Bridges between Logic and Algebra
Part 3: Interpolation and Amalgamation

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**Lemma**

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1. $\Sigma \models_\mathcal{V} \Delta$
2. $\Delta \subseteq \text{Cg}_{\mathcal{F}}(\Sigma)$
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**Lemma**

The following are equivalent for any sets of $\mathcal{L}$-equations $\Sigma(\overline{x})$, $\Delta(\overline{x})$:

1. $\Sigma \models_{\mathcal{V}} \Delta$ i.e., for any $A \in \mathcal{V}$ and homomorphism $e : \text{Tm}(\overline{x}) \rightarrow A$,
   
   $\Sigma \subseteq \ker(e) \implies \Delta \subseteq \ker(e)$.
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Lemma

The following are equivalent for any sets of $\mathcal{L}$-equations $\Sigma(\overline{x})$, $\Delta(\overline{x})$:

1. $\Sigma \models_{\mathcal{V}} \Delta$ i.e., for any $A \in \mathcal{V}$ and homomorphism $e : \text{TM}(\overline{x}) \to A$, $\Sigma \subseteq \ker(e) \implies \Delta \subseteq \ker(e)$.

2. $\Delta \subseteq C_{g_{\text{F}(\overline{x})}}(\Sigma)$. 
∀ admits deductive interpolation
\[\mathcal{V}\] admits **deductive interpolation** if whenever \(\Sigma(\overline{x}, \overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z})\), there exists a set of equations \(\Delta(\overline{y})\) such that

\[
\Sigma(\overline{x}, \overline{y}) \models_{\mathcal{V}} \Delta(\overline{y}) \quad \text{and} \quad \Delta(\overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}).
\]
\( \forall \) admits **deductive interpolation** if whenever \( \Sigma(\overline{x}, \overline{y}) \models_{\forall} \varepsilon(\overline{y}, \overline{z}) \), there exists a set of equations \( \Delta(\overline{y}) \) such that

\[
\Sigma(\overline{x}, \overline{y}) \models_{\forall} \Delta(\overline{y}) \quad \text{and} \quad \Delta(\overline{y}) \models_{\forall} \varepsilon(\overline{y}, \overline{z}).
\]

**Lemma**

\( \forall \) admits deductive interpolation if and only if for any set of equations \( \Sigma(\overline{x}, \overline{y}) \), there exists a set of equations \( \Delta(\overline{y}) \) such that

\[
\Sigma(\overline{x}, \overline{y}) \models_{\forall} \varepsilon(\overline{y}, \overline{z}) \iff \Delta(\overline{y}) \models_{\forall} \varepsilon(\overline{y}, \overline{z}).
\]
\( \forall \) admits **deductive interpolation** if whenever \( \Sigma(\overline{x}, \overline{y}) \models \vee(\overline{y}, \overline{z}) \), there exists a set of equations \( \Delta(\overline{y}) \) such that

\[
\Sigma(\overline{x}, \overline{y}) \models \forall \Delta(\overline{y}) \quad \text{and} \quad \Delta(\overline{y}) \models \forall \vee(\overline{y}, \overline{z}).
\]

**Lemma**

\( \forall \) admits deductive interpolation if and only if for any set of equations \( \Sigma(\overline{x}, \overline{y}) \), there exists a set of equations \( \Delta(\overline{y}) \) such that

\[
\Sigma(\overline{x}, \overline{y}) \models \forall \vee(\overline{y}, \overline{z}) \iff \Delta(\overline{y}) \models \forall \vee(\overline{y}, \overline{z}).
\]

**Proof hint.** Consider \( \Delta(\overline{y}) := \{ \vee(\overline{y}) \mid \Sigma(\overline{x}, \overline{y}) \models \forall \vee(\overline{y}) \} \).
The inclusion map \( i : F(\overline{y}) \rightarrow F(\overline{x}, \overline{y}); \ \alpha \mapsto \alpha \)
The inclusion map $i : F(\bar{y}) \to F(\bar{x}, \bar{y})$; $\alpha \mapsto \alpha$ “lifts” to the maps

$$i^* : \text{Con } F(\bar{y}) \to \text{Con } F(\bar{x}, \bar{y}); \ \Theta \mapsto \text{Cg}_{F(\bar{x}, \bar{y})}(i[\Theta])$$
Lifting Inclusions

The inclusion map \( i : F(\overline{y}) \to F(\overline{x}, \overline{y}) \); \( \alpha \mapsto \alpha \) “lifts” to the maps

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i^* : \text{Con} F(\overline{y}) \to \text{Con} F(\overline{x}, \overline{y}); \quad \Theta \mapsto Cg_{F(\overline{x}, \overline{y})}(i[\Theta])
\]

\[
i^{-1} : \text{Con} F(\overline{x}, \overline{y}) \to \text{Con} F(\overline{y}); \quad \Psi \mapsto i^{-1}[\Psi] = \Psi \cap F(\overline{y})^2.
\]
The inclusion map \( i : F(\bar{y}) \to F(\bar{x}, \bar{y}) \); \( \alpha \mapsto \alpha \) “lifts” to the maps

\[
\begin{align*}
i^* : \text{Con} F(\bar{y}) & \to \text{Con} F(\bar{x}, \bar{y}); \quad \Theta \mapsto C_{g_{F(\bar{x}, \bar{y})}}(i[\Theta]) \\
i^{-1} : \text{Con} F(\bar{x}, \bar{y}) & \to \text{Con} F(\bar{y}); \quad \Psi \mapsto i^{-1}[\Psi] = \Psi \cap F(\bar{y})^2.
\end{align*}
\]

Note that the pair \( \langle i^*, i^{-1} \rangle \) is an adjunction, i.e.,

\[
i^*(\Theta) \subseteq \Psi \iff \Theta \subseteq i^{-1}(\Psi).
\]
The following are equivalent:

1) $\mathcal{V}$ admits **deductive interpolation**, i.e., for any set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a set of equations $\Delta(\overline{y})$ such that

$$\Sigma(\overline{x}, \overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}) \iff \Delta(\overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}).$$
Deductive Interpolation Again

The following are equivalent:

1. \( \mathcal{V} \) admits **deductive interpolation**, i.e., for any set of equations \( \Sigma(x, \bar{y}) \), there exists a set of equations \( \Delta(\bar{y}) \) such that

   \[
   \Sigma(x, \bar{y}) \models \mathcal{V} \varepsilon(\bar{y}, \bar{z}) \iff \Delta(\bar{y}) \models \mathcal{V} \varepsilon(\bar{y}, \bar{z}).
   \]

2. The following diagram commutes (where \( i, j, k, l \) are inclusion maps):

   \[
   \begin{array}{ccc}
   \text{Con } \mathcal{F}(x, \bar{y}) & \xrightarrow{i^{-1}} & \text{Con } \mathcal{F}(\bar{y}) \\
   j^* & \downarrow & l^* \\
   \text{Con } \mathcal{F}(x, \bar{y}, \bar{z}) & \xrightarrow{k^{-1}} & \text{Con } \mathcal{F}(\bar{y}, \bar{z})
   \end{array}
   \]
Deductive Interpolation Again

The following are equivalent:

(1) $\mathcal{V}$ admits **deductive interpolation**, i.e., for any set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a set of equations $\Delta(\overline{y})$ such that

$$\Sigma(\overline{x}, \overline{y}) \models_\mathcal{V} \varepsilon(\overline{y}, \overline{z}) \iff \Delta(\overline{y}) \models_\mathcal{V} \varepsilon(\overline{y}, \overline{z}).$$

(2) The following diagram commutes (where $i, j, k, l$ are inclusion maps):

$$\begin{array}{c}
\text{Con } F(\overline{x}, \overline{y}) \xrightarrow{i^{-1}} \text{Con } F(\overline{y}) \\
i^* \downarrow \quad \quad \quad \quad \quad \downarrow i^*
\end{array}$$

$$\begin{array}{c}
\text{Con } F(\overline{x}, \overline{y}, \overline{z}) \xrightarrow{k^{-1}} \text{Con } F(\overline{y}, \overline{z}) \\
k^* \downarrow \quad \quad \quad \quad \quad \downarrow k^*
\end{array}$$

That is, for any $\Theta \in \text{Con } F(\overline{x}, \overline{y})$,

$$\mathcal{C}g_{F(\overline{x}, \overline{y}, \overline{z})}(\Theta) \cap F(\overline{y}, \overline{z})^2 = \mathcal{C}g_{F(\overline{y}, \overline{z})}(\Theta \cap F(\overline{y})^2).$$
But now...

