cup \{\alpha\}$ in IL. For $\beta = \alpha$, note that $T \vdash_{\mathrm{IL}} \alpha \to \alpha$. If β is in T or an axiom, then $T \vdash_{\mathrm{IL}} \beta$ and, since $T \vdash_{\mathrm{IL}} \beta \to (\alpha \to \beta)$, also $T \vdash_{\mathrm{IL}} \alpha \to \beta$. For the induction step, suppose that $T \cup \{\alpha\} \vdash_{\mathrm{IL}} \gamma$ and $T \cup \{\alpha\} \vdash_{\mathrm{IL}} \gamma \to \beta$. By the induction hypothesis, $T \vdash_{\mathrm{IL}} \alpha \to \gamma$ and $T \vdash_{\mathrm{IL}} \alpha \to (\gamma \to \beta)$.

A Deduction Theorem

Theorem

For any set of formulas $T \cup \{\alpha, \beta\}$,

$$T \vdash_{IL} \alpha \to \beta \iff T \cup \{\alpha\} \vdash_{IL} \beta.$$

Proof.

($\Rightarrow$) Suppose that $T \vdash_{IL} \alpha \to \beta$. By monotonicity, $T \cup \{\alpha\} \vdash_{IL} \alpha \to \beta$ and, by reflexivity, $T \cup \{\alpha\} \vdash_{IL} \alpha$. So, by modus ponens, $T \cup \{\alpha\} \vdash_{IL} \beta$.

($\Leftarrow$) By induction on the length of a derivation of β from $T \cup \{\alpha\}$ in IL. For $\beta = \alpha$, note that $T \vdash_{IL} \alpha \to \alpha$. If β is in T or an axiom, then $T \vdash_{IL} \beta$ and, since $T \vdash_{IL} \beta \to (\alpha \to \beta)$, also $T \vdash_{IL} \alpha \to \beta$. For the induction step, suppose that $T \cup \{\alpha\} \vdash_{IL} \gamma$ and $T \cup \{\alpha\} \vdash_{IL} \gamma \to \beta$. By the induction hypothesis, $T \vdash_{IL} \alpha \to \gamma$ and $T \vdash_{IL} \alpha \to (\gamma \to \beta)$. Since also $T \vdash_{IL} (\alpha \to (\gamma \to \beta)) \to ((\alpha \to \gamma) \to (\alpha \to \beta))$, we get $T \vdash_{IL} \alpha \to \beta$. □

Heyting Algebras

A **Heyting algebra** is an algebraic structure $\langle A, \wedge, \vee, \to, \bot, \top \rangle$ such that

Heyting Algebras

A **Heyting algebra** is an algebraic structure $\langle A, \wedge, \vee, \to, \bot, \top \rangle$ such that

(i) $\langle A, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice with $a \leq b :\Longleftrightarrow a \wedge b = a$;

Heyting Algebras

A **Heyting algebra** is an algebraic structure $\langle A, \wedge, \vee, \to, \bot, \top \rangle$ such that

(i) $\langle A, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice with $a \le b :\Longleftrightarrow a \wedge b = a$;

(ii) $a \le b \to c \iff a \wedge b \le c$ for all $a, b, c \in A$.

Heyting Algebras

A **Heyting algebra** is an algebraic structure $\langle A, \wedge, \vee, \to, \bot, \top \rangle$ such that

(i) $\langle A, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice with $a \le b :\Longleftrightarrow a \wedge b = a$;

(ii) $a \le b \to c \iff a \wedge b \le c$ for all $a, b, c \in A$.

The class $\mathcal{HA}$ of Heyting algebras forms a **variety**.

Heyting Algebras

A **Heyting algebra** is an algebraic structure $\langle A, \wedge, \vee, \to, \bot, \top \rangle$ such that

(i) $\langle A, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice with $a \leq b :\Longleftrightarrow a \wedge b = a$;

(ii) $a \leq b \to c \iff a \wedge b \leq c$ for all $a, b, c \in A$.

The class $\mathcal{HA}$ of Heyting algebras forms a **variety**.

Examples:

1. any Boolean algebra;

Heyting Algebras

A **Heyting algebra** is an algebraic structure $\langle A, \wedge, \vee, \to, \bot, \top\rangle$ such that

(i) $\langle A, \wedge, \vee, \bot, \top\rangle$ is a bounded lattice with $a \leq b :\Longleftrightarrow a \wedge b = a$;

(ii) $a \leq b \to c \iff a \wedge b \leq c$ for all $a, b, c \in A$.

The class $\mathcal{HA}$ of Heyting algebras forms a **variety**.

Examples:

1. any Boolean algebra;
2. letting $\mathcal{U}$ be the set of upsets of a poset $\langle X, \leq\rangle$,
   $\langle \mathcal{U}, \cap, \cup, \to, \emptyset, X\rangle$ where $Y \to Z = \{a \in X \mid a \leq b \in Y \implies b \in Z\}$;

Heyting Algebras

A **Heyting algebra** is an algebraic structure $\langle A, \wedge, \vee, \to, \bot, \top \rangle$ such that

(i) $\langle A, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice with $a \le b :\Longleftrightarrow a \wedge b = a$;

(ii) $a \le b \to c \iff a \wedge b \le c$ for all $a, b, c \in A$.

The class $\mathcal{HA}$ of Heyting algebras forms a **variety**.

Examples:

1. any Boolean algebra;
2. letting $\mathcal{U}$ be the set of upsets of a poset $\langle X, \le \rangle$, $\langle \mathcal{U}, \cap, \cup, \to, \emptyset, X \rangle$ where $Y \to Z = \{a \in X \mid a \le b \in Y \implies b \in Z\}$;
3. letting $\mathcal{O}$ be the set of open subsets of $\mathbb{R}$ with the usual topology,
$$\langle \mathcal{O}, \cap, \cup, \to, \emptyset, \mathbb{R} \rangle \text{ where } Y \to Z = \mathrm{int}(Y^c \cup Z).$$

The Lindenbaum-Tarski Construction

Given any set of formulas T, define a binary relation on formulas by

$$\alpha \, \Theta_T \beta \; :\Longleftrightarrow \; T \vdash_{\mathsf{IL}} \alpha \leftrightarrow \beta.$$

The Lindenbaum-Tarski Construction

Given any set of formulas T, define a binary relation on formulas by

$$\alpha \, \Theta_T \beta \;\; :\Longleftrightarrow \;\; T \vdash_{\mathsf{IL}} \alpha \leftrightarrow \beta.$$

Then Θ_T is an equivalence relation

The Lindenbaum-Tarski Construction

Given any set of formulas T, define a binary relation on formulas by

$$\alpha \, \Theta_T \beta \;:\Longleftrightarrow\; T \vdash_{\mathsf{IL}} \alpha \leftrightarrow \beta.$$

Then Θ_T is an equivalence relation satisfying for $\star \in \{\wedge, \vee, \to\}$,

$$\alpha_1 \, \Theta_T \beta_1 \text{ and } \alpha_2 \, \Theta_T \beta_2 \quad \Longrightarrow \quad \alpha_1 \star \alpha_2 \, \Theta_T \, \beta_1 \star \beta_2,$$

The Lindenbaum-Tarski Construction

Given any set of formulas T, define a binary relation on formulas by

$$\alpha \, \Theta_T \beta \;:\Longleftrightarrow\; T \vdash_{\mathsf{IL}} \alpha \leftrightarrow \beta.$$

Then Θ_T is an equivalence relation satisfying for $\star \in \{\wedge, \vee, \to\}$,

$$\alpha_1 \, \Theta_T \beta_1 \text{ and } \alpha_2 \, \Theta_T \beta_2 \quad \Longrightarrow \quad \alpha_1 \star \alpha_2 \, \Theta_T \, \beta_1 \star \beta_2,$$

and we obtain a Heyting algebra

$$\mathbf{A}_T = \langle A_T, \wedge_T, \vee_T, \to_T, [\bot]_T, [\top]_T \rangle$$

where A_T is the set of Θ_T-equivalence classes $[\alpha]_T$

The Lindenbaum-Tarski Construction

Given any set of formulas T, define a binary relation on formulas by

$$\alpha \, \Theta_T \beta \;\; :\Longleftrightarrow \;\; T \vdash_{\mathsf{IL}} \alpha \leftrightarrow \beta.$$

Then Θ_T is an equivalence relation satisfying for $\star \in \{\wedge, \vee, \to\}$,

$$\alpha_1 \, \Theta_T \beta_1 \text{ and } \alpha_2 \, \Theta_T \beta_2 \quad \Longrightarrow \quad \alpha_1 \star \alpha_2 \, \Theta_T \, \beta_1 \star \beta_2,$$

and we obtain a Heyting algebra

$$\mathbf{A}_T = \langle A_T, \wedge_T, \vee_T, \to_T, [\bot]_T, [\top]_T \rangle$$

where A_T is the set of Θ_T-equivalence classes $[\alpha]_T$ and for $\star \in \{\wedge, \vee, \to\}$,

$$[\alpha]_T \star_T [\beta]_T = [\alpha \star \beta]_T.$$

The Lindenbaum-Tarski Construction

Given any set of formulas T, define a binary relation on formulas by

$$\alpha \, \Theta_T \beta \;\; :\Longleftrightarrow \;\; T \vdash_{\mathsf{IL}} \alpha \leftrightarrow \beta.$$

Then Θ_T is an equivalence relation satisfying for $\star \in \{\wedge, \vee, \to\}$,

$$\alpha_1 \, \Theta_T \beta_1 \;\text{ and }\; \alpha_2 \, \Theta_T \beta_2 \quad \Longrightarrow \quad \alpha_1 \star \alpha_2 \, \Theta_T \, \beta_1 \star \beta_2,$$

and we obtain a Heyting algebra

$$\mathbf{A}_T = \langle A_T, \wedge_T, \vee_T, \to_T, [\bot]_T, [\top]_T \rangle$$

where A_T is the set of Θ_T-equivalence classes $[\alpha]_T$ and for $\star \in \{\wedge, \vee, \to\}$,

$$[\alpha]_T \star_T [\beta]_T = [\alpha \star \beta]_T.$$

In particular, $\vdash_{\mathsf{IL}} \alpha$ if and only if $\mathbf{A}_\emptyset \models \alpha \approx \top$.

Equational Consequence

For any set of equations $\Sigma \cup \{\alpha \approx \beta\}$, we write

$$\Sigma \models_{\mathcal{HA}} \alpha \approx \beta$$

Equational Consequence

For any set of equations $\Sigma \cup \{\alpha \approx \beta\}$, we write

$$\Sigma \models_{\mathcal{HA}} \alpha \approx \beta$$

if for any homomorphism e from the formula algebra to a Heyting algebra,

$$e(\gamma) = e(\delta) \text{ for all } \gamma \approx \delta \in \Sigma \implies e(\alpha) = e(\beta).$$

Equational Consequence

For any set of equations $\Sigma \cup \{\alpha \approx \beta\}$, we write

$$\Sigma \models_{\mathcal{HA}} \alpha \approx \beta$$

if for any homomorphism e from the formula algebra to a Heyting algebra,

$$e(\gamma) = e(\delta) \text{ for all } \gamma \approx \delta \in \Sigma \implies e(\alpha) = e(\beta).$$

Note. $\models_{\mathcal{HA}}$ is a finitary structural equational consequence relation.

A First Bridge Theorem

Theorem

$\mathcal{HA}$ *is an* **equivalent algebraic semantics** *for* IL

A First Bridge Theorem

Theorem

$\mathcal{HA}$ *is an* **equivalent algebraic semantics** *for* IL *with* **transformers**

$$\tau(\alpha) = \alpha \approx \top \quad \textit{and} \quad \rho(\alpha \approx \beta) = \alpha \leftrightarrow \beta.$$

A First Bridge Theorem

Theorem

$\mathcal{HA}$ *is an* **equivalent algebraic semantics** *for* IL *with* **transformers**

$$\tau(\alpha) = \alpha \approx \top \quad \textit{and} \quad \rho(\alpha \approx \beta) = \alpha \leftrightarrow \beta.$$

(i) *For any set of formulas* $T \cup \{\alpha\}$,

$$T \vdash_{\mathsf{IL}} \alpha \iff \tau[T] \models_{\mathcal{HA}} \tau(\alpha).$$

A First Bridge Theorem

Theorem

$\mathcal{HA}$ *is an* **equivalent algebraic semantics** *for* IL *with* **transformers**

$$\tau(\alpha) = \alpha \approx \top \quad \textit{and} \quad \rho(\alpha \approx \beta) = \alpha \leftrightarrow \beta.$$

(i) *For any set of formulas* $T \cup \{\alpha\}$,

$$T \vdash_{\mathsf{IL}} \alpha \iff \tau[T] \models_{\mathcal{HA}} \tau(\alpha).$$

(ii) *For any set of equations* $\Sigma \cup \{\alpha \approx \beta\}$,

$$\Sigma \models_{\mathcal{HA}} \alpha \approx \beta \iff \rho[T] \vdash_{\mathsf{IL}} \rho(\alpha \approx \beta).$$

A First Bridge Theorem

Theorem

$\mathcal{HA}$ *is an* **equivalent algebraic semantics** *for* IL *with* **transformers**

$$\tau(\alpha) = \alpha \approx \top \quad \textit{and} \quad \rho(\alpha \approx \beta) = \alpha \leftrightarrow \beta.$$

(i) *For any set of formulas* $T \cup \{\alpha\}$,

$$T \vdash_{\mathsf{IL}} \alpha \iff \tau[T] \models_{\mathcal{HA}} \tau(\alpha).$$

(ii) *For any set of equations* $\Sigma \cup \{\alpha \approx \beta\}$,

$$\Sigma \models_{\mathcal{HA}} \alpha \approx \beta \iff \rho[T] \vdash_{\mathsf{IL}} \rho(\alpha \approx \beta).$$

(iii) *For any formulas* α, β,

$$\alpha \dashv\vdash_{\mathsf{IL}} \rho(\tau(\alpha)) \quad \textit{and} \quad \alpha \approx \beta \;=\!\!|\!\!\models_{\mathcal{HA}} \tau(\rho(\alpha \approx \beta)).$$

Proof Sketch

For (i), we need to prove

$$T \vdash_{\mathsf{IL}} \alpha \iff \{\gamma \approx \top \mid \gamma \in T\} \models_{\mathcal{HA}} \alpha \approx \top.$$

Proof Sketch

For (i), we need to prove

$$T \vdash_{\text{IL}} \alpha \quad \Longleftrightarrow \quad \{\gamma \approx \top \mid \gamma \in T\} \models_{\mathcal{HA}} \alpha \approx \top.$$

($\Rightarrow$) A straightforward induction on the length of a derivation of α from T in IL using properties of Heyting algebras.

Proof Sketch

For (i), we need to prove

$$T \vdash_{\mathsf{IL}} \alpha \iff \{\gamma \approx \top \mid \gamma \in T\} \models_{\mathcal{HA}} \alpha \approx \top.$$

($\Rightarrow$) A straightforward induction on the length of a derivation of α from T in IL using properties of Heyting algebras.

($\Leftarrow$) Suppose that $T \nvdash_{\mathsf{IL}} \alpha$

Proof Sketch

For (i), we need to prove

$$T \vdash_{\mathrm{IL}} \alpha \iff \{\gamma \approx \top \mid \gamma \in T\} \models_{\mathcal{HA}} \alpha \approx \top.$$

($\Rightarrow$) A straightforward induction on the length of a derivation of α from T in IL using properties of Heyting algebras.

($\Leftarrow$) Suppose that $T \nvdash_{\mathrm{IL}} \alpha$ and consider the homomorphism e from the formula algebra to the Heyting algebra $\mathbf{A}_T$ given by $e(\gamma) = [\gamma]_T$.

Proof Sketch

For (i), we need to prove

$$T \vdash_{\text{IL}} \alpha \iff \{\gamma \approx \top \mid \gamma \in T\} \models_{\mathcal{HA}} \alpha \approx \top.$$

($\Rightarrow$) A straightforward induction on the length of a derivation of α from T in IL using properties of Heyting algebras.

($\Leftarrow$) Suppose that $T \nvdash_{\text{IL}} \alpha$ and consider the homomorphism e from the formula algebra to the Heyting algebra $\mathbf{A}_T$ given by $e(\gamma) = [\gamma]_T$. Since

$$T \vdash_{\text{IL}} \gamma \iff [\gamma]_T = [\top]_T,$$

Proof Sketch

For (i), we need to prove

$$T \vdash_{\mathsf{IL}} \alpha \quad \Longleftrightarrow \quad \{\gamma \approx \top \mid \gamma \in T\} \models_{\mathcal{HA}} \alpha \approx \top.$$

($\Rightarrow$) A straightforward induction on the length of a derivation of α from T in IL using properties of Heyting algebras.

($\Leftarrow$) Suppose that $T \not\vdash_{\mathsf{IL}} \alpha$ and consider the homomorphism e from the formula algebra to the Heyting algebra $\mathbf{A}_T$ given by $e(\gamma) = [\gamma]_T$. Since

$$T \vdash_{\mathsf{IL}} \gamma \quad \Longleftrightarrow \quad [\gamma]_T = [\top]_T,$$

we have $[\gamma]_T = [\top]_T$ for all $\gamma \in T$ and $[\alpha]_T \neq [\top]_T$,

Proof Sketch

For (i), we need to prove

$$T \vdash_{\mathsf{IL}} \alpha \iff \{\gamma \approx \top \mid \gamma \in T\} \models_{\mathcal{HA}} \alpha \approx \top.$$

($\Rightarrow$) A straightforward induction on the length of a derivation of α from T in IL using properties of Heyting algebras.

($\Leftarrow$) Suppose that $T \nvdash_{\mathsf{IL}} \alpha$ and consider the homomorphism e from the formula algebra to the Heyting algebra $\mathbf{A}_T$ given by $e(\gamma) = [\gamma]_T$. Since

$$T \vdash_{\mathsf{IL}} \gamma \iff [\gamma]_T = [\top]_T,$$

we have $[\gamma]_T = [\top]_T$ for all $\gamma \in T$ and $[\alpha]_T \neq [\top]_T$, so

$$\{\gamma \approx \top \mid \gamma \in T\} \not\models_{\mathcal{HA}} \alpha \approx \top.$$

Proof Sketch

For (i), we need to prove

$$T \vdash_{\mathrm{IL}} \alpha \iff \{\gamma \approx \top \mid \gamma \in T\} \models_{\mathcal{HA}} \alpha \approx \top.$$

($\Rightarrow$) A straightforward induction on the length of a derivation of α from T in IL using properties of Heyting algebras.

($\Leftarrow$) Suppose that $T \nvdash_{\mathrm{IL}} \alpha$ and consider the homomorphism e from the formula algebra to the Heyting algebra $\mathbf{A}_T$ given by $e(\gamma) = [\gamma]_T$. Since

$$T \vdash_{\mathrm{IL}} \gamma \iff [\gamma]_T = [\top]_T,$$

we have $[\gamma]_T = [\top]_T$ for all $\gamma \in T$ and $[\alpha]_T \neq [\top]_T$, so

$$\{\gamma \approx \top \mid \gamma \in T\} \not\models_{\mathcal{HA}} \alpha \approx \top.$$

(iii) is easy to check, and (ii) follows directly from (i) and (iii). □

Sequent Calculi

- The first sequent calculi for (first-order) classical and intuitionistic logic were introduced by Gentzen in the 1930s.

Sequent Calculi

- The first sequent calculi for (first-order) classical and intuitionistic logic were introduced by Gentzen in the 1930s.
- Proof-search-oriented variants of Gentzen's sequent calculus for intuitionistic logic were later developed by Ketonen, Kleene, Ono, Vorob'ev, Dragalin, Troelstra, Dyckhoff, Hudelmeier...

Sequent Calculi

- The first sequent calculi for (first-order) classical and intuitionistic logic were introduced by Gentzen in the 1930s.
- Proof-search-oriented variants of Gentzen's sequent calculus for intuitionistic logic were later developed by Ketonen, Kleene, Ono, Vorob'ev, Dragalin, Troelstra, Dyckhoff, Hudelmeier...
- Sequent calculi (and many variants thereof) have been introduced for many other non-classical logics and classes of algebraic structures.

Sequents

A **sequent** is an ordered pair consisting of a finite multiset of formulas Γ and a formula α, written $\Gamma \Rightarrow \alpha$.

Sequents

A **sequent** is an ordered pair consisting of a finite multiset of formulas Γ and a formula α, written $\Gamma \Rightarrow \alpha$.

We typically write Γ, Π for the multiset sum of Γ and Π, and omit brackets.

Sequents

A **sequent** is an ordered pair consisting of a finite multiset of formulas Γ and a formula α, written $\Gamma \Rightarrow \alpha$.

We typically write Γ, Π for the multiset sum of Γ and Π, and omit brackets.

A **sequent calculus** GL consists of a set of rules with instances

$$\frac{S_1 \quad \dots \quad S_n}{S_0} \qquad \text{where } S_0, S_1, \dots, S_n \text{ are sequents.}$$

Sequents

A **sequent** is an ordered pair consisting of a finite multiset of formulas Γ and a formula α, written $\Gamma \Rightarrow \alpha$.

We typically write Γ, Π for the multiset sum of Γ and Π, and omit brackets.