What does deductive interpolation mean algebraically?
A variety $\mathcal{V}$ has the **amalgamation property** if for any $A, B, C \in \mathcal{V}$ and embeddings $i: A \to B$ and $j: A \to C$, there exist $D \in \mathcal{V}$ and embeddings $h: B \to D$ and $k: C \to D$ satisfying $hi = kj$. 
A variety $\mathcal{V}$ has the **amalgamation property** if for any $A, B, C \in \mathcal{V}$ and embeddings $i : A \to B$ and $j : A \to C$, there exist $D \in \mathcal{V}$ and embeddings $h : B \to D$ and $k : C \to D$ satisfying $hi = kj$. 
A variety $\mathcal{V}$ has the **amalgamation property** if for any $A, B, C \in \mathcal{V}$ and embeddings $i : A \to B$ and $j : A \to C$, there exist $D \in \mathcal{V}$ and embeddings $h : B \to D$ and $k : C \to D$ satisfying $hi = kj$. 
Lemma (Pigozzi 1972)

\( \forall \) has the amalgamation property if and only if for any \( \Theta \in \text{Con} F(\overline{x}, \overline{y}) \), \( \Psi \in \text{Con} F(\overline{y}, \overline{z}) \) satisfying

\[
\Theta \cap F(\overline{y})^2 = \Psi \cap F(\overline{y})^2,
\]

This property of congruences of free algebras can be reformulated in terms of consequence as the so-called Robinson property.
A Key Lemma

Lemma (Pigozzi 1972)

\( \mathcal{V} \) has the amalgamation property if and only if for any \( \Theta \in \text{Con } F(x, y) \), \( \Psi \in \text{Con } F(y, z) \) satisfying

\[
\Theta \cap F(y)^2 = \Psi \cap F(y)^2,
\]

there exists \( \Phi \in \text{Con } F(x, y, z) \) satisfying

\[
\Theta = \Phi \cap F(x, y)^2 \quad \text{and} \quad \Psi = \Phi \cap F(y, z)^2.
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Lemma (Pigozzi 1972)

\( \mathcal{V} \) has the amalgamation property if and only if for any \( \Theta \in \text{Con} F(\overline{x}, \overline{y}) \), \( \Psi \in \text{Con} F(\overline{y}, \overline{z}) \) satisfying

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there exists \( \Phi \in \text{Con} F(\overline{x}, \overline{y}, \overline{z}) \) satisfying

\[
\Theta = \Phi \cap F(\overline{x}, \overline{y})^2 \quad \text{and} \quad \Psi = \Phi \cap F(\overline{y}, \overline{z})^2.
\]

This property of congruences of free algebras can be reformulated in terms of consequence as the so-called Robinson property.
Proof Sketch (⇒)

Suppose that $\mathcal{V}$ has the amalgamation property and $\Theta \in \text{Con } F(\bar{x}, \bar{y})$, $\Psi \in \text{Con } F(\bar{y}, \bar{z})$ satisfy $\Phi_0 := \Theta \cap F(\bar{y})^2 = \Psi \cap F(\bar{y})^2$. 

$\Phi := \ker(g)$ for some surjective homomorphism $g : F(x, y, z) \rightarrow D$ with $\Phi := \ker(g)$ satisfying $\Theta = \Phi \cap F(x, y)$ and $\Psi = \Phi \cap F(y, z)$. 

$F(x, y)$ $F(x, y, z)$ $F(y)$ $F(y, z)$ $B$ $D$ $A$ $C$ $g$
Proof Sketch ($\Rightarrow$)

Suppose that $\mathcal{V}$ has the amalgamation property and $\Theta \in \text{Con } F(\overline{x}, \overline{y})$, $\Psi \in \text{Con } F(\overline{y}, \overline{z})$ satisfy $\Phi_0 := \Theta \cap F(\overline{y})^2 = \Psi \cap F(\overline{y})^2$. We define

$$A = F(\overline{y})/\Phi_0, \quad B = F(\overline{x}, \overline{y})/\Theta, \quad \text{and} \quad C = F(\overline{y}, \overline{z})/\Psi,$$

yielding an amalgam $D$ and a surjective homomorphism $g : F(\overline{x}, \overline{y}, \overline{z}) \to D$ with $\Phi := \ker(g)$ satisfying $\Theta = \Phi \cap F(\overline{x}, \overline{y})^2$ and $\Psi = \Phi \cap F(\overline{y}, \overline{z})^2$. 
Suppose that $\mathcal{V}$ has the amalgamation property and $\Theta \in \text{Con} F(\bar{x}, \bar{y})$, $\Psi \in \text{Con} F(\bar{y}, \bar{z})$ satisfy $\Phi_0 := \Theta \cap F(\bar{y})^2 = \Psi \cap F(\bar{y})^2$. We define

$$A = F(\bar{y})/\Phi_0,$$  $$B = F(\bar{x}, \bar{y})/\Theta,$$  $$C = F(\bar{y}, \bar{z})/\Psi,$$

yielding an amalgam $D$. 

\begin{center}
\begin{tikzcd}
F(\bar{x}, \bar{y}) & F(\bar{x}, \bar{y}, \bar{z}) \\
F(\bar{y}) & F(\bar{y}, \bar{z})\\
B & D\\
A & C
\end{tikzcd}
\end{center}
Suppose that $\mathcal{V}$ has the amalgamation property and $\Theta \in \text{Con} \ F(\overline{x}, \overline{y})$, $\Psi \in \text{Con} \ F(\overline{y}, \overline{z})$ satisfy $\Phi_0 := \Theta \cap F(\overline{y})^2 = \Psi \cap F(\overline{y})^2$. We define

$$A = F(\overline{y})/\Phi_0, \quad B = F(\overline{x}, \overline{y})/\Theta, \quad \text{and} \quad C = F(\overline{y}, \overline{z})/\Psi,$$

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Suppose that $\mathcal{V}$ has the amalgamation property and $\Theta \in \text{Con} \ F(\overline{x}, \overline{y})$, $\Psi \in \text{Con} \ F(\overline{y}, \overline{z})$ satisfy $\Phi_0 := \Theta \cap F(\overline{y})^2 = \Psi \cap F(\overline{y})^2$. We define

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\[
\begin{array}{ccc}
F(\overline{x}, \overline{y}) & \hookrightarrow & F(\overline{x}, \overline{y}, \overline{z}) \\
\uparrow & & \uparrow \\
F(\overline{y}) & \hookrightarrow & F(\overline{y}, \overline{z}) \\
\downarrow & & \downarrow \\
B & \xrightarrow{g} & D \\
\downarrow & & \downarrow \\
A & \xleftarrow{} & C
\end{array}
\]
Proof Sketch (⇐)

Let $B, C \in \mathcal{V}$ be generated by $\overline{x}, \overline{y}$ and $\overline{y}, \overline{z}$, respectively, with a common subalgebra $A$ generated by $\overline{y}$. 

Consider the surjective homomorphisms $\pi_A: F(\overline{y}) \rightarrow A$, $\pi_B: F(x, \overline{y}) \rightarrow B$, and $\pi_C: F(\overline{y}, \overline{z}) \rightarrow C$.

Then $\Theta = \ker(\pi_B)$, $\Psi = \ker(\pi_C)$ satisfy $\Theta \cap F(\overline{y})^2 = \Psi \cap F(\overline{y})^2$ so, by assumption, there exists $\Phi \in \text{Con} F(x, \overline{y}, \overline{z})$ such that $\Phi \cap F(x, \overline{y})^2 = \Theta$ and $\Phi \cap F(\overline{y}, \overline{z})^2 = \Psi$.

The required amalgam is then $D = F(x, \overline{y}, \overline{y})/\Phi$. 

$F(x, \overline{y})$ $F(x, \overline{y}, \overline{z})$ $F(\overline{y})$ $F(\overline{y}, \overline{z})$ $B$ $D$ $A$ $C$
Proof Sketch ($\iff$)

Let $B, C \in \mathcal{V}$ be generated by $\bar{x}, \bar{y}$ and $\bar{y}, \bar{z}$, respectively, with a common subalgebra $A$ generated by $\bar{y}$. Consider the surjective homomorphisms

$$
\pi_A : F(\bar{y}) \to A, \quad \pi_B : F(\bar{x}, \bar{y}) \to B, \quad \text{and} \quad \pi_C : F(\bar{y}, \bar{z}) \to C.
$$

Then $\Theta = \ker(\pi_B)$, $\Psi = \ker(\pi_C)$ satisfy $\Theta \cap F(y)^2 = \Psi \cap F(y)^2$ so, by assumption, there exists $\Phi \in \text{Con} F(x, y, z)$ such that $\Phi \cap F(x, y)^2 = \Theta$ and $\Phi \cap F(y, z)^2 = \Psi$.