A **sequent calculus** GL consists of a set of rules with instances

$$\frac{S_1 \quad \dots \quad S_n}{S_0} \quad \text{where } S_0, S_1, \dots, S_n \text{ are sequents.}$$

A GL-**derivation** of a sequent S is a finite tree of sequents with root S built using the rules of GL.

Sequents

A **sequent** is an ordered pair consisting of a finite multiset of formulas Γ and a formula α, written $\Gamma \Rightarrow \alpha$.

We typically write Γ, Π for the multiset sum of Γ and Π, and omit brackets.

A **sequent calculus** GL consists of a set of rules with instances

$$\frac{S_1 \quad \ldots \quad S_n}{S_0} \quad \text{where } S_0, S_1, \ldots, S_n \text{ are sequents.}$$

A GL-**derivation** of a sequent S is a finite tree of sequents with root S built using the rules of GL. If there exists a GL-derivation of a sequent S of height at most n, we write $\vdash^n_{\mathrm{GL}} S$ or just $\vdash_{\mathrm{GL}} S$.

A Sequent Calculus GIL for Intuitionistic Logic

Identity Axioms

$$\overline{\Gamma, x \Rightarrow x}\ \text{(id)}$$

A Sequent Calculus GIL for Intuitionistic Logic

Identity Axioms

$$\frac{}{\Gamma, x \Rightarrow x}\ (\text{id})$$

Left Operation Rules

$$\frac{}{\Gamma, \bot \Rightarrow \delta}\ (\bot\Rightarrow)$$

Right Operation Rules

$$\frac{}{\Gamma \Rightarrow \top}\ (\Rightarrow\top)$$

A Sequent Calculus GIL for Intuitionistic Logic

Identity Axioms

$$\frac{}{\Gamma, x \Rightarrow x}\ (\text{id})$$

Left Operation Rules

$$\frac{}{\Gamma, \bot \Rightarrow \delta}\ (\bot\Rightarrow)$$

$$\frac{\Gamma, \alpha, \beta \Rightarrow \delta}{\Gamma, \alpha \wedge \beta \Rightarrow \delta}\ (\wedge\Rightarrow)$$

Right Operation Rules

$$\frac{}{\Gamma \Rightarrow \top}\ (\Rightarrow\top)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\Rightarrow\wedge)$$

A Sequent Calculus GIL for Intuitionistic Logic

Identity Axioms

$$\frac{}{\Gamma, x \Rightarrow x}\ (\text{id})$$

Left Operation Rules

$$\frac{}{\Gamma, \bot \Rightarrow \delta}\ (\bot\Rightarrow)$$

$$\frac{\Gamma, \alpha, \beta \Rightarrow \delta}{\Gamma, \alpha \wedge \beta \Rightarrow \delta}\ (\wedge\Rightarrow)$$

$$\frac{\Gamma, \alpha \Rightarrow \delta \quad \Gamma, \beta \Rightarrow \delta}{\Gamma, \alpha \vee \beta \Rightarrow \delta}\ (\vee\Rightarrow)$$

Right Operation Rules

$$\frac{}{\Gamma \Rightarrow \top}\ (\Rightarrow\top)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\Rightarrow\wedge)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow\vee)_l \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow\vee)_r$$

A Sequent Calculus GIL for Intuitionistic Logic

Identity Axioms

$$\frac{}{\Gamma, x \Rightarrow x}\ (\text{id})$$

Left Operation Rules

$$\frac{}{\Gamma, \bot \Rightarrow \delta}\ (\bot\Rightarrow)$$

$$\frac{\Gamma, \alpha, \beta \Rightarrow \delta}{\Gamma, \alpha \wedge \beta \Rightarrow \delta}\ (\wedge\Rightarrow)$$

$$\frac{\Gamma, \alpha \Rightarrow \delta \quad \Gamma, \beta \Rightarrow \delta}{\Gamma, \alpha \vee \beta \Rightarrow \delta}\ (\vee\Rightarrow)$$

$$\frac{\Gamma, \alpha \to \beta \Rightarrow \alpha \quad \Gamma, \beta \Rightarrow \delta}{\Gamma, \alpha \to \beta \Rightarrow \delta}\ (\to\Rightarrow)$$

Right Operation Rules

$$\frac{}{\Gamma \Rightarrow \top}\ (\Rightarrow\top)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\Rightarrow\wedge)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow\vee)_l \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow\vee)_r$$

$$\frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\ (\Rightarrow\to)$$

A Sequent Calculus GIL for Intuitionistic Logic

Identity Axioms

$$\frac{}{\Gamma, x \Rightarrow x} \text{ (id)}$$

Cut Rule

$$\frac{\Gamma \Rightarrow \alpha \quad \Pi, \alpha \Rightarrow \delta}{\Gamma, \Pi \Rightarrow \delta} \text{ (cut)}$$

Left Operation Rules

$$\frac{}{\Gamma, \bot \Rightarrow \delta} \; (\bot\Rightarrow)$$

$$\frac{\Gamma, \alpha, \beta \Rightarrow \delta}{\Gamma, \alpha \wedge \beta \Rightarrow \delta} \; (\wedge\Rightarrow)$$

$$\frac{\Gamma, \alpha \Rightarrow \delta \quad \Gamma, \beta \Rightarrow \delta}{\Gamma, \alpha \vee \beta \Rightarrow \delta} \; (\vee\Rightarrow)$$

$$\frac{\Gamma, \alpha \to \beta \Rightarrow \alpha \quad \Gamma, \beta \Rightarrow \delta}{\Gamma, \alpha \to \beta \Rightarrow \delta} \; (\to\Rightarrow)$$

Right Operation Rules

$$\frac{}{\Gamma \Rightarrow \top} \; (\Rightarrow\top)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta} \; (\Rightarrow\wedge)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta} \; (\Rightarrow\vee)_l \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta} \; (\Rightarrow\vee)_r$$

$$\frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta} \; (\Rightarrow\to)$$

An Example Derivation

$$\dfrac{}{\Rightarrow ((x \to y) \land (x \lor z)) \to (y \lor z)}\ (\Rightarrow\to)$$

An Example Derivation

$$\dfrac{\dfrac{}{(x \to y) \land (x \lor z) \Rightarrow y \lor z}\ (\land\Rightarrow)}{\Rightarrow ((x \to y) \land (x \lor z)) \to (y \lor z)}\ (\Rightarrow\to)$$

An Example Derivation

$$\dfrac{\dfrac{\dfrac{\overline{}\;(\vee\Rightarrow)}{x \to y, x \vee z \Rightarrow y \vee z}}{(x \to y) \wedge (x \vee z) \Rightarrow y \vee z}\;(\wedge\Rightarrow)}{\Rightarrow ((x \to y) \wedge (x \vee z)) \to (y \vee z)}\;(\Rightarrow\to)$$

An Example Derivation

$$\dfrac{\dfrac{\overline{x \to y, x \Rightarrow y \vee z}\ (\to\Rightarrow)}{x \to y, x \vee z \Rightarrow y \vee z}\ (\vee\Rightarrow)}{\dfrac{(x \to y) \wedge (x \vee z) \Rightarrow y \vee z}{\Rightarrow ((x \to y) \wedge (x \vee z)) \to (y \vee z)}\ (\Rightarrow\to)}\ (\wedge\Rightarrow)$$

An Example Derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x \to y, x \Rightarrow x}\,(\text{id})}{x \to y, x \Rightarrow y \vee z}\,(\to\Rightarrow)}{x \to y, x \vee z \Rightarrow y \vee z}\,(\vee\Rightarrow)}{(x \to y) \wedge (x \vee z) \Rightarrow y \vee z}\,(\wedge\Rightarrow)}{\Rightarrow ((x \to y) \wedge (x \vee z)) \to (y \vee z)}\,(\Rightarrow\to)$$

An Example Derivation

$$\dfrac{\dfrac{\dfrac{}{x \to y, x \Rightarrow x}\ (\mathrm{id}) \quad \dfrac{}{y, x \Rightarrow y \vee z}\ (\Rightarrow\vee)_l}{x \to y, x \Rightarrow y \vee z}\ (\to\Rightarrow)}{\dfrac{\dfrac{x \to y, x \vee z \Rightarrow y \vee z}{(x \to y) \wedge (x \vee z) \Rightarrow y \vee z}\ (\wedge\Rightarrow)}{\Rightarrow ((x \to y) \wedge (x \vee z)) \to (y \vee z)}\ (\Rightarrow\to)}\ (\vee\Rightarrow)$$

$$
\dfrac{
  \dfrac{
    \dfrac{}{x \to y, x \Rightarrow x}\,(\mathrm{id})
    \qquad
    \dfrac{
      \dfrac{}{y, x \Rightarrow y}\,(\mathrm{id})
    }{y, x \Rightarrow y \vee z}\,(\Rightarrow\vee)_l
  }{x \to y, x \Rightarrow y \vee z}\,(\to\Rightarrow)
}{
  \dfrac{
    \dfrac{x \to y, x \vee z \Rightarrow y \vee z}{(x \to y) \wedge (x \vee z) \Rightarrow y \vee z}\,(\wedge\Rightarrow)
  }{\Rightarrow ((x \to y) \wedge (x \vee z)) \to (y \vee z)}\,(\Rightarrow\to)
}\,(\vee\Rightarrow)
$$

An Example Derivation

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{x \to y, x \Rightarrow x}\ (\text{id})
      \qquad
      \dfrac{
        \dfrac{}{y, x \Rightarrow y}\ (\text{id})
      }{y, x \Rightarrow y \vee z}\ (\Rightarrow\vee)_l
    }{x \to y, x \Rightarrow y \vee z}\ (\to\Rightarrow)
    \qquad
    \dfrac{}{x \to y, z \Rightarrow y \vee z}\ (\Rightarrow\vee)_r
  }{
    \dfrac{x \to y, x \vee z \Rightarrow y \vee z}{(x \to y) \wedge (x \vee z) \Rightarrow y \vee z}\ (\wedge\Rightarrow)
  }\ (\vee\Rightarrow)
}{\Rightarrow ((x \to y) \wedge (x \vee z)) \to (y \vee z)}\ (\Rightarrow\to)
$$

An Example Derivation

$$
\dfrac{
  \dfrac{
    \dfrac{}{x \to y, x \Rightarrow x}\,(\mathrm{id})
    \qquad
    \dfrac{\dfrac{}{y, x \Rightarrow y}\,(\mathrm{id})}{y, x \Rightarrow y \vee z}\,(\Rightarrow\vee)_l
  }{x \to y, x \Rightarrow y \vee z}\,(\to\Rightarrow)
  \qquad
  \dfrac{\dfrac{}{x \to y, z \Rightarrow z}\,(\mathrm{id})}{x \to y, z \Rightarrow y \vee z}\,(\Rightarrow\vee)_r
}{
  \dfrac{
    \dfrac{x \to y, x \vee z \Rightarrow y \vee z}{(x \to y) \wedge (x \vee z) \Rightarrow y \vee z}\,(\wedge\Rightarrow)
  }{\Rightarrow ((x \to y) \wedge (x \vee z)) \to (y \vee z)}\,(\Rightarrow\to)
}\,(\vee\Rightarrow)
$$

Identity Lemma

Let GIL$^\circ$ be the sequent calculus GIL *without* the cut rule.

Lemma

For any finite multiset of formulas Γ *and any formula* α*,*

$$\vdash_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \alpha.$$

Identity Lemma

Let GIL° be the sequent calculus GIL *without* the cut rule.

Lemma

For any finite multiset of formulas Γ *and any formula* α,

$$\vdash_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \alpha.$$

Proof.

By induction on the **size** (number of occurrences of connectives) $|\alpha|$ of α.

Identity Lemma

Let GIL° be the sequent calculus GIL *without* the cut rule.

Lemma

For any finite multiset of formulas Γ *and any formula* α,

$$\vdash_{\text{GIL}^\circ} \Gamma, \alpha \Rightarrow \alpha.$$

Proof.

By induction on the **size** (number of occurrences of connectives) $|\alpha|$ of α. The base case $\Gamma, x \Rightarrow x$ is an instance of (id).

Identity Lemma

Let GIL$^\circ$ be the sequent calculus GIL *without* the cut rule.

Lemma

For any finite multiset of formulas Γ *and any formula* α*,*

$$\vdash_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \alpha.$$

Proof.

By induction on the **size** (number of occurrences of connectives) $|\alpha|$ of α. The base case $\Gamma, x \Rightarrow x$ is an instance of (id). For the inductive step, we consider the principal connective of α;

Identity Lemma

Let GIL$^\circ$ be the sequent calculus GIL *without* the cut rule.

Lemma

For any finite multiset of formulas Γ *and any formula* α*,*

$$\vdash_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \alpha.$$

Proof.

By induction on the **size** (number of occurrences of connectives) $|\alpha|$ of α. The base case $\Gamma, x \Rightarrow x$ is an instance of (id). For the inductive step, we consider the principal connective of α; e.g., for $\alpha = \alpha_1 \wedge \alpha_2$, we obtain

Identity Lemma

Let GIL$^\circ$ be the sequent calculus GIL *without* the cut rule.

Lemma

For any finite multiset of formulas Γ *and any formula* α,

$$\vdash_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \alpha.$$

Proof.

By induction on the **size** (number of occurrences of connectives) $|\alpha|$ of α. The base case $\Gamma, x \Rightarrow x$ is an instance of (id). For the inductive step, we consider the principal connective of α; e.g., for $\alpha = \alpha_1 \wedge \alpha_2$, we obtain

$$\frac{}{\Gamma, \alpha_1 \wedge \alpha_2 \Rightarrow \alpha_1 \wedge \alpha_2}\ (\wedge\Rightarrow)$$

Identity Lemma

Let GIL$^\circ$ be the sequent calculus GIL *without* the cut rule.

Lemma

For any finite multiset of formulas Γ *and any formula* α,

$$\vdash_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \alpha.$$

Proof.

By induction on the **size** (number of occurrences of connectives) $|\alpha|$ of α. The base case $\Gamma, x \Rightarrow x$ is an instance of (id). For the inductive step, we consider the principal connective of α; e.g., for $\alpha = \alpha_1 \wedge \alpha_2$, we obtain

$$\cfrac{\cfrac{}{\Gamma, \alpha_1, \alpha_2 \Rightarrow \alpha_1 \wedge \alpha_2}\;(\Rightarrow\wedge)}{\Gamma, \alpha_1 \wedge \alpha_2 \Rightarrow \alpha_1 \wedge \alpha_2}\;(\wedge\Rightarrow)$$

Identity Lemma

Let GIL$^\circ$ be the sequent calculus GIL *without* the cut rule.

Lemma

For any finite multiset of formulas Γ *and any formula* α,

$$\vdash_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \alpha.$$

Proof.

By induction on the **size** (number of occurrences of connectives) $|\alpha|$ of α. The base case $\Gamma, x \Rightarrow x$ is an instance of (id). For the inductive step, we consider the principal connective of α; e.g., for $\alpha = \alpha_1 \land \alpha_2$, we obtain

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, \alpha_1, \alpha_2 \Rightarrow \alpha_1} \quad \dfrac{\vdots}{\Gamma, \alpha_1, \alpha_2 \Rightarrow \alpha_2}}{\Gamma, \alpha_1, \alpha_2 \Rightarrow \alpha_1 \land \alpha_2}\ (\Rightarrow\land)}{\Gamma, \alpha_1 \land \alpha_2 \Rightarrow \alpha_1 \land \alpha_2}\ (\land\Rightarrow)$$

□

Soundness and Completeness

Theorem

$$\vdash_{\mathsf{GIL}} \alpha_1, \ldots, \alpha_n \Rightarrow \beta \iff \vdash_{\mathsf{IL}} (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta.$$

Soundness and Completeness

Theorem

$$\vdash_{\mathsf{GIL}} \alpha_1, \ldots, \alpha_n \Rightarrow \beta \iff \vdash_{\mathsf{IL}} (\alpha_1 \land \ldots \land \alpha_n) \to \beta.$$

Proof.

($\Rightarrow$) By induction on the height of a derivation of $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ in GIL.

Soundness and Completeness

Theorem

$$\vdash_{\mathsf{GIL}} \alpha_1, \ldots, \alpha_n \Rightarrow \beta \iff \vdash_{\mathsf{IL}} (\alpha_1 \land \ldots \land \alpha_n) \to \beta.$$

Proof.

($\Rightarrow$) By induction on the height of a derivation of $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ in GIL. It suffices to check that the rules of GIL preserve derivability in IL;

Soundness and Completeness

Theorem

$$\vdash_{\mathsf{GIL}} \alpha_1, \ldots, \alpha_n \Rightarrow \beta \iff \vdash_{\mathsf{IL}} (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta.$$

Proof.

($\Rightarrow$) By induction on the height of a derivation of $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ in GIL. It suffices to check that the rules of GIL preserve derivability in IL; e.g.,

$$\vdash_{\mathsf{IL}} (\gamma \wedge \alpha) \to \delta \text{ and } \vdash_{\mathsf{IL}} (\gamma \wedge \beta) \to \delta \implies \vdash_{\mathsf{IL}} (\gamma \wedge (\alpha \vee \beta)) \to \delta.$$

Soundness and Completeness

Theorem

$$\vdash_{\mathsf{GIL}} \alpha_1, \ldots, \alpha_n \Rightarrow \beta \iff \vdash_{\mathsf{IL}} (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta.$$

Proof.

($\Rightarrow$) By induction on the height of a derivation of $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ in GIL. It suffices to check that the rules of GIL preserve derivability in IL; e.g.,

$$\vdash_{\mathsf{IL}} (\gamma \wedge \alpha) \to \delta \text{ and } \vdash_{\mathsf{IL}} (\gamma \wedge \beta) \to \delta \implies \vdash_{\mathsf{IL}} (\gamma \wedge (\alpha \vee \beta)) \to \delta.$$

($\Leftarrow$) We prove that $\vdash_{\mathsf{IL}} \alpha$ implies $\vdash_{\mathsf{GIL}} \Rightarrow \alpha$ by induction on the length of a derivation of α in IL,

Soundness and Completeness

Theorem

$$\vdash_{\mathsf{GIL}} \alpha_1, \ldots, \alpha_n \Rightarrow \beta \iff \vdash_{\mathsf{IL}} (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta.$$

Proof.

($\Rightarrow$) By induction on the height of a derivation of $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ in GIL. It suffices to check that the rules of GIL preserve derivability in IL; e.g.,

$$\vdash_{\mathsf{IL}} (\gamma \wedge \alpha) \to \delta \text{ and } \vdash_{\mathsf{IL}} (\gamma \wedge \beta) \to \delta \quad \Longrightarrow \quad \vdash_{\mathsf{IL}} (\gamma \wedge (\alpha \vee \beta)) \to \delta.$$

($\Leftarrow$) We prove that $\vdash_{\mathsf{IL}} \alpha$ implies $\vdash_{\mathsf{GIL}} \Rightarrow \alpha$ by induction on the length of a derivation of α in IL, showing that the axioms are derivable in GIL and that modus ponens preserves derivability,

Soundness and Completeness

Theorem

$$\vdash_{\mathsf{GIL}} \alpha_1, \ldots, \alpha_n \Rightarrow \beta \iff \vdash_{\mathsf{IL}} (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta.$$

Proof.

($\Rightarrow$) By induction on the height of a derivation of $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ in GIL. It suffices to check that the rules of GIL preserve derivability in IL; e.g.,

$$\vdash_{\mathsf{IL}} (\gamma \wedge \alpha) \to \delta \text{ and } \vdash_{\mathsf{IL}} (\gamma \wedge \beta) \to \delta \quad \Longrightarrow \quad \vdash_{\mathsf{IL}} (\gamma \wedge (\alpha \vee \beta)) \to \delta.$$

($\Leftarrow$) We prove that $\vdash_{\mathsf{IL}} \alpha$ implies $\vdash_{\mathsf{GIL}} \Rightarrow \alpha$ by induction on the length of a derivation of α in IL, showing that the axioms are derivable in GIL and that modus ponens preserves derivability, i.e., cutting twice with $\beta, \beta \to \gamma \Rightarrow \gamma$,

$$\vdash_{\mathsf{GIL}} \Rightarrow \beta \text{ and } \vdash_{\mathsf{GIL}} \Rightarrow \beta \to \gamma \quad \Longrightarrow \quad \vdash_{\mathsf{GIL}} \Rightarrow \gamma.$$

Soundness and Completeness

Theorem

$$\vdash_{\mathsf{GIL}} \alpha_1, \ldots, \alpha_n \Rightarrow \beta \iff \vdash_{\mathsf{IL}} (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta.$$

Proof.