The required amalgam is then $D = F(x, y, y) / \Phi$. 

\[ \begin{array}{c}
F(\bar{x}, \bar{y}) \leftrightarrow F(\bar{x}, \bar{y}, \bar{z}) \\
F(\bar{y}) \leftrightarrow F(\bar{y}, \bar{z}) \\
A \leftarrow \pi_A \\
B \leftarrow \pi_B \\
C \leftarrow \pi_C \\
D \\
\end{array} \]
Proof Sketch (⇐)

Let \( B, C \in \mathcal{V} \) be generated by \( \overline{x}, \overline{y} \) and \( \overline{y}, \overline{z} \), respectively, with a common subalgebra \( A \) generated by \( \overline{y} \). Consider the surjective homomorphisms

\[
\pi_A : F(\overline{y}) \rightarrow A, \quad \pi_B : F(\overline{x}, \overline{y}) \rightarrow B, \quad \text{and} \quad \pi_C : F(\overline{y}, \overline{z}) \rightarrow C.
\]

Then \( \Theta = \ker(\pi_B) \), \( \Psi = \ker(\pi_C) \) satisfy \( \Theta \cap F(\overline{y})^2 = \Psi \cap F(\overline{y})^2 \).
Proof Sketch (⇐)

Let $B, C \in \mathcal{V}$ be generated by $\overline{x}, \overline{y}$ and $\overline{y}, \overline{z}$, respectively, with a common subalgebra $A$ generated by $\overline{y}$. Consider the surjective homomorphisms

$$\pi_A : F(\overline{y}) \to A, \quad \pi_B : F(\overline{x}, \overline{y}) \to B, \quad \text{and} \quad \pi_C : F(\overline{y}, \overline{z}) \to C.$$ 

Then $\Theta = \ker(\pi_B)$, $\Psi = \ker(\pi_C)$ satisfy $\Theta \cap F(\overline{y})^2 = \Psi \cap F(\overline{y})^2$ so, by assumption, there exists $\Phi \in \text{Con} F(\overline{x}, \overline{y}, \overline{z})$ such that $\Phi \cap F(\overline{x}, \overline{y})^2 = \Theta$ and $\Phi \cap F(\overline{y}, \overline{z})^2 = \Psi$. 

\[
\begin{array}{ccc}
F(\overline{x}, \overline{y}) & \leftrightarrow & F(\overline{x}, \overline{y}, \overline{z}) \\
\uparrow & & \uparrow \\
F(\overline{y}) & \leftrightarrow & F(\overline{y}, \overline{z}) \\
\downarrow \pi_B & & \downarrow \\
B & \leftarrow & D \\
\downarrow \pi_A & & \downarrow \\
A & \rightarrow & C \\
\end{array}
\]
Proof Sketch ($\iff$)

Let $B, C \in \mathcal{V}$ be generated by $\overline{x}, \overline{y}$ and $\overline{y}, \overline{z}$, respectively, with a common subalgebra $A$ generated by $\overline{y}$. Consider the surjective homomorphisms

$$\pi_A : F(\overline{y}) \to A, \quad \pi_B : F(\overline{x}, \overline{y}) \to B, \quad \text{and} \quad \pi_C : F(\overline{y}, \overline{z}) \to C.$$ 

Then $\Theta = \ker(\pi_B)$, $\Psi = \ker(\pi_C)$ satisfy $\Theta \cap F(\overline{y})^2 = \Psi \cap F(\overline{y})^2$ so, by assumption, there exists $\Phi \in \text{Con} F(\overline{x}, \overline{y}, \overline{z})$ such that $\Phi \cap F(\overline{x}, \overline{y})^2 = \Theta$ and $\Phi \cap F(\overline{y}, \overline{z})^2 = \Psi$. The required amalgam is then $D = F(\overline{x}, \overline{y}, \overline{y})/\Phi$. 

\[
\begin{array}{c}
F(\overline{x}, \overline{y}) \leftarrow F(\overline{x}, \overline{y}, \overline{z}) \\
\uparrow \quad \quad \quad \uparrow \\
F(\overline{y}) \quad \quad \quad F(\overline{y}, \overline{z}) \\
\downarrow \pi_B \quad \downarrow \pi_C \\
B \quad \quad \quad D \\
\downarrow \pi_A \quad \quad \quad \downarrow \\
A \leftarrow \quad \quad \quad C
\end{array}
\]
Theorem

If $\mathcal{V}$ has the amalgamation property, then $\mathcal{V}$ admits deductive interpolation.
From Amalgamation to Deductive Interpolation

Theorem

If $\mathcal{V}$ has the amalgamation property, then $\mathcal{V}$ admits deductive interpolation.

Proof.

Suppose that $\mathcal{V}$ has the amalgamation property.
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If $\mathcal{V}$ has the amalgamation property, then $\mathcal{V}$ admits deductive interpolation.

Proof.

Suppose that $\mathcal{V}$ has the amalgamation property. Given $\Sigma(\overline{x}, \overline{y})$, define

$$\Theta = C_{g_{F(\overline{x}, \overline{y})}}(\Sigma),$$

and

$$\Pi = \Theta \cap F(\overline{y}).$$

Since $\Theta \cap F(\overline{y}) = \Psi \cap F(\overline{y})$, there exists $\Phi \in \text{Con} F(\overline{x}, \overline{y}, \overline{z})$ satisfying $\Theta = \Phi \cap F(\overline{x}, \overline{y})$ and $\Psi = \Phi \cap F(\overline{y}, \overline{z})$.

But $\Sigma \models_{\mathcal{V}} \Pi$ and for any $\epsilon(\overline{y}, \overline{z})$, $\Sigma \models_{\mathcal{V}} \epsilon = \Rightarrow \epsilon \in \Psi = \Phi \cap F(\overline{y}, \overline{z}) = \Rightarrow \Pi \models_{\mathcal{V}} \epsilon.$
Theorem

If $\mathcal{V}$ has the amalgamation property, then $\mathcal{V}$ admits deductive interpolation.

Proof.

Suppose that $\mathcal{V}$ has the amalgamation property. Given $\Sigma(\overline{x}, \overline{y})$, define

$$\Theta = C_{g_{F(\overline{x}, \overline{y})}}(\Sigma), \quad \Pi = \Theta \cap F(\overline{y})^2,$$

where

$$\Pi = \Theta \cap F(\overline{y})^2.$$
From Amalgamation to Deductive Interpolation

**Theorem**

*If* $\mathcal{V}$ *has the amalgamation property, then* $\mathcal{V}$ *admits deductive interpolation.*

**Proof.**

Suppose that $\mathcal{V}$ has the amalgamation property. Given $\Sigma(x, y)$, define

$$
\Theta = Cg_{F(x,y)}(\Sigma), \quad \Pi = \Theta \cap F(y)^2, \quad \text{and} \quad \Psi = Cg_{F(y,z)}(\Pi).
$$
Theorem

If $\mathcal{V}$ has the amalgamation property, then $\mathcal{V}$ admits deductive interpolation.

Proof.

Suppose that $\mathcal{V}$ has the amalgamation property. Given $\Sigma(\bar{x}, \bar{y})$, define

$$
\Theta = \mathcal{C}_{\mathcal{F}(\bar{x}, \bar{y})}(\Sigma), \quad \Pi = \Theta \cap F(\bar{y})^2, \quad \text{and} \quad \Psi = \mathcal{C}_{\mathcal{F}(\bar{y}, \bar{z})}(\Pi).
$$

Since $\Theta \cap F(\bar{y})^2 = \Psi \cap F(\bar{y})^2$, 

From Amalgamation to Deductive Interpolation

Theorem

If $\mathcal{V}$ has the amalgamation property, then $\mathcal{V}$ admits deductive interpolation.

Proof.

Suppose that $\mathcal{V}$ has the amalgamation property. Given $\Sigma(x, y)$, define

$$\Theta = Cg_{F(x, y)}(\Sigma), \quad \Pi = \Theta \cap F(y)^2, \quad \text{and} \quad \Psi = Cg_{F(y, z)}(\Pi).$$

Since $\Theta \cap F(y)^2 = \Psi \cap F(y)^2$, there exists $\Phi \in Con F(x, y, z)$ satisfying

$$\Theta = \Phi \cap F(x, y)^2 \quad \text{and} \quad \Psi = \Phi \cap F(y, z)^2.$$
Theorem

If $\mathcal{V}$ has the amalgamation property, then $\mathcal{V}$ admits deductive interpolation.

Proof.

Suppose that $\mathcal{V}$ has the amalgamation property. Given $\Sigma(\bar{x}, \bar{y})$, define

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\Theta = C_{g_{F(\bar{x}, \bar{y})}}(\Sigma), \quad \Pi = \Theta \cap F(\bar{y})^2, \quad \text{and} \quad \Psi = C_{g_{F(\bar{y}, \bar{z})}}(\Pi).
$$

Since $\Theta \cap F(\bar{y})^2 = \Psi \cap F(\bar{y})^2$, there exists $\Phi \in \text{Con} F(\bar{x}, \bar{y}, \bar{z})$ satisfying

$$
\Theta = \Phi \cap F(\bar{x}, \bar{y})^2 \quad \text{and} \quad \Psi = \Phi \cap F(\bar{y}, \bar{z})^2.
$$

But $\Sigma \models_{\mathcal{V}} \Pi$
From Amalgamation to Deductive Interpolation

Theorem

If $\mathcal{V}$ has the amalgamation property, then $\mathcal{V}$ admits deductive interpolation.

Proof.

Suppose that $\mathcal{V}$ has the amalgamation property. Given $\Sigma(x, y)$, define

$$
\Theta = C_g_{F(x, y)}(\Sigma), \quad \Pi = \Theta \cap F(y)^2, \quad \text{and} \quad \Psi = C_g_{F(y, z)}(\Pi).
$$

Since $\Theta \cap F(y)^2 = \Psi \cap F(y)^2$, there exists $\Phi \in \text{Con } F(x, y, z)$ satisfying

$$
\Theta = \Phi \cap F(x, y)^2 \quad \text{and} \quad \Psi = \Phi \cap F(y, z)^2.
$$

But $\Sigma \models \Pi$ and for any $\varepsilon(y, z)$,

$$
\Sigma \models \varepsilon \quad \Rightarrow
$$
From Amalgamation to Deductive Interpolation

**Theorem**

If $\mathcal{V}$ has the amalgamation property, then $\mathcal{V}$ admits deductive interpolation.

**Proof.**

Suppose that $\mathcal{V}$ has the amalgamation property. Given $\Sigma(x, y)$, define

$$\Theta = Cg_{F(x, y)}(\Sigma), \quad \Pi = \Theta \cap F(y)^2, \quad \text{and} \quad \Psi = Cg_{F(y, z)}(\Pi).$$

Since $\Theta \cap F(y)^2 = \Psi \cap F(y)^2$, there exists $\Phi \in \text{Con } F(x, y, z)$ satisfying

$$\Theta = \Phi \cap F(x, y)^2 \quad \text{and} \quad \Psi = \Phi \cap F(y, z)^2.$$

But $\Sigma \models_{\mathcal{V}} \Pi$ and for any $\varepsilon(y, z)$,

$$\Sigma \models_{\mathcal{V}} \varepsilon \implies \varepsilon \in \Psi = \Phi \cap F(y, z)^2.$$
From Amalgamation to Deductive Interpolation

**Theorem**

If $\mathcal{V}$ has the amalgamation property, then $\mathcal{V}$ admits deductive interpolation.

**Proof.**

Suppose that $\mathcal{V}$ has the amalgamation property. Given $\Sigma(x, y)$, define

$$\Theta = C_{g_{F(x, y)}}(\Sigma), \quad \Pi = \Theta \cap F(y)^2, \quad \text{and} \quad \Psi = C_{g_{F(y, z)}}(\Pi).$$

Since $\Theta \cap F(y)^2 = \Psi \cap F(y)^2$, there exists $\Phi \in \text{Con} F(x, y, z)$ satisfying

$$\Theta = \Phi \cap F(x, y)^2 \quad \text{and} \quad \Psi = \Phi \cap F(y, z)^2.$$ 

But $\Sigma \models \Pi$ and for any $\varepsilon(y, z)$,

$$\Sigma \models \varepsilon \quad \Longrightarrow \quad \varepsilon \in \Psi = \Phi \cap F(y, z)^2 \quad \Longrightarrow \quad \Pi \models \varepsilon.$$
$\mathcal{V}$ has the **extension property** if whenever $\Sigma(\overline{x}, \overline{y}), \Pi(\overline{y}, \overline{z}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z})$, there exists a set of equations $\Delta(\overline{y}, \overline{z})$ such that $\Sigma(\overline{x}, \overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z})$ and $\Delta(\overline{y}, \overline{z}), \Pi(\overline{y}, \overline{z}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z})$.

Note. The extension property may be viewed as a local deduction theorem where $\Delta$ is understood as "$\Pi \rightarrow \varepsilon$".

E.g., for Heyting algebras, if $\Sigma(\overline{x}, \overline{y}), \Pi(\overline{y}, \overline{z}) \models_{\mathcal{V}} \alpha(\overline{y}, \overline{z}) \approx \beta(\overline{y}, \overline{z})$, then we can assume that $\Pi$ is finite and let $\Delta$ consist of the single equation $\top \approx \bigwedge \{ \gamma \leftrightarrow \delta | \gamma \approx \delta \in \Pi \} \rightarrow (\alpha \leftrightarrow \beta)$.
The Extension Property

\[ V \text{ has the extension property if whenever } \Sigma(\bar{x}, \bar{y}), \Pi(\bar{y}, \bar{z}) \models_V \epsilon(\bar{y}, \bar{z}), \]
there exists a set of equations \( \Delta(\bar{y}, \bar{z}) \) such that

\[
\Sigma(\bar{x}, \bar{y}) \models_V \Delta(\bar{y}, \bar{z}) \quad \text{and} \quad \Delta(\bar{y}, \bar{z}), \Pi(\bar{y}, \bar{z}) \models_V \epsilon(\bar{y}, \bar{z}).
\]
\( \mathcal{V} \) has the \textbf{extension property} if whenever \( \Sigma(x, y), \Pi(y, z) \models \varepsilon(y, z) \), there exists a set of equations \( \Delta(y, z) \) such that

\[
\Sigma(x, y) \models \Delta(y, z) \quad \text{and} \quad \Delta(y, z), \Pi(y, z) \models \varepsilon(y, z).
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\textbf{Note.} The extension property may be viewed as a local deduction theorem where \( \Delta \) is understood as \( \Pi \rightarrow \varepsilon \).
The Extension Property

\( \mathcal{V} \) has the \textbf{extension property} if whenever \( \Sigma(\overline{x}, \overline{y}), \Pi(\overline{y}, \overline{z}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}) \), there exists a set of equations \( \Delta(\overline{y}, \overline{z}) \) such that

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\textbf{Note.} The extension property may be viewed as a local deduction theorem where \( \Delta \) is understood as “\( \Pi \rightarrow \varepsilon \)”. E.g., for Heyting algebras, if

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\Sigma(\overline{x}, \overline{y}), \Pi(\overline{y}, \overline{z}) \models_{\mathcal{H}_{\mathcal{A}}} \alpha(\overline{y}, \overline{z}) \approx \beta(\overline{y}, \overline{z}),
\]
\( \mathcal{V} \) has the **extension property** if whenever \( \Sigma(\overline{x}, \overline{y}), \Pi(\overline{y}, \overline{z}) \models_\mathcal{V} \varepsilon(\overline{y}, \overline{z}) \), there exists a set of equations \( \Delta(\overline{y}, \overline{z}) \) such that

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**Note.** The extension property may be viewed as a local deduction theorem where \( \Delta \) is understood as “\( \Pi \rightarrow \varepsilon \)”. E.g., for Heyting algebras, if

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\( \mathcal{V} \) has the **extension property** if whenever \( \Sigma(\overline{x}, \overline{y}), \Pi(\overline{y}, \overline{z}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}) \), there exists a set of equations \( \Delta(\overline{y}, \overline{z}) \) such that

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**Note.** The extension property may be viewed as a local deduction theorem where \( \Delta \) is understood as “\( \Pi \rightarrow \varepsilon \)”. E.g., for Heyting algebras, if

\[
\Sigma(\overline{x}, \overline{y}), \Pi(\overline{y}, \overline{z}) \models_{\mathcal{HA}} \alpha(\overline{y}, \overline{z}) \approx \beta(\overline{y}, \overline{z}),
\]

then we can assume that \( \Pi \) is finite and let \( \Delta \) consist of the single equation

\[
\top \approx \bigwedge \{ \gamma \leftrightarrow \delta \mid \gamma \approx \delta \in \Pi \} \rightarrow (\alpha \leftrightarrow \beta).
\]
Theorem (Bacsich, Czelakowski, Pigozzi, Ono, . . .)