($\Rightarrow$) By induction on the height of a derivation of $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ in GIL. It suffices to check that the rules of GIL preserve derivability in IL; e.g.,

$$\vdash_{\mathsf{IL}} (\gamma \wedge \alpha) \to \delta \text{ and } \vdash_{\mathsf{IL}} (\gamma \wedge \beta) \to \delta \implies \vdash_{\mathsf{IL}} (\gamma \wedge (\alpha \vee \beta)) \to \delta.$$

($\Leftarrow$) We prove that $\vdash_{\mathsf{IL}} \alpha$ implies $\vdash_{\mathsf{GIL}} \Rightarrow \alpha$ by induction on the length of a derivation of α in IL, showing that the axioms are derivable in GIL and that modus ponens preserves derivability, i.e., cutting twice with $\beta, \beta \to \gamma \Rightarrow \gamma$,

$$\vdash_{\mathsf{GIL}} \Rightarrow \beta \text{ and } \vdash_{\mathsf{GIL}} \Rightarrow \beta \to \gamma \implies \vdash_{\mathsf{GIL}} \Rightarrow \gamma.$$

Hence if $\vdash_{\mathsf{IL}} (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta$, then $\vdash_{\mathsf{GIL}} \Rightarrow (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta$,

Soundness and Completeness

Theorem

$$\vdash_{\mathsf{GIL}} \alpha_1, \ldots, \alpha_n \Rightarrow \beta \iff \vdash_{\mathsf{IL}} (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta.$$

Proof.

($\Rightarrow$) By induction on the height of a derivation of $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ in GIL. It suffices to check that the rules of GIL preserve derivability in IL; e.g.,

$$\vdash_{\mathsf{IL}} (\gamma \wedge \alpha) \to \delta \text{ and } \vdash_{\mathsf{IL}} (\gamma \wedge \beta) \to \delta \implies \vdash_{\mathsf{IL}} (\gamma \wedge (\alpha \vee \beta)) \to \delta.$$

($\Leftarrow$) We prove that $\vdash_{\mathsf{IL}} \alpha$ implies $\vdash_{\mathsf{GIL}} \Rightarrow \alpha$ by induction on the length of a derivation of α in IL, showing that the axioms are derivable in GIL and that modus ponens preserves derivability, i.e., cutting twice with $\beta, \beta \to \gamma \Rightarrow \gamma$,

$$\vdash_{\mathsf{GIL}} \Rightarrow \beta \text{ and } \vdash_{\mathsf{GIL}} \Rightarrow \beta \to \gamma \implies \vdash_{\mathsf{GIL}} \Rightarrow \gamma.$$

Hence if $\vdash_{\mathsf{IL}} (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta$, then $\vdash_{\mathsf{GIL}} \Rightarrow (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta$, and the result follows by cutting with $\alpha_1, \ldots, \alpha_n, (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta \Rightarrow \beta$. □

Weakening, Invertibility, Contraction

Lemma

(a) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \delta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \delta$.

Weakening, Invertibility, Contraction

Lemma

(a) $\vdash^{n}_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \delta \implies \vdash^{n}_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \delta.$

(b) $\vdash^{n}_{\mathsf{GIL}^\circ} \Gamma, \alpha \wedge \beta \Rightarrow \delta \implies \vdash^{n}_{\mathsf{GIL}^\circ} \Gamma, \alpha, \beta \Rightarrow \delta.$

Weakening, Invertibility, Contraction

Lemma

(a) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \delta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \delta.$

(b) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \wedge \beta \Rightarrow \delta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha, \beta \Rightarrow \delta.$

(c) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \alpha \wedge \beta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \alpha$ *and* $\vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \beta.$

Weakening, Invertibility, Contraction

Lemma

(a) $\vdash^{n}_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \delta \implies \vdash^{n}_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \delta.$

(b) $\vdash^{n}_{\mathsf{GIL}^\circ} \Gamma, \alpha \wedge \beta \Rightarrow \delta \implies \vdash^{n}_{\mathsf{GIL}^\circ} \Gamma, \alpha, \beta \Rightarrow \delta.$

(c) $\vdash^{n}_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \alpha \wedge \beta \implies \vdash^{n}_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \alpha$ *and* $\vdash^{n}_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \beta.$

(d) $\vdash^{n}_{\mathsf{GIL}^\circ} \Gamma, \alpha \vee \beta \Rightarrow \delta \implies \vdash^{n}_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \delta$ *and* $\vdash^{n}_{\mathsf{GIL}^\circ} \Gamma, \beta \Rightarrow \delta.$

Weakening, Invertibility, Contraction

Lemma

(a) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \delta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \delta$.

(b) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \wedge \beta \Rightarrow \delta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha, \beta \Rightarrow \delta$.

(c) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \alpha \wedge \beta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \alpha$ *and* $\vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \beta$.

(d) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \vee \beta \Rightarrow \delta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \delta$ *and* $\vdash^n_{\mathsf{GIL}^\circ} \Gamma, \beta \Rightarrow \delta$.

(e) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \alpha \to \beta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \beta$.

Weakening, Invertibility, Contraction

Lemma

(a) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \delta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \delta.$

(b) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \wedge \beta \Rightarrow \delta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha, \beta \Rightarrow \delta.$

(c) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \alpha \wedge \beta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \alpha$ *and* $\vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \beta.$

(d) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \vee \beta \Rightarrow \delta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \delta$ *and* $\vdash^n_{\mathsf{GIL}^\circ} \Gamma, \beta \Rightarrow \delta.$

(e) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \alpha \to \beta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \beta.$

(f) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \to \beta \Rightarrow \delta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \beta \Rightarrow \delta.$

Weakening, Invertibility, Contraction

Lemma

(a) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \delta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \delta$.

(b) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \wedge \beta \Rightarrow \delta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha, \beta \Rightarrow \delta$.

(c) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \alpha \wedge \beta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \alpha$ *and* $\vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \beta$.

(d) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \vee \beta \Rightarrow \delta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \delta$ *and* $\vdash^n_{\mathsf{GIL}^\circ} \Gamma, \beta \Rightarrow \delta$.

(e) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \alpha \to \beta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \beta$.

(f) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \to \beta \Rightarrow \delta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \beta \Rightarrow \delta$.

(g) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha, \alpha \Rightarrow \delta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \delta$.

Weakening, Invertibility, Contraction

Lemma

(a) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \delta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \delta$.

(b) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \land \beta \Rightarrow \delta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha, \beta \Rightarrow \delta$.

(c) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \alpha \land \beta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \alpha$ *and* $\vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \beta$.

(d) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \lor \beta \Rightarrow \delta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \delta$ *and* $\vdash^n_{\mathsf{GIL}^\circ} \Gamma, \beta \Rightarrow \delta$.

(e) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma \Rightarrow \alpha \to \beta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \beta$.

(f) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \to \beta \Rightarrow \delta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \beta \Rightarrow \delta$.

(g) $\vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha, \alpha \Rightarrow \delta \implies \vdash^n_{\mathsf{GIL}^\circ} \Gamma, \alpha \Rightarrow \delta$.

Proof.

Each claim can be proved by a simple (if rather tedious) induction on *n*. □

Cut Elimination

Theorem

Any GIL*-derivable sequent is* GIL°*-derivable.*

Cut Elimination

Theorem

Any GIL*-derivable sequent is* GIL°*-derivable.*

Proof idea. We push uppermost cuts upwards in GIL-derivations until they reach axioms and disappear, e.g.. . .

Cut Elimination

Theorem

Any GIL*-derivable sequent is* GIL$^\circ$*-derivable.*

Proof idea. We push uppermost cuts upwards in GIL-derivations until they reach axioms and disappear, e.g....

$$\dfrac{\dfrac{\vdots}{\Sigma \Rightarrow \delta} \quad \dfrac{}{\Pi, \delta \Rightarrow \delta}\,(\text{id})}{\Sigma, \Pi \Rightarrow \delta}\,(\text{cut})$$

Cut Elimination

Theorem

Any GIL*-derivable sequent is* GIL$^\circ$*-derivable.*

Proof idea. We push uppermost cuts upwards in GIL-derivations until they reach axioms and disappear, e.g. . . .

$$\dfrac{\dfrac{\vdots}{\Sigma \Rightarrow \delta} \quad \dfrac{}{\Pi, \delta \Rightarrow \delta}\,(\text{id})}{\Sigma, \Pi \Rightarrow \delta}\,(\text{cut}) \quad \Longrightarrow \quad \dfrac{\vdots}{\Sigma, \Pi \Rightarrow \delta}$$

Cut Elimination

Theorem

Any GIL*-derivable sequent is* GIL$^\circ$*-derivable.*

Proof idea. We push uppermost cuts upwards in GIL-derivations until they reach axioms and disappear, e.g. . . .

$$\dfrac{\dfrac{\vdots}{\Sigma \Rightarrow \delta} \quad \dfrac{}{\Pi, \delta \Rightarrow \delta}\,(\text{id})}{\Sigma, \Pi \Rightarrow \delta}\,(\text{cut}) \quad \Longrightarrow \quad \dfrac{\vdots}{\Sigma, \Pi \Rightarrow \delta}$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\Sigma \Rightarrow \alpha}}{\Sigma \Rightarrow \alpha \vee \beta}\,(\Rightarrow\vee)_l \quad \dfrac{\dfrac{\vdots}{\Pi, \alpha \Rightarrow \delta} \quad \dfrac{\vdots}{\Pi, \beta \Rightarrow \delta}}{\Pi, \alpha \vee \beta \Rightarrow \delta}\,(\vee\Rightarrow)}{\Sigma, \Pi \Rightarrow \delta}\,(\text{cut})$$

Cut Elimination

Theorem

Any GIL*-derivable sequent is* GIL$^\circ$*-derivable.*

Proof idea. We push uppermost cuts upwards in GIL-derivations until they reach axioms and disappear, e.g. . . .

$$\dfrac{\dfrac{\vdots}{\Sigma \Rightarrow \delta} \quad \dfrac{}{\Pi, \delta \Rightarrow \delta}\,(\mathrm{id})}{\Sigma, \Pi \Rightarrow \delta}\,(\mathrm{cut}) \quad \Longrightarrow \quad \dfrac{\vdots}{\Sigma, \Pi \Rightarrow \delta}$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\Sigma \Rightarrow \alpha}}{\Sigma \Rightarrow \alpha \vee \beta}\,(\Rightarrow\vee)_l \quad \dfrac{\dfrac{\vdots}{\Pi, \alpha \Rightarrow \delta} \quad \dfrac{\vdots}{\Pi, \beta \Rightarrow \delta}}{\Pi, \alpha \vee \beta \Rightarrow \delta}\,(\vee\Rightarrow)}{\Sigma, \Pi \Rightarrow \delta}\,(\mathrm{cut}) \quad \Longrightarrow \quad \dfrac{\dfrac{\vdots}{\Sigma \Rightarrow \alpha} \quad \dfrac{\vdots}{\Pi, \alpha \Rightarrow \delta}}{\Sigma, \Pi \Rightarrow \delta}\,(\mathrm{cut})$$

More formally...

We prove (constructively) that

$$\vdash^{m}_{\mathsf{GIL}^\circ} \Sigma \Rightarrow \alpha \ \text{ and } \ \vdash^{n}_{\mathsf{GIL}^\circ} \Pi, \alpha \Rightarrow \delta \implies \vdash_{\mathsf{GIL}^\circ} \Sigma, \Pi \Rightarrow \delta,$$

More formally...