The following are equivalent:

(1) \( \mathcal{V} \) has the extension property: whenever \( \Sigma(\overline{x}, \overline{y}), \Pi(\overline{y}, \overline{z}) \vdash_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}) \), there exists a set of equations \( \Delta(\overline{y}, \overline{z}) \) such that

\[
\Sigma(\overline{x}, \overline{y}) \vdash_{\mathcal{V}} \Delta(\overline{y}, \overline{z}) \quad \text{and} \quad \Delta(\overline{y}, \overline{z}), \Pi(\overline{y}, \overline{z}) \vdash_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}).
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The following are equivalent:

1. \( \forall \) has the extension property: whenever \( \Sigma(\bar{x}, \bar{y}), \Pi(\bar{y}, \bar{z}) \models_\forall \varepsilon(\bar{y}, \bar{z}) \), there exists a set of equations \( \Delta(\bar{y}, \bar{z}) \) such that

\[
\Sigma(\bar{x}, \bar{y}) \models_\forall \Delta(\bar{y}, \bar{z}) \quad \text{and} \quad \Delta(\bar{y}, \bar{z}), \Pi(\bar{y}, \bar{z}) \models_\forall \varepsilon(\bar{y}, \bar{z}).
\]

2. For any \( \Theta \in \text{Con } F(\bar{x}, \bar{y}) \) and \( \Psi \in \text{Con } F(\bar{y}, \bar{z}) \),

\[
C_{g_{F(\bar{x}, \bar{y}, \bar{z})}}(\Theta \cup \Psi) \cap F(\bar{y}, \bar{z})^2 = C_{g_{F(\bar{y}, \bar{z})}}((C_{g_{F(\bar{x}, \bar{y}, \bar{z})}}(\Theta) \cap F(\bar{y}, \bar{z})^2) \cup \Psi).
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Theorem (Bacsich, Czelakowski, Pigozzi, Ono, …)

The following are equivalent:

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   \]

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   \[
   \text{Cg}_{\mathbf{F}(\overline{x}, \overline{y}, \overline{z})}(\Theta \cup \Psi) \cap \mathbf{F}(\overline{y}, \overline{z})^2 = \text{Cg}_{\mathbf{F}(\overline{y}, \overline{z})}((\text{Cg}_{\mathbf{F}(\overline{x}, \overline{y}, \overline{z})}(\Theta) \cap \mathbf{F}(\overline{y}, \overline{z})^2) \cup \Psi).
   \]

3. \( \mathcal{V} \) has the congruence extension property: for any subalgebra \( \mathbf{B} \) of \( \mathbf{A} \in \mathcal{V} \) and \( \Theta \in \text{Con } \mathbf{B} \), there exists \( \Phi \in \text{Con } \mathbf{A} \) with \( \Theta = \Phi \cap \mathbf{B}^2 \).
Theorem

If $\mathcal{V}$ admits deductive interpolation and has the extension property, then it has the amalgamation property.
From Deductive Interpolation to Amalgamation

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Let $\mathcal{V}$ admit deductive interpolation and have the extension property, and consider $\Theta \in \text{Con} \mathbf{F}(x,y)$, $\Psi \in \text{Con} \mathbf{F}(y,z)$ with $\Theta \cap F(y)^2 = \Psi \cap F(y)^2$. Define $\Phi = C_{\mathbf{F}(x,y,z)}(\Theta \cup \Psi)$. Then by the extension property,

$$\Phi \cap F(y,z)^2 = C_{\mathbf{F}(y,z)}\left((C_{\mathbf{F}(x,y,z)}(\Theta) \cap F(y,z)^2) \cup \Psi\right).$$
**Theorem**

*If $\mathcal{V}$ admits deductive interpolation and has the extension property, then it has the amalgamation property.*

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Let $\mathcal{V}$ admit deductive interpolation and have the extension property, and consider $\Theta \in \text{Con} \mathbf{F}(\bar{x}, \bar{y})$, $\Psi \in \text{Con} \mathbf{F}(\bar{y}, \bar{z})$ with $\Theta \cap F(\bar{y})^2 = \Psi \cap F(\bar{y})^2$. Define $\Phi = \text{Cg}_{\mathbf{F}(\bar{x}, \bar{y}, \bar{z})}(\Theta \cup \Psi)$. Then by the extension property,

$$
\Phi \cap F(\bar{y}, \bar{z})^2 = \text{Cg}_{\mathbf{F}(\bar{x}, \bar{y}, \bar{z})} \left( (\text{Cg}_{\mathbf{F}(\bar{x}, \bar{y}, \bar{z})}(\Theta) \cap F(\bar{y}, \bar{z})^2) \cup \Psi \right).
$$

But then, using deductive interpolation,

$$
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From Deductive Interpolation to Amalgamation

**Theorem**

If \( \mathcal{V} \) admits deductive interpolation and has the extension property, then it has the amalgamation property.

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Let \( \mathcal{V} \) admit deductive interpolation and have the extension property, and consider \( \Theta \in \text{Con} \mathbf{F}(\bar{x}, \bar{y}) \), \( \Psi \in \text{Con} \mathbf{F}(\bar{y}, \bar{z}) \) with \( \Theta \cap F(\bar{y})^2 = \Psi \cap F(\bar{y})^2 \).

Define \( \Phi = Cg_{F(\bar{x}, \bar{y}, \bar{z})}(\Theta \cup \Psi) \). Then by the extension property,

\[
\Phi \cap F(\bar{y}, \bar{z})^2 = Cg_{F(\bar{y}, \bar{z})}((Cg_{F(\bar{x}, \bar{y}, \bar{z})}(\Theta) \cap F(\bar{y}, \bar{z})^2) \cup \Psi).
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\]
From Deductive Interpolation to Amalgamation

**Theorem**

If $\mathcal{V}$ admits deductive interpolation and has the extension property, then it has the amalgamation property.

**Proof.**

Let $\mathcal{V}$ admit deductive interpolation and have the extension property, and consider $\Theta \in \text{Con } \mathbf{F}(\overline{x}, \overline{y})$, $\Psi \in \text{Con } \mathbf{F}(\overline{y}, \overline{z})$ with $\Theta \cap F(\overline{y})^2 = \Psi \cap F(\overline{y})^2$.

Define $\Phi = \text{Cg}_{\mathbf{F}(\overline{x}, \overline{y}, \overline{z})} (\Theta \cup \Psi)$. Then by the extension property,

$$\Phi \cap F(\overline{y}, \overline{z})^2 = \text{Cg}_{\mathbf{F}(\overline{y}, \overline{z})} ((\text{Cg}_{\mathbf{F}(\overline{x}, \overline{y}, \overline{z})} (\Theta) \cap F(\overline{y}, \overline{z})^2) \cup \Psi).$$

But then, using deductive interpolation,

$$\Phi \cap F(\overline{y}, \overline{z})^2 = \text{Cg}_{\mathbf{F}(\overline{y}, \overline{z})} (\text{Cg}_{\mathbf{F}(\overline{y}, \overline{z})} (\Theta \cap F(\overline{y})^2) \cup \Psi) = \Psi \cap F(\overline{y})^2,$$

and symmetrically, $\Phi \cap F(\overline{x}, \overline{y})^2 = \Theta \cap F(\overline{z})^2$.  

\[\square\]
A Bridge Theorem

Theorem (Jónsson, Pigozzi, Bacsich, Czelakowski . . . )

A variety with the congruence extension property admits deductive interpolation if and only if it has the amalgamation property.
We can cross this bridge in both directions,
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- interpolation has been proved for many intermediate and modal logics by establishing the amalgamation property (often using dualities) for corresponding varieties of Heyting and modal algebras;
We can cross this bridge in both directions, e.g.,

- interpolation has been proved for many intermediate and modal logics by establishing the amalgamation property (often using dualities) for corresponding varieties of Heyting and modal algebras;

- the amalgamation property has been established for many varieties of residuated lattices by proving interpolation (often using proof theory) for corresponding substructural logics.
Further Relationships... 

\[
\begin{align*}
\text{CEP} + \text{FAP} \\
\Downarrow \\
\text{TIP} \quad \implies \quad \text{AP} \quad \implies \quad \text{WAP} \quad \implies \quad \text{FAP} \\
\Downarrow \\
\text{MIP} \quad \implies \quad \text{RP} \quad \implies \quad \text{CDIP} \quad \implies \quad \text{DIP} \\
\Downarrow \\
\text{DIP} + \text{EP}
\end{align*}
\]
Can we describe uniform interpolation algebraically?
A logic $\mathcal{V}$ admits **deductive interpolation** if for any set of equations $\Sigma(x, y)$, there exists a set of equations $\Delta(y)$ such that

$$\Sigma(x, y) \models_{\mathcal{V}} \varepsilon(y, z) \iff \Delta(y) \models_{\mathcal{V}} \varepsilon(y, z).$$
\( \mathcal{V} \) admits **right uniform deductive interpolation** if for any *finite* set of equations \( \Sigma(\overline{x}, \overline{y}) \), there exists a *finite* set of equations \( \Delta(\overline{y}) \) such that

\[
\Sigma(\overline{x}, \overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}) \iff \Delta(\overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}).
\]

But what does the extra ingredient in (ii) mean algebraically?
$\mathcal{V}$ admits **right uniform deductive interpolation** if for any *finite* set of equations $\Sigma(x, \overline{y})$, there exists a *finite* set of equations $\Delta(\overline{y})$ such that

$$\Sigma(x, \overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}) \iff \Delta(\overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z})$$.