We prove (constructively) that

$$\vdash^m_{\mathsf{GIL}^\circ} \Sigma \Rightarrow \alpha \;\text{ and }\; \vdash^n_{\mathsf{GIL}^\circ} \Pi, \alpha \Rightarrow \delta \;\Longrightarrow\; \vdash_{\mathsf{GIL}^\circ} \Sigma, \Pi \Rightarrow \delta,$$

by induction on the lexicographically ordered pair $\langle |\alpha|, m + n \rangle$.

More formally...

We prove (constructively) that

$$\vdash^{m}_{\mathsf{GIL}^\circ} \Sigma \Rightarrow \alpha \quad \text{and} \quad \vdash^{n}_{\mathsf{GIL}^\circ} \Pi, \alpha \Rightarrow \delta \quad \Longrightarrow \quad \vdash_{\mathsf{GIL}^\circ} \Sigma, \Pi \Rightarrow \delta,$$

by induction on the lexicographically ordered pair $\langle |\alpha|, m + n \rangle$.

E.g., suppose that α is $\beta \to \gamma$ and the derivations of the premises end with

$$\dfrac{\dfrac{\vdots}{\Sigma, \beta \Rightarrow \gamma}}{\Sigma \Rightarrow \beta \to \gamma}\ (\to\Rightarrow) \qquad \text{and} \qquad \dfrac{\dfrac{\vdots}{\Pi, \beta \to \gamma \Rightarrow \beta} \quad \dfrac{\vdots}{\Pi, \gamma \Rightarrow \delta}}{\Pi, \beta \to \gamma \Rightarrow \delta}\ (\to\Rightarrow)$$

More formally...

We prove (constructively) that

$$\vdash^m_{\mathsf{GIL}^\circ} \Sigma \Rightarrow \alpha \quad \text{and} \quad \vdash^n_{\mathsf{GIL}^\circ} \Pi, \alpha \Rightarrow \delta \quad \Longrightarrow \quad \vdash_{\mathsf{GIL}^\circ} \Sigma, \Pi \Rightarrow \delta,$$

by induction on the lexicographically ordered pair $\langle |\alpha|, m+n \rangle$.

E.g., suppose that α is $\beta \to \gamma$ and the derivations of the premises end with

$$\dfrac{\dfrac{\vdots}{\Sigma, \beta \Rightarrow \gamma}}{\Sigma \Rightarrow \beta \to \gamma}\ (\to\Rightarrow) \qquad \text{and} \qquad \dfrac{\dfrac{\vdots}{\Pi, \beta \to \gamma \Rightarrow \beta} \quad \dfrac{\vdots}{\Pi, \gamma \Rightarrow \delta}}{\Pi, \beta \to \gamma \Rightarrow \delta}\ (\to\Rightarrow)$$

We apply the induction hypothesis three times:

More formally. . .

We prove (constructively) that

$$\vdash^m_{\mathsf{GIL}^\circ} \Sigma \Rightarrow \alpha \quad \text{and} \quad \vdash^n_{\mathsf{GIL}^\circ} \Pi, \alpha \Rightarrow \delta \quad \Longrightarrow \quad \vdash_{\mathsf{GIL}^\circ} \Sigma, \Pi \Rightarrow \delta,$$

by induction on the lexicographically ordered pair $\langle |\alpha|, m+n \rangle$.

E.g., suppose that α is $\beta \to \gamma$ and the derivations of the premises end with

$$\dfrac{\dfrac{\vdots}{\Sigma, \beta \Rightarrow \gamma}}{\Sigma \Rightarrow \beta \to \gamma}\,(\to\Rightarrow) \qquad \text{and} \qquad \dfrac{\dfrac{\vdots}{\Pi, \beta \to \gamma \Rightarrow \beta} \quad \dfrac{\vdots}{\Pi, \gamma \Rightarrow \delta}}{\Pi, \beta \to \gamma \Rightarrow \delta}\,(\to\Rightarrow)$$

We apply the induction hypothesis three times:

1. $\vdash^m_{\mathsf{GIL}^\circ} \Sigma \Rightarrow \beta \to \gamma$ and $\vdash^{n-1}_{\mathsf{GIL}^\circ} \Pi, \beta \to \gamma \Rightarrow \beta$ yields $\vdash_{\mathsf{GIL}^\circ} \Sigma, \Pi \Rightarrow \beta$

More formally...

We prove (constructively) that

$$\vdash^m_{\mathsf{GIL}^\circ} \Sigma \Rightarrow \alpha \quad \text{and} \quad \vdash^n_{\mathsf{GIL}^\circ} \Pi, \alpha \Rightarrow \delta \quad \Longrightarrow \quad \vdash_{\mathsf{GIL}^\circ} \Sigma, \Pi \Rightarrow \delta,$$

by induction on the lexicographically ordered pair $\langle |\alpha|, m + n \rangle$.

E.g., suppose that α is $\beta \to \gamma$ and the derivations of the premises end with

$$\dfrac{\dfrac{\vdots}{\Sigma, \beta \Rightarrow \gamma}}{\Sigma \Rightarrow \beta \to \gamma}\ (\to\Rightarrow) \qquad \text{and} \qquad \dfrac{\dfrac{\vdots}{\Pi, \beta \to \gamma \Rightarrow \beta} \quad \dfrac{\vdots}{\Pi, \gamma \Rightarrow \delta}}{\Pi, \beta \to \gamma \Rightarrow \delta}\ (\to\Rightarrow)$$

We apply the induction hypothesis three times:

1. $\vdash^m_{\mathsf{GIL}^\circ} \Sigma \Rightarrow \beta \to \gamma$ and $\vdash^{n-1}_{\mathsf{GIL}^\circ} \Pi, \beta \to \gamma \Rightarrow \beta$ yields $\vdash_{\mathsf{GIL}^\circ} \Sigma, \Pi \Rightarrow \beta$
2. $\vdash_{\mathsf{GIL}^\circ} \Sigma, \Pi \Rightarrow \beta$ and $\Sigma, \beta \Rightarrow \gamma$ yields $\vdash_{\mathsf{GIL}^\circ} \Sigma, \Sigma, \Pi \Rightarrow \gamma$

More formally...

We prove (constructively) that

$$\vdash^m_{\mathsf{GIL}^\circ} \Sigma \Rightarrow \alpha \quad \text{and} \quad \vdash^n_{\mathsf{GIL}^\circ} \Pi, \alpha \Rightarrow \delta \quad \Longrightarrow \quad \vdash_{\mathsf{GIL}^\circ} \Sigma, \Pi \Rightarrow \delta,$$

by induction on the lexicographically ordered pair $\langle |\alpha|, m+n \rangle$.

E.g., suppose that α is $\beta \to \gamma$ and the derivations of the premises end with

$$\dfrac{\dfrac{\vdots}{\Sigma, \beta \Rightarrow \gamma}}{\Sigma \Rightarrow \beta \to \gamma}\;(\to\Rightarrow) \qquad \text{and} \qquad \dfrac{\dfrac{\vdots}{\Pi, \beta \to \gamma \Rightarrow \beta} \quad \dfrac{\vdots}{\Pi, \gamma \Rightarrow \delta}}{\Pi, \beta \to \gamma \Rightarrow \delta}\;(\to\Rightarrow)$$

We apply the induction hypothesis three times:

1. $\vdash^m_{\mathsf{GIL}^\circ} \Sigma \Rightarrow \beta \to \gamma$ and $\vdash^{n-1}_{\mathsf{GIL}^\circ} \Pi, \beta \to \gamma \Rightarrow \beta$ yields $\vdash_{\mathsf{GIL}^\circ} \Sigma, \Pi \Rightarrow \beta$
2. $\vdash_{\mathsf{GIL}^\circ} \Sigma, \Pi \Rightarrow \beta$ and $\Sigma, \beta \Rightarrow \gamma$ yields $\vdash_{\mathsf{GIL}^\circ} \Sigma, \Sigma, \Pi \Rightarrow \gamma$
3. $\vdash_{\mathsf{GIL}^\circ} \Sigma, \Sigma, \Pi \Rightarrow \gamma$ and $\Pi, \gamma \Rightarrow \delta$ yields $\vdash_{\mathsf{GIL}^\circ} \Sigma, \Sigma, \Pi, \Pi \Rightarrow \delta$.

More formally...

We prove (constructively) that

$$\vdash^m_{\mathsf{GIL}^\circ} \Sigma \Rightarrow \alpha \quad \text{and} \quad \vdash^n_{\mathsf{GIL}^\circ} \Pi, \alpha \Rightarrow \delta \quad \Longrightarrow \quad \vdash_{\mathsf{GIL}^\circ} \Sigma, \Pi \Rightarrow \delta,$$

by induction on the lexicographically ordered pair $\langle |\alpha|, m+n \rangle$.

E.g., suppose that α is $\beta \to \gamma$ and the derivations of the premises end with

$$\dfrac{\dfrac{\vdots}{\Sigma, \beta \Rightarrow \gamma}}{\Sigma \Rightarrow \beta \to \gamma}\ (\to\Rightarrow) \qquad \text{and} \qquad \dfrac{\dfrac{\vdots}{\Pi, \beta \to \gamma \Rightarrow \beta} \quad \dfrac{\vdots}{\Pi, \gamma \Rightarrow \delta}}{\Pi, \beta \to \gamma \Rightarrow \delta}\ (\to\Rightarrow)$$

We apply the induction hypothesis three times:

1. $\vdash^m_{\mathsf{GIL}^\circ} \Sigma \Rightarrow \beta \to \gamma$ and $\vdash^{n-1}_{\mathsf{GIL}^\circ} \Pi, \beta \to \gamma \Rightarrow \beta$ yields $\vdash_{\mathsf{GIL}^\circ} \Sigma, \Pi \Rightarrow \beta$
2. $\vdash_{\mathsf{GIL}^\circ} \Sigma, \Pi \Rightarrow \beta$ and $\Sigma, \beta \Rightarrow \gamma$ yields $\vdash_{\mathsf{GIL}^\circ} \Sigma, \Sigma, \Pi \Rightarrow \gamma$
3. $\vdash_{\mathsf{GIL}^\circ} \Sigma, \Sigma, \Pi \Rightarrow \gamma$ and $\Pi, \gamma \Rightarrow \delta$ yields $\vdash_{\mathsf{GIL}^\circ} \Sigma, \Sigma, \Pi, \Pi \Rightarrow \delta$.