**Lemma**

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$\mathcal{V}$ admits **right uniform deductive interpolation** if for any *finite* set of equations $\Sigma(x, y)$, there exists a *finite* set of equations $\Delta(y)$ such that

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$$

**Lemma**

$\mathcal{V}$ admits right uniform deductive interpolation if and only if

1. $\mathcal{V}$ admits deductive interpolation;
Right Uniform Deductive Interpolation

\( \mathcal{V} \) admits right uniform deductive interpolation if for any finite set of equations \( \Sigma(\overline{x}, \overline{y}) \), there exists a finite set of equations \( \Delta(\overline{y}) \) such that

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Lemma

\( \mathcal{V} \) admits right uniform deductive interpolation if and only if

(i) \( \mathcal{V} \) admits deductive interpolation;

(ii) for any finite set of equations \( \Sigma(\overline{x}, \overline{y}) \), there exists a finite set of equations \( \Delta(\overline{y}) \) such that

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**Lemma**

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But what does the extra ingredient in (ii) mean *algebraically*?
Finitely Generated Congruences

The finitely generated congruences of an algebra $A$ always form a join-semilattice $K\text{Con} A$. 
The finitely generated congruences of an algebra $A$ always form a join-semilattice $K\text{Con} A$.

Recall that the inclusion map $i: F(\overline{y}) \to F(\overline{x}, \overline{y})$ “lifts” to the maps

\[
i^*: \text{Con} F(\overline{y}) \to \text{Con} F(\overline{x}, \overline{y}); \quad \Theta \mapsto C_{g_{F(\overline{x}, \overline{y})}}(i[\Theta])
\]

\[
i^{-1}: \text{Con} F(\overline{x}, \overline{y}) \to \text{Con} F(\overline{y}); \quad \psi \mapsto i^{-1}[\psi] = \psi \cap F(\overline{y})^2.
\]
The finitely generated congruences of an algebra $A$ always form a join-semilattice $K\text{Con} A$.

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$$i^{-1}: \text{Con} F(\overline{x}, \overline{y}) \to \text{Con} F(\overline{y}); \quad \Psi \mapsto i^{-1}[\Psi] = \Psi \cap F(\overline{y})^2.$$ 

The compact lifting of $i$ restricts $i^*$ to $K\text{Con} F(\overline{y}) \to K\text{Con} F(\overline{x}, \overline{y})$;
The **finitely generated** congruences of an algebra $A$ always form a join-semilattice $K\text{Con} A$.

Recall that the inclusion map $i: F(y) \to F(x, y)$ “lifts” to the maps

\[
i^*: \text{Con} F(y) \to \text{Con} F(x, y); \quad \Theta \mapsto C_{g_{F(x, y)}}(i[\Theta])
\]

\[
i^{-1}: \text{Con} F(x, y) \to \text{Con} F(y); \quad \psi \mapsto i^{-1}[\psi] = \psi \cap F(y)^2.
\]

The **compact lifting of $i$** restricts $i^*$ to $K\text{Con} F(y) \to K\text{Con} F(x, y)$; it has a right adjoint if $i^{-1}$ restricts to $K\text{Con} F(x, y) \to K\text{Con} F(y)$. 
Lemma

The following are equivalent:

1. For any finite set of equations $\Sigma(x, y)$, there is a finite set of equations $\Delta(y)$ such that

$$\Sigma(x, y) \models \varepsilon(y) \iff \Delta(y) \models \varepsilon(y).$$
Lemma

The following are equivalent:

1. For any finite set of equations \( \Sigma(\overline{x}, \overline{y}) \), there is a finite set of equations \( \Delta(\overline{y}) \) such that

\[
\Sigma(\overline{x}, \overline{y}) \models \nu \varepsilon(\overline{y}) \iff \Delta(\overline{y}) \models \nu \varepsilon(\overline{y}).
\]

2. For finite \( \overline{x}, \overline{y} \), the compact lifting of \( F(\overline{y}) \hookrightarrow F(\overline{x}, \overline{y}) \) has a right adjoint;
The Missing Ingredient

Lemma

The following are equivalent:

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   $\Sigma(\overline{x}, \overline{y}) \models \nu \varepsilon(\overline{y}) \iff \Delta(\overline{y}) \models \nu \varepsilon(\overline{y})$.

2. For finite $\overline{x}, \overline{y}$, the compact lifting of $F(\overline{y}) \hookrightarrow F(\overline{x}, \overline{y})$ has a right adjoint; that is,

   $\Theta \in KCon F(\overline{x}, \overline{y}) \implies \Theta \cap F(\overline{y})^2 \in KCon F(\overline{y})$. 
An algebra $A \in \mathcal{V}$ is called

- **finitely generated** if it is generated by a finite subset of $A$;
An algebra $A \in \mathcal{V}$ is called

- **finitely generated** if it is generated by a finite subset of $A$;

- **finitely presented** if it is isomorphic to $F(\overline{x})/\Theta$ for some finite set $\overline{x}$ and finitely generated congruence $\Theta$ on $F(\overline{x})$. 
An algebra $A \in \mathcal{V}$ is called

- **finitely generated** if it is generated by a finite subset of $A$;
- **finitely presented** if it is isomorphic to $F(\overline{x})/\Theta$ for some finite set $\overline{x}$ and finitely generated congruence $\Theta$ on $F(\overline{x})$.

**Useful Lemma**

*If $A \in \mathcal{V}$ is finitely presented and isomorphic to $F(\overline{y})/\Psi$ for some finite set $\overline{y}$ and congruence $\Psi$ on $F(\overline{y})$, then $\Psi$ is finitely generated.*
Coherence

Theorem (Kowalski and Metcalfe 2019)

The following are equivalent:

1. For finite \( \overline{x}, \overline{y} \), the compact lifting of \( F(\overline{y}) \hookrightarrow F(\overline{x}, \overline{y}) \) has a right adjoint; that is, \( \Theta \in K\text{Con} F(\overline{x}, \overline{y}) \implies \Theta \cap F(\overline{y})^2 \in K\text{Con} F(\overline{y}). \)

2. \( V \) is coherent: every finitely generated subalgebra of a finitely presented member of \( V \) is finitely presented.

3. The compact lifting of any homomorphism between finitely presented algebras in \( V \) has a right adjoint.

Note. Every locally finite variety is coherent.
Theorem (Kowalski and Metcalfe 2019)

The following are equivalent:

(1) For finite $\bar{x}$, $\bar{y}$, the compact lifting of $F(\bar{y}) \hookrightarrow F(\bar{x}, \bar{y})$ has a right adjoint; that is, $\Theta \in \text{KCon } F(\bar{x}, \bar{y}) \Rightarrow \Theta \cap F(\bar{y})^2 \in \text{KCon } F(\bar{y})$.

(2) $\mathcal{V}$ is coherent:

Coherence

Theorem (Kowalski and Metcalfe 2019)

The following are equivalent:

1. For finite $\bar{x}$, $\bar{y}$, the compact lifting of $F(\bar{y}) \hookrightarrow F(\bar{x}, \bar{y})$ has a right adjoint; that is, $\Theta \in \text{KCon } F(\bar{x}, \bar{y}) \implies \Theta \cap F(\bar{y})^2 \in \text{KCon } F(\bar{y})$.

2. $\mathcal{V}$ is coherent: every finitely generated subalgebra of a finitely presented member of $\mathcal{V}$ is finitely presented.

Note. Every locally finite variety is coherent.
Coherence

Theorem (Kowalski and Metcalfe 2019)

The following are equivalent:

1. For finite \( \overline{x}, \overline{y} \), the compact lifting of \( F(\overline{y}) \hookrightarrow F(\overline{x}, \overline{y}) \) has a right adjoint; that is, \( \Theta \in K\text{Con} F(\overline{x}, \overline{y}) \implies \Theta \cap F(\overline{y})^2 \in K\text{Con} F(\overline{y}) \).

2. \( \mathcal{V} \) is coherent: every finitely generated subalgebra of a finitely presented member of \( \mathcal{V} \) is finitely presented.

3. The compact lifting of any homomorphism between finitely presented algebras in \( \mathcal{V} \) has a right adjoint.

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The following are equivalent:

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2. $\mathcal{V}$ is coherent: every finitely generated subalgebra of a finitely presented member of $\mathcal{V}$ is finitely presented.