Finally, using the previous lemma, $\vdash_{\mathsf{GIL}^\circ} \Sigma, \Pi \Rightarrow \delta$. □

A First Quiz

1. Explain how GIL° can be used to **decide** if $\Gamma \vdash_{\mathrm{IL}} \alpha$ for Γ finite.

A First Quiz

1. Explain how GIL° can be used to **decide** if $\Gamma \vdash_{\mathsf{IL}} \alpha$ for Γ finite.
2. Use GIL° to prove the **disjunction property** for intuitionistic logic

$$\vdash_{\mathsf{IL}} \alpha \vee \beta \implies \vdash_{\mathsf{IL}} \alpha \text{ or } \vdash_{\mathsf{IL}} \beta.$$

A First Quiz

1. Explain how GIL° can be used to **decide** if $\Gamma \vdash_{\mathrm{IL}} \alpha$ for Γ finite.
2. Use GIL° to prove the **disjunction property** for intuitionistic logic

$$\vdash_{\mathrm{IL}} \alpha \vee \beta \implies \vdash_{\mathrm{IL}} \alpha \text{ or } \vdash_{\mathrm{IL}} \beta.$$

3. Give an algorithm to check if formulas $\alpha_1, \ldots, \alpha_n$ are **independent** in intuitionistic logic, that is, to check if for any formula $\beta(y_1, \ldots, y_n)$,

$$\vdash_{\mathrm{IL}} \beta(\alpha_1, \ldots, \alpha_n) \implies \vdash_{\mathrm{IL}} \beta.$$

Decidability

Theorem (Gentzen 1935)

Finitary consequence in intuitionistic logic is decidable.

Decidability

Theorem (Gentzen 1935)

Finitary consequence in intuitionistic logic is decidable.

Proof.

To decide $\Gamma \vdash_{\mathsf{IL}} \alpha$ for a finite set of formulas $\Gamma \cup \{\alpha\}$, we search for a derivation of $\Gamma \Rightarrow \alpha$ in GIL°.

Decidability

Theorem (Gentzen 1935)

Finitary consequence in intuitionistic logic is decidable.

Proof.

To decide $\Gamma \vdash_{\mathsf{IL}} \alpha$ for a finite set of formulas $\Gamma \cup \{\alpha\}$, we search for a derivation of $\Gamma \Rightarrow \alpha$ in GIL°. If the left hand sides of sequents are viewed as *sets*,

Decidability

Theorem (Gentzen 1935)

Finitary consequence in intuitionistic logic is decidable.

Proof.

To decide $\Gamma \vdash_{\mathsf{IL}} \alpha$ for a finite set of formulas $\Gamma \cup \{\alpha\}$, we search for a derivation of $\Gamma \Rightarrow \alpha$ in GIL°. If the left hand sides of sequents are viewed as *sets*, which is possible because of the admissibility of contraction,

Decidability

Theorem (Gentzen 1935)

Finitary consequence in intuitionistic logic is decidable.

Proof.

To decide $\Gamma \vdash_{\mathsf{IL}} \alpha$ for a finite set of formulas $\Gamma \cup \{\alpha\}$, we search for a derivation of $\Gamma \Rightarrow \alpha$ in GIL°. If the left hand sides of sequents are viewed as *sets*, which is possible because of the admissibility of contraction, then there are only finitely many sequents that can occur in such a derivation.

Decidability

Theorem (Gentzen 1935)

Finitary consequence in intuitionistic logic is decidable.

Proof.

To decide $\Gamma \vdash_{\mathrm{IL}} \alpha$ for a finite set of formulas $\Gamma \cup \{\alpha\}$, we search for a derivation of $\Gamma \Rightarrow \alpha$ in GIL°. If the left hand sides of sequents are viewed as *sets*, which is possible because of the admissibility of contraction, then there are only finitely many sequents that can occur in such a derivation. Hence, loop-checking and backtracking can be used to check all potential derivations. $\square$

Decidability

Theorem (Gentzen 1935)

Finitary consequence in intuitionistic logic is decidable.

Proof.

To decide $\Gamma \vdash_{\mathsf{IL}} \alpha$ for a finite set of formulas $\Gamma \cup \{\alpha\}$, we search for a derivation of $\Gamma \Rightarrow \alpha$ in GIL°. If the left hand sides of sequents are viewed as *sets*, which is possible because of the admissibility of contraction, then there are only finitely many sequents that can occur in such a derivation. Hence, loop-checking and backtracking can be used to check all potential derivations. □

Corollary

The quasi-equational theory of Heyting algebras is decidable.

The Disjunction Property

Corollary

For any formulas α, β,

$$\vdash_{\mathsf{IL}} \alpha \vee \beta \implies \vdash_{\mathsf{IL}} \alpha \text{ or } \vdash_{\mathsf{IL}} \beta.$$

The Disjunction Property

Corollary

For any formulas α, β,

$$\vdash_{\mathsf{IL}} \alpha \vee \beta \implies \vdash_{\mathsf{IL}} \alpha \text{ or } \vdash_{\mathsf{IL}} \beta.$$

Proof.

Just consider the last step of a derivation of $\Rightarrow \alpha \vee \beta$ in GIL°. □

The Disjunction Property

Corollary

For any formulas α, β,

$$\vdash_{\text{IL}} \alpha \vee \beta \implies \vdash_{\text{IL}} \alpha \text{ or } \vdash_{\text{IL}} \beta.$$

Proof.

Just consider the last step of a derivation of $\Rightarrow \alpha \vee \beta$ in GIL°. □

Note. It follows similarly that the following **Visser rules** are admissible in intuitionistic logic:

$$\frac{\bigwedge_{i=1}^{n}(\alpha_i \to \beta_i) \to (\alpha_{n+1} \vee \alpha_{n+2})}{\bigvee_{j=1}^{n+2}(\bigwedge_{i=1}^{n}(\alpha_i \to \beta_i) \to \alpha_j)} \quad n = 0, 1, 2, \ldots$$

Interpolation

Theorem (Schütte 1962)

If $\alpha(\overline{x}, \overline{y})$ *and* $\beta(\overline{y}, \overline{z})$ *are formulas such that* $\alpha \vdash_{\mathsf{IL}} \beta$*, then there exists a formula* $\gamma(\overline{y})$ *such that* $\alpha \vdash_{\mathsf{IL}} \gamma$ *and* $\gamma \vdash_{\mathsf{IL}} \beta$*.*

Interpolation

Theorem (Schütte 1962)

If $\alpha(\overline{x}, \overline{y})$ and $\beta(\overline{y}, \overline{z})$ are formulas such that $\alpha \vdash_{\text{IL}} \beta$, then there exists a formula $\gamma(\overline{y})$ such that $\alpha \vdash_{\text{IL}} \gamma$ and $\gamma \vdash_{\text{IL}} \beta$.

Proof sketch. We prove that for any sequent $\Sigma(\overline{x}, \overline{y}), \Pi(\overline{y}, \overline{z}) \Rightarrow \delta(\overline{y}, \overline{z})$,

Interpolation

Theorem (Schütte 1962)

If $\alpha(\overline{x}, \overline{y})$ and $\beta(\overline{y}, \overline{z})$ are formulas such that $\alpha \vdash_{\mathsf{IL}} \beta$, then there exists a formula $\gamma(\overline{y})$ such that $\alpha \vdash_{\mathsf{IL}} \gamma$ and $\gamma \vdash_{\mathsf{IL}} \beta$.

Proof sketch. We prove that for any sequent $\Sigma(\overline{x}, \overline{y}), \Pi(\overline{y}, \overline{z}) \Rightarrow \delta(\overline{y}, \overline{z})$,

$$\vdash^{n}_{\mathsf{GIL}^\circ} \Sigma, \Pi \Rightarrow \delta \quad \Longrightarrow \quad \begin{array}{c}\text{there exists a formula } \gamma(\overline{y}) \text{ such that} \\ \vdash_{\mathsf{GIL}^\circ} \Sigma \Rightarrow \gamma \ \text{ and } \ \vdash_{\mathsf{GIL}^\circ} \Pi, \gamma \Rightarrow \delta,\end{array}$$

Theorem (Schütte 1962)

If $\alpha(\overline{x}, \overline{y})$ and $\beta(\overline{y}, \overline{z})$ are formulas such that $\alpha \vdash_{\mathrm{IL}} \beta$, then there exists a formula $\gamma(\overline{y})$ such that $\alpha \vdash_{\mathrm{IL}} \gamma$ and $\gamma \vdash_{\mathrm{IL}} \beta$.

Proof sketch. We prove that for any sequent $\Sigma(\overline{x}, \overline{y}), \Pi(\overline{y}, \overline{z}) \Rightarrow \delta(\overline{y}, \overline{z})$,

$$\vdash^{n}_{\mathrm{GIL}^{\circ}} \Sigma, \Pi \Rightarrow \delta \quad \Longrightarrow \quad \begin{array}{c}\text{there exists a formula } \gamma(\overline{y}) \text{ such that} \\ \vdash_{\mathrm{GIL}^{\circ}} \Sigma \Rightarrow \gamma \ \text{ and } \ \vdash_{\mathrm{GIL}^{\circ}} \Pi, \gamma \Rightarrow \delta,\end{array}$$

by induction on n.

Interpolation

Theorem (Schütte 1962)

If $\alpha(\overline{x}, \overline{y})$ and $\beta(\overline{y}, \overline{z})$ are formulas such that $\alpha \vdash_{\mathrm{IL}} \beta$, then there exists a formula $\gamma(\overline{y})$ such that $\alpha \vdash_{\mathrm{IL}} \gamma$ and $\gamma \vdash_{\mathrm{IL}} \beta$.

Proof sketch. We prove that for any sequent $\Sigma(\overline{x}, \overline{y}), \Pi(\overline{y}, \overline{z}) \Rightarrow \delta(\overline{y}, \overline{z})$,

$$\vdash^{n}_{\mathrm{GIL}^\circ} \Sigma, \Pi \Rightarrow \delta \quad \Longrightarrow \quad \text{there exists a formula } \gamma(\overline{y}) \text{ such that } \vdash_{\mathrm{GIL}^\circ} \Sigma \Rightarrow \gamma \ \text{ and } \ \vdash_{\mathrm{GIL}^\circ} \Pi, \gamma \Rightarrow \delta,$$

by induction on n.

Base case. E.g., if Σ is Σ', δ, let $\gamma = \delta$;

Interpolation

Theorem (Schütte 1962)

If $\alpha(\overline{x}, \overline{y})$ and $\beta(\overline{y}, \overline{z})$ are formulas such that $\alpha \vdash_{\text{IL}} \beta$, then there exists a formula $\gamma(\overline{y})$ such that $\alpha \vdash_{\text{IL}} \gamma$ and $\gamma \vdash_{\text{IL}} \beta$.

Proof sketch. We prove that for any sequent $\Sigma(\overline{x}, \overline{y}), \Pi(\overline{y}, \overline{z}) \Rightarrow \delta(\overline{y}, \overline{z})$,

$$\vdash^{n}_{\text{GIL}^\circ} \Sigma, \Pi \Rightarrow \delta \quad \Longrightarrow \quad \begin{array}{c} \text{there exists a formula } \gamma(\overline{y}) \text{ such that} \\ \vdash_{\text{GIL}^\circ} \Sigma \Rightarrow \gamma \ \text{ and } \vdash_{\text{GIL}^\circ} \Pi, \gamma \Rightarrow \delta, \end{array}$$

by induction on n.

Base case. E.g., if Σ is Σ', δ, let $\gamma = \delta$; if Π is Π', δ, let $\gamma = \top$.

Interpolation

Inductive step. E.g., if Σ is $\Sigma', \alpha \to \beta$ and the derivation ends with

$$\dfrac{\dfrac{\vdots}{\Sigma', \alpha \to \beta, \Pi \Rightarrow \alpha} \quad \dfrac{\vdots}{\Sigma', \beta, \Pi \Rightarrow \delta}}{\Sigma', \alpha \to \beta, \Pi \Rightarrow \delta}\ (\to\Rightarrow)$$

Interpolation

Inductive step. E.g., if Σ is $\Sigma', \alpha \to \beta$ and the derivation ends with

$$\dfrac{\dfrac{\vdots}{\Sigma', \alpha \to \beta, \Pi \Rightarrow \alpha} \quad \dfrac{\vdots}{\Sigma', \beta, \Pi \Rightarrow \delta}}{\Sigma', \alpha \to \beta, \Pi \Rightarrow \delta}\ (\to\Rightarrow)$$

then by the induction hypothesis twice, there exist formulas $\gamma_1(\overline{y}), \gamma_2(\overline{y})$ such that the following sequents are GIL-derivable:

Inductive step. E.g., if Σ is $\Sigma', \alpha \to \beta$ and the derivation ends with

$$\dfrac{\dfrac{\vdots}{\Sigma', \alpha \to \beta, \Pi \Rightarrow \alpha} \quad \dfrac{\vdots}{\Sigma', \beta, \Pi \Rightarrow \delta}}{\Sigma', \alpha \to \beta, \Pi \Rightarrow \delta}\ (\to\Rightarrow)$$

then by the induction hypothesis twice, there exist formulas $\gamma_1(\overline{y}), \gamma_2(\overline{y})$ such that the following sequents are GIL-derivable:

$$\Pi \Rightarrow \gamma_1; \quad \Sigma', \alpha \to \beta, \gamma_1 \Rightarrow \alpha;$$

Interpolation

Inductive step. E.g., if Σ is $\Sigma', \alpha \to \beta$ and the derivation ends with

$$\dfrac{\dfrac{\vdots}{\Sigma', \alpha \to \beta, \Pi \Rightarrow \alpha} \quad \dfrac{\vdots}{\Sigma', \beta, \Pi \Rightarrow \delta}}{\Sigma', \alpha \to \beta, \Pi \Rightarrow \delta} \ (\to\Rightarrow)$$

then by the induction hypothesis twice, there exist formulas $\gamma_1(\overline{y}), \gamma_2(\overline{y})$ such that the following sequents are GIL-derivable:

$$\Pi \Rightarrow \gamma_1; \quad \Sigma', \alpha \to \beta, \gamma_1 \Rightarrow \alpha; \quad \Sigma', \beta \Rightarrow \gamma_2; \quad \text{and} \quad \Pi, \gamma_2 \Rightarrow \delta.$$

Interpolation

Inductive step. E.g., if Σ is $\Sigma', \alpha \to \beta$ and the derivation ends with

$$\dfrac{\dfrac{\vdots}{\Sigma', \alpha \to \beta, \Pi \Rightarrow \alpha} \quad \dfrac{\vdots}{\Sigma', \beta, \Pi \Rightarrow \delta}}{\Sigma', \alpha \to \beta, \Pi \Rightarrow \delta} \ (\to\Rightarrow)$$

then by the induction hypothesis twice, there exist formulas $\gamma_1(\overline{y}), \gamma_2(\overline{y})$ such that the following sequents are GIL-derivable:

$$\Pi \Rightarrow \gamma_1; \quad \Sigma', \alpha \to \beta, \gamma_1 \Rightarrow \alpha; \quad \Sigma', \beta \Rightarrow \gamma_2; \quad \text{and} \quad \Pi, \gamma_2 \Rightarrow \delta.$$

We obtain an interpolant $\gamma_1 \to \gamma_2$ with derivations ending with

Interpolation

Inductive step. E.g., if Σ is $\Sigma', \alpha \to \beta$ and the derivation ends with

$$\dfrac{\dfrac{\vdots}{\Sigma', \alpha \to \beta, \Pi \Rightarrow \alpha} \quad \dfrac{\vdots}{\Sigma', \beta, \Pi \Rightarrow \delta}}{\Sigma', \alpha \to \beta, \Pi \Rightarrow \delta} \; (\to\Rightarrow)$$

then by the induction hypothesis twice, there exist formulas $\gamma_1(\overline{y}), \gamma_2(\overline{y})$ such that the following sequents are GIL-derivable:

$$\Pi \Rightarrow \gamma_1; \quad \Sigma', \alpha \to \beta, \gamma_1 \Rightarrow \alpha; \quad \Sigma', \beta \Rightarrow \gamma_2; \quad \text{and} \quad \Pi, \gamma_2 \Rightarrow \delta.$$

We obtain an interpolant $\gamma_1 \to \gamma_2$ with derivations ending with

$$\dfrac{\dfrac{\dfrac{\vdots}{\Sigma', \alpha \to \beta, \gamma_1 \Rightarrow \alpha} \quad \dfrac{\vdots}{\Sigma', \beta, \gamma_1 \Rightarrow \gamma_2}}{\Sigma', \alpha \to \beta, \gamma_1 \Rightarrow \gamma_2} \; (\to\Rightarrow)}{\Sigma', \alpha \to \beta \Rightarrow \gamma_1 \to \gamma_2} \; (\Rightarrow\to)$$

Interpolation

Inductive step. E.g., if Σ is $\Sigma', \alpha \to \beta$ and the derivation ends with

$$\dfrac{\dfrac{\vdots}{\Sigma', \alpha \to \beta, \Pi \Rightarrow \alpha} \quad \dfrac{\vdots}{\Sigma', \beta, \Pi \Rightarrow \delta}}{\Sigma', \alpha \to \beta, \Pi \Rightarrow \delta} \ (\to\Rightarrow)$$

then by the induction hypothesis twice, there exist formulas $\gamma_1(\overline{y}), \gamma_2(\overline{y})$ such that the following sequents are GIL-derivable:

$$\Pi \Rightarrow \gamma_1; \quad \Sigma', \alpha \to \beta, \gamma_1 \Rightarrow \alpha; \quad \Sigma', \beta \Rightarrow \gamma_2; \quad \text{and} \quad \Pi, \gamma_2 \Rightarrow \delta.$$

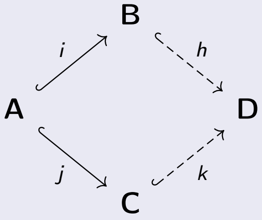
We obtain an interpolant $\gamma_1 \to \gamma_2$ with derivations ending with

$$\dfrac{\dfrac{\dfrac{\vdots}{\Sigma', \alpha \to \beta, \gamma_1 \Rightarrow \alpha} \quad \dfrac{\vdots}{\Sigma', \beta, \gamma_1 \Rightarrow \gamma_2}}{\Sigma', \alpha \to \beta, \gamma_1 \Rightarrow \gamma_2} \ (\to\Rightarrow)}{\Sigma', \alpha \to \beta \Rightarrow \gamma_1 \to \gamma_2} \ (\Rightarrow\to) \qquad \dfrac{\dfrac{\vdots}{\Pi, \gamma_1 \to \gamma_2 \Rightarrow \gamma_1} \quad \dfrac{\vdots}{\Pi, \gamma_2 \Rightarrow \delta}}{\Pi, \gamma_1 \to \gamma_2 \Rightarrow \delta} \ (\to\Rightarrow) \quad \square$$

An Algebraic Consequence

Theorem (Day 1972)

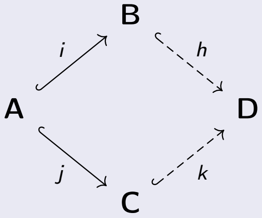
$\mathcal{HA}$ admits the **amalgamation property***; that is, for any* $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{HA}$ *and embeddings* $i \colon \mathbf{A} \to \mathbf{B}$ *and* $j \colon \mathbf{A} \to \mathbf{C}$*, there exist* $\mathbf{D} \in \mathcal{HA}$ *and embeddings* $h \colon \mathbf{B} \to \mathbf{D}$ *and* $k \colon \mathbf{C} \to \mathbf{D}$ *satisfying* $hi = kj$*.*



An Algebraic Consequence

Theorem (Day 1972)

$\mathcal{HA}$ admits the **amalgamation property***; that is, for any* $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{HA}$ *and embeddings* $i\colon \mathbf{A} \to \mathbf{B}$ *and* $j\colon \mathbf{A} \to \mathbf{C}$*, there exist* $\mathbf{D} \in \mathcal{HA}$ *and embeddings* $h\colon \mathbf{B} \to \mathbf{D}$ *and* $k\colon \mathbf{C} \to \mathbf{D}$ *satisfying* $hi = kj$*.*



Proof.

By construction or as a consequence of interpolation (shown later). □

Tomorrow

We will. . .

- consider Pitts' uniform interpolation theorem for intuitionistic logic

Tomorrow

We will. . .

- consider Pitts' uniform interpolation theorem for intuitionistic logic
- explain some of the nuts and bolts of universal algebra

Tomorrow

We will...

- consider Pitts' uniform interpolation theorem for intuitionistic logic
- explain some of the nuts and bolts of universal algebra
- define and interpret consequence in classes of algebraic structures

Tomorrow

We will...

- consider Pitts' uniform interpolation theorem for intuitionistic logic
- explain some of the nuts and bolts of universal algebra
- define and interpret consequence in classes of algebraic structures

References

A. Day, Varieties of Heyting algebras, II (Amalgamation and injectivity). Unpublished note (1972).

D. de Jongh and L.A. Chagrova. The decidability of dependency in intuitionistic propositional logic. *Journal of Symbolic Logic* 60 (1995), no. 2, 498–504.

R. Dyckhoff. Intuitionistic decision procedures since Gentzen. *Advances in Proof Theory*, Birkhäuser (2016), 245–267.

S. Ghilardi and M. Zawadowski. *Sheaves, Games and Model Completions*, Kluwer (2002).

A.M. Pitts. On an interpretation of second-order quantification in first-order intuitionistic propositional logic. *Journal of Symbolic Logic* 57 (1992), 33–52.

K. Schütte. Der Interpolationssatz der intuitionistischen Pradikatenlogik. *Mathematische Annalen* 148 (1962), 192–200.