3. The compact lifting of any homomorphism between finitely presented algebras in $\mathcal{V}$ has a right adjoint.

Note. Every locally finite variety is coherent.
Proof of (1) $\iff$ (2)

We prove that $\mathcal{V}$ is coherent if and only if for any finite $\overline{x}$, $\overline{y}$,

$$\Theta \in \text{KCon } F(\overline{x}, \overline{y}) \implies \Theta \cap F(\overline{y})^2 \in \text{KCon } F(\overline{y}).$$
Proof of (1) ⇔ (2)

We prove that $\mathcal{V}$ is coherent if and only if for any finite $\overline{x}, \overline{y}$,

$$\Theta \in K\text{Con} F(\overline{x}, \overline{y}) \implies \Theta \cap F(\overline{y})^2 \in K\text{Con} F(\overline{y}).$$

($\Rightarrow$) Let $\mathcal{V}$ be coherent and consider finite $\overline{x}, \overline{y}$ and $\Theta \in K\text{Con} F(\overline{x}, \overline{y})$. 

($\Leftarrow$) Let $\mathcal{V}$ be coherent and consider finite $\overline{x}, \overline{y}$ and $\Theta \in K\text{Con} F(\overline{x}, \overline{y})$. 

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($\Rightarrow$) Let $\mathcal{V}$ be coherent and consider finite $\overline{x}$, $\overline{y}$ and $\Theta \in K\text{Con} \ F(\overline{x}, \overline{y})$. Then $F(\overline{x}, \overline{y})/\Theta$ is finitely presented.
Proof of \((1) \iff (2)\)

We prove that \(\mathcal{V}\) is coherent if and only if for any finite \(\bar{x}, \bar{y}\),

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\Theta \in \text{KCon } F(\bar{x}, \bar{y}) \implies \Theta \cap F(\bar{y})^2 \in \text{KCon } F(\bar{y}).
\]

\((\Rightarrow)\) Let \(\mathcal{V}\) be coherent and consider finite \(\bar{x}, \bar{y}\) and \(\Theta \in \text{KCon } F(\bar{x}, \bar{y})\). Then \(F(\bar{x}, \bar{y})/\Theta\) is finitely presented and, by coherence, \(F(\bar{y})/(\Theta \cap F(\bar{y})^2)\) is also finitely presented.
Proof of $(1) \iff (2)$

We prove that $\mathcal{V}$ is coherent if and only if for any finite $\bar{x}, \bar{y}$,

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($\Leftarrow$)
Proof of (1) $\Leftrightarrow$ (2)

We prove that $\mathcal{V}$ is coherent if and only if for any finite $\overline{x}, \overline{y}$,

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$(\Leftarrow)$ Let $B$ be a finitely generated subalgebra of a finitely presented $A \in \mathcal{V}$. 

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Proof of (1) $\iff$ (2)

We prove that $\mathcal{V}$ is coherent if and only if for any finite $\bar{x}$, $\bar{y}$,

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\Theta \in \text{KCon } \mathbf{F}(\bar{x}, \bar{y}) \implies \Theta \cap F(\bar{y})^2 \in \text{KCon } F(\bar{y}).
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($\Leftarrow$) Let $\mathcal{B}$ be a finitely generated subalgebra of a finitely presented $\mathbf{A} \in \mathcal{V}$. Let $\bar{x}$, $\bar{y}$ and $\bar{y}$ be finite sets generating $\mathbf{A}$ and $\mathcal{B}$, respectively.
Proof of (1) ⇔ (2)

We prove that \( \mathcal{V} \) is coherent if and only if for any finite \( \bar{x}, \bar{y} \),

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(⇒) Let \( \mathcal{V} \) be coherent and consider finite \( \bar{x}, \bar{y} \) and \( \Theta \in \text{KCon } F(\bar{x}, \bar{y}) \). Then \( F(\bar{x}, \bar{y})/\Theta \) is finitely presented and, by coherence, \( F(\bar{y})/(\Theta \cap F(\bar{y})^2) \) is also finitely presented. So, by the useful lemma, \( \Theta \cap F(\bar{y})^2 \in \text{KCon } F(\bar{y}) \).

(⇐) Let \( B \) be a finitely generated subalgebra of a finitely presented \( A \in \mathcal{V} \). Let \( \bar{x}, \bar{y} \) and \( \bar{y} \) be finite sets generating \( A \) and \( B \), respectively. The natural onto homomorphism \( h: F(\bar{x}, \bar{y}) \to A \) restricts to \( k: F(\bar{y}) \to B \),
Proof of $ (1) \iff (2) $ 

We prove that $ \mathcal{V} $ is coherent if and only if for any finite $ \bar{x}, \bar{y} $,

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($ \Rightarrow $) Let $ \mathcal{V} $ be coherent and consider finite $ \bar{x}, \bar{y} $ and $ \Theta \in \text{KCon } F(\bar{x}, \bar{y}) $. Then $ F(\bar{x}, \bar{y})/\Theta $ is finitely presented and, by coherence, $ F(\bar{y})/(\Theta \cap F(\bar{y})^2) $ is also finitely presented. So, by the useful lemma, $ \Theta \cap F(\bar{y})^2 \in \text{KCon } F(\bar{y}) $.

($ \Leftarrow $) Let $ \mathcal{B} $ be a finitely generated subalgebra of a finitely presented $ \mathcal{A} \in \mathcal{V} $. Let $ \bar{x}, \bar{y} $ and $ \bar{y} $ be finite sets generating $ \mathcal{A} $ and $ \mathcal{B} $, respectively. The natural onto homomorphism $ h: F(\bar{x}, \bar{y}) \to \mathcal{A} $ restricts to $ k: F(\bar{y}) \to \mathcal{B} $, which must also be onto.
We prove that $\mathcal{V}$ is coherent if and only if for any finite $\overline{x}$, $\overline{y}$,

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($\Leftarrow$) Let $B$ be a finitely generated subalgebra of a finitely presented $A \in \mathcal{V}$. Let $\overline{x}$, $\overline{y}$ and $\overline{y}$ be finite sets generating $A$ and $B$, respectively. The natural onto homomorphism $h: F(\overline{x}, \overline{y}) \to A$ restricts to $k: F(\overline{y}) \to B$, which must also be onto. But $\ker h \in \text{KCon } F(\overline{x}, \overline{y})$ by the useful lemma,
We prove that $\mathcal{V}$ is coherent if and only if for any finite $\bar{x}$, $\bar{y}$,

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($\Rightarrow$) Let $\mathcal{V}$ be coherent and consider finite $\bar{x}$, $\bar{y}$ and $\Theta \in \text{KCon } F(\bar{x}, \bar{y})$. Then $F(\bar{x}, \bar{y})/\Theta$ is finitely presented and, by coherence, $F(\bar{y})/(\Theta \cap F(\bar{y})^2)$ is also finitely presented. So, by the useful lemma, $\Theta \cap F(\bar{y})^2 \in \text{KCon } F(\bar{y})$.

($\Leftarrow$) Let $B$ be a finitely generated subalgebra of a finitely presented $A \in \mathcal{V}$. Let $\bar{x}$, $\bar{y}$ and $\bar{y}$ be finite sets generating $A$ and $B$, respectively. The natural onto homomorphism $h: F(\bar{x}, \bar{y}) \to A$ restricts to $k: F(\bar{y}) \to B$, which must also be onto. But $\ker h \in \text{KCon } F(\bar{x}, \bar{y})$ by the useful lemma, so, using the assumption, $\ker k = \ker h \cap F(\bar{y})^2 \in \text{KCon } F(\bar{y})$. 
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$(\Leftarrow)$ Let $B$ be a finitely generated subalgebra of a finitely presented $A \in \mathcal{V}$. Let $\bar{x}$, $\bar{y}$ and $\bar{y}$ be finite sets generating $A$ and $B$, respectively. The natural onto homomorphism $h: F(\bar{x}, \bar{y}) \to A$ restricts to $k: F(\bar{y}) \to B$, which must also be onto. But $\ker h \in \text{KCon } F(\bar{x}, \bar{y})$ by the useful lemma, so, using the assumption, $\ker k = \ker h \cap F(\bar{y})^2 \in \text{KCon } F(\bar{y})$. Hence, since $B$ is isomorphic to $F(\bar{y})/\ker k$, it is finitely presented. \qed
Theorem (Kowalski and Metcalfe 2019)

A variety with the congruence extension property admits right uniform deductive interpolation if and only if it has the amalgamation property and is coherent.
$\mathcal{V}$ has **left uniform deductive interpolation** if for any finite set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a finite set of equations $\Delta(\overline{y})$ such that

$$\Pi(\overline{y}, \overline{z}) \models_{\mathcal{V}} \Sigma(\overline{x}, \overline{y}) \iff \Pi(\overline{y}, \overline{z}) \models_{\mathcal{V}} \Delta(\overline{y}).$$
\( \mathcal{V} \) has **left uniform deductive interpolation** if for any finite set of equations \( \Sigma(\bar{x}, \bar{y}) \), there exists a finite set of equations \( \Delta(\bar{y}) \) such that

\[
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\]

### Lemma

**The following are equivalent:**

1. \( \mathcal{V} \) has **left uniform deductive interpolation**.
2. \( \mathcal{V} \) has deductive interpolation, and for finite sets \( \bar{x}, \bar{y} \), the compact lifting of \( F(\bar{y}) \hookrightarrow F(\bar{x}, \bar{y}) \) has a left adjoint.
3. If \( \mathcal{V} \) is locally finite, these are equivalent to
   - \( \mathcal{V} \) has deductive interpolation, is congruence distributive, and for finite sets \( \bar{x}, \bar{y} \), the compact lifting of \( F(\bar{y}) \hookrightarrow F(\bar{x}, \bar{y}) \) preserves meets.
Left Uniform Deductive Interpolation

\( \mathcal{V} \) has **left uniform deductive interpolation** if for any finite set of equations \( \Sigma(\overline{x}, \overline{y}) \), there exists a finite set of equations \( \Delta(\overline{y}) \) such that

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\]

**Lemma**

The following are equivalent:

1. \( \mathcal{V} \) has left uniform deductive interpolation.

2. \( \mathcal{V} \) has deductive interpolation, and for finite sets \( \overline{x}, \overline{y} \), the compact lifting of \( F(\overline{y}) \hookrightarrow F(\overline{x}, \overline{y}) \) has a left adjoint.

Moreover, if \( \mathcal{V} \) is locally finite, these are equivalent to

3. \( \mathcal{V} \) has deductive interpolation, is congruence distributive, and for finite sets \( \overline{x}, \overline{y} \), the compact lifting of \( F(\overline{y}) \hookrightarrow F(\overline{x}, \overline{y}) \) preserves meets.
An **implicative semilattice** is an algebraic structure \( \langle A, \land, \rightarrow \rangle \) satisfying

(i) \( \langle A, \land \rangle \) is a semilattice;
(ii) \( a \land b \leq c \iff a \leq b \rightarrow c \) for all \( a, b, c \in A \).
An **implicative semilattice** is an algebraic structure \( \langle A, \wedge, \rightarrow \rangle \) satisfying

(i) \( \langle A, \wedge \rangle \) is a semilattice;

(ii) \( a \wedge b \leq c \iff a \leq b \rightarrow c \) for all \( a, b, c \in A \).

The variety \( \mathcal{ISL} \) of implicative semilattices forms an equivalent algebraic semantics for the implication-conjunction fragment of intuitionistic logic.
An implicative semilattice is an algebraic structure $\langle A, \wedge, \rightarrow \rangle$ satisfying
(i) $\langle A, \wedge \rangle$ is a semilattice; (ii) $a \wedge b \leq c \iff a \leq b \rightarrow c$ for all $a, b, c \in A$.

The variety $\mathcal{ISL}$ of implicative semilattices forms an equivalent algebraic semantics for the implication-conjunction fragment of intuitionistic logic that admits right but not left uniform deductive interpolation.
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Consider \( \Sigma = \{ \top \approx ((y_1 \rightarrow x) \land (y_2 \rightarrow x)) \rightarrow x \} \)
An Example

An **implicative semilattice** is an algebraic structure \( \langle A, \land, \rightarrow \rangle \) satisfying

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Consider \( \Sigma = \{ \top \approx ((y_1 \rightarrow x) \land (y_2 \rightarrow x)) \rightarrow x \} \) and observe that

\[
\{ \top \approx y_1 \} \models_{\mathcal{ISL}} \Sigma \quad \text{and} \quad \{ \top \approx y_2 \} \models_{\mathcal{ISL}} \Sigma,
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An **implicative semilattice** is an algebraic structure $\langle A, \land, \rightarrow \rangle$ satisfying

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\]

but there is no finite $\Delta(y_1, y_2)$ satisfying

\[
\Delta \models_{\mathcal{ISL}} \Sigma, \quad \{ \top \approx y_1 \} \models_{\mathcal{ISL}} \Delta, \quad \text{and} \quad \{ \top \approx y_2 \} \models_{\mathcal{ISL}} \Delta,
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An Example

An implicative semilattice is an algebraic structure $\langle A, \land, \to \rangle$ satisfying
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The variety $\mathcal{ISL}$ of implicative semilattices forms an equivalent algebraic semantics for the implication-conjunction fragment of intuitionistic logic that admits right but not left uniform deductive interpolation.

Consider $\Sigma = \{ \top \approx ((y_1 \to x) \land (y_2 \to x)) \to x \}$ and observe that
$$\{ \top \approx y_1 \} \models_{\mathcal{ISL}} \Sigma$$
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but there is no finite $\Delta(y_1, y_2)$ satisfying
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since such a $\Delta$ would give a definition of $y_1 \lor y_2$ for implicative semilattices.
Lemma

The following are equivalent:

1. The compact lifting of any homomorphism between finitely presented algebras in $\mathcal{V}$ has a left adjoint.
Lemma

The following are equivalent:

(1) The compact lifting of any homomorphism between finitely presented algebras in $\mathcal{V}$ has a left adjoint.

(2) The compact lifting of any inclusion $F(y) \hookrightarrow F(x, y)$ has a left adjoint, and for any finite $\overline{x}$, $\text{KCon } F(\overline{x})$ is a Brouwerian join-semilattice (i.e., $\lor$ is residuated).
Lemma

The following are equivalent:

1. The compact lifting of any homomorphism between finitely presented algebras in $\mathcal{V}$ has a left adjoint.

2. The compact lifting of any inclusion $\mathcal{F}(\bar{y}) \hookrightarrow \mathcal{F}(\bar{x}, \bar{y})$ has a left adjoint, and for any finite $\bar{x}$, $K_{\text{Con}} \mathcal{F}(\bar{x})$ is a Brouwerian join-semilattice (i.e., $\lor$ is residuated).

Note. The condition that $K_{\text{Con}} \mathcal{F}(\omega)$ is a Brouwerian join-semilattice is equivalent to the property of equationally definable principal congruences.
Theorem (Wheeler 1976)

The theory of $\mathcal{V}$ has a model completion if and only if $\mathcal{V}$ is coherent, admits the amalgamation property, and has the conservative congruence extension property for its finitely presented members.
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The theory of $\mathcal{V}$ has a model completion if and only if $\mathcal{V}$ is coherent, admits the amalgamation property, and has the conservative congruence extension property for its finitely presented members.

Theorem (Ghilardi and Zawadowski 2002)

Suppose that

(i) $\mathcal{V}$ is coherent and has the amalgamation property;

(ii) for finite sets $x$, $y$, the compact lifting of $F(y) \hookrightarrow F(x, y)$ has a left adjoint, and $K\text{Con} F(x)$ is dually Brouwerian.
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Model Completions

Theorem (Wheeler 1976)

The theory of $\mathcal{V}$ has a model completion if and only if $\mathcal{V}$ is coherent, admits the amalgamation property, and has the conservative congruence extension property for its finitely presented members.

Theorem (Ghilardi and Zawadowski 2002)

Suppose that

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Then the theory of $\mathcal{V}$ has a model completion.
Suppose that
(i) $V$ has left and right uniform interpolation;
(ii) For any finite $x$ and finite set of equations $\Sigma(x)$, $\Delta(x)$ with $x$ finite, there exists a finite set of equations $\Pi(x)$ such that for any finite set of equations $\Gamma(x)$,
$\Gamma, \Sigma \models V \Delta \iff \Gamma \models V \Pi$.
Then the theory of $V$ has a model completion.
Theorem (van Gool, Metcalfe, and Tsinakis 2017)

Suppose that

(i) \( \forall \) has left and right uniform interpolation;

Then the theory of \( \forall \) has a model completion.
Theorem (van Gool, Metcalfe, and Tsinakis 2017)

Suppose that

(i) \( \mathcal{V} \) has left and right uniform interpolation;

(ii) For any finite \( \overline{x} \) and finite set of equations \( \Sigma(\overline{x}) \), \( \Delta(\overline{x}) \) with \( \overline{x} \) finite, there exists a finite set of equations \( \Pi(\overline{x}) \) such that for any finite set of equations \( \Gamma(\overline{x}) \),

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Theorem (van Gool, Metcalfe, and Tsinakis 2017)

Suppose that

(i) $\mathcal{V}$ has left and right uniform interpolation;

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Tomorrow

We will...
We will…

- investigate uniform interpolation for some particular case studies
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- investigate uniform interpolation for some particular case studies
- provide a general criterion for establishing the failure of coherence
- pose some open problems and challenges.
S. Ghilardi and M. Zawadowski.  

Uniform interpolation and compact congruences.  


G. Metcalfe, F. Montagna, and C. Tsinakis.  
Amalgamation and interpolation in ordered algebras.  

W.H. Wheeler.  
Model-companions and definability in existentially complete structures.  