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Bivariate Hermite Interpolation

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Nice
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Hermite Interpolation

- Choose a number of variables n (default value $n := 2$)
- Choose a degree d, a number of points r
- Choose a multiplicity sequence $\underline{m} := (m_1, \cdots, m_r)$
- Choose r points $\underline{p} := (p_1, ..., p_r)$ in $\mathbb{R}^n$ (very general)
- Search a polynomial of degree at most d with prescribed values and derivatives up to order $m_i - 1$ at p_i for each i.
- This amounts to search a section of the generalized evaluation $ev : \mathbb{R}[x_1, x_2]_d \to \mathbb{R}_{\underline{p},\underline{m}}$

Easy remarks

- We are only interested in the existence of a section
- hence in the surjectivity of our map $ev : \mathbb{R}[x_1, x_2]_d \to \mathbb{R}_{\underline{p},\underline{m}}$.
- Necessary condition on dimensions : $\chi := \frac{(d+1)(d+2)}{2} - \sum_i \frac{m_i(m_i+1)}{2} \geq 0$.
- Is it sufficient?
- In other words, is the dimension of Ker ev as expected?

The kernel of *ev*

- consists of equations of curves of degree d
- passing through p_i with multiplicity m_i.
- Thus we search for the dimension of this "linear" system of curves.

Obstructions I

- $r = 2, m_1 + m_2 \geq d + 2$
(perfectly compatible with $\chi \geq 0$):
- interpolation for (d, m_1, m_2) is not possible on the line (p_1, p_2).

Obstructions II

- $r = 5, m_1 + \cdots + m_5 \geq 2d + 2$
  (perfectly compatible with $\chi \geq 0$):
- interpolation for $(d, \underline{m})$ is not possible on the conic through $(p_1, \cdots, p_5)$.

Analysis or geometry?

Four surveys around 2000 : 75 percent geometry!

- Miranda (Notices 1999)
- Ciliberto (European Congress 2000)
- Harbourne (Napoli 2000)
- Lorentz : Hermite interpolation by algebraic polynomials: A survey,
  in Numerical Analysis 2000 (Elsevier).

The point of view of geometry

- Algebraic geometers give a highly abstract interpretation involving
- punctual schemes, compactification, coherent modules, line bundles, blow-up, genericity, cohomology
- beyond language, is all this stuff helpful?

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Blowing-up

- The line and the conic above are (-1)-*curves* on the surface obtained by *blowing-up* $(p_1, \cdots, p_r)$ in the projective plane.
- Blowing-up p_i replaces p_i by a (-1)-curve E_i.
- Two lines through p_i do not meet anymore on the blown-up surface:
- they meet E_i in different points.
- Two curves of degree d and d' through the p_i's have dd' common points in the plane (Bezout)
- and only $dd' - r$ points on the blown-up plane.

(-1)-curves

- A (-1)-curve is a smooth rational curve of "self-intersection" -1.
- The line $p_i p_j$ yields a (-1)-curve (self-intersection $d^2 - r$ with $d := 1, r := 2$).
- The conic through $(p_1, \cdots, p_5)$ yields a (-1)-curve (self-intersection $d^2 - r$ with $d := 2, r := 5$).
- There are 27 (-1)-curves on the general cubic surface (plane blown-up at six points, $k = \mathbb{C}$)
- There are infinitely many (-1)-curves on the plane blown-up at r points if $r \geq 9$

The generic rational surface

- So we work on the (generic) surface S_r obtained by blowing-up the projective plane at r generic points.

The Neron-Severi group of S_r

- This is the group of equivalence classes of "virtual curves" on S_r (divisors)
- Two curves are equivalent iff they are in the same linear system
- The class of a curve is determined by its degree d and its multiplicities at the p_i's(the m_i's)
- $NS(S_r)$ is isomorphic to $\mathbb{Z}^{r+1}$
- each class C has its linear system (possibly "empty"), with a dimension denoted $H^0(C)$.
- We search for this dimension.

Examples of classes

- The class of the line p_2p_3 on S_5 : $(1, 0, 1, 1, 0, 0)$.
- The class of the conic through $p_1, \cdots, p_5$ on S_6 : $(2, 1, 1, 1, 1, 1, 0)$
- The so-called canonical class K is $(-3, -1, \cdots, -1)$.

The intersection form

- The intersection form
  gives the "right" number of intersection points of two curves (generalizing Bezout formula)
- $(d, m_1, \cdots, m_r)\dot{}(e, n_1, \cdots, n_r) := de - m_1 n_1 - \cdots - m_r n_r.$
- $(1, 1, 1, 0, \cdots, 0)^2 = -1, \quad (1, 1, 1, 0, \cdots, 0).K = -1$
- $(2, 1, 1, 1, 1, 1, 0, \cdots, 0)^2 = -1,$
  $(2, 1, 1, 1, 1, 1, 0, \cdots, 0).K = -1$
- the "expected" dimension of a class D is
  $[\frac{D.D - D.K}{2} + 1]_+$ (Riemann-Roch).

Speciality

- A class c (with positive degree) is said non-special if its dimension is the expected one, special otherwise
- We want to recognize special systems
- A class $(d, \underline{m})$ (with d positive) is non-special iff the corresponding evaluation
  $\mathbb{R}[x_1, x_2]_d \to \mathbb{R}_{\underline{p},\underline{m}}$
  has maximal rank.

(-1)-classes

- Definition : D is a (-1)-class : $\Leftrightarrow D^2 = D.K = -1$.
- The class of a (-1)-curve is a (-1)-class.
- Conversely each (-1)-class comes from a (-1)-curve.

The conjecture I

- Easy observation: a class D $(d > 0)$ is special as soon as for some (-1)-class C, $D.C \leq -2$ $(d > 0)$
- Conjecture: conversely, if a class D $(d > 0)$is special, it is because it is "obstructed" by some (-1)-class C (namely $D.C \leq -2$).

The Weyl group

- The Weyl group is the group W of orthogonal transformations of $NS(S_r)$ fixing K.
- It permutes transitively (-1)-classes.
- It is finite iff $r \leq 8$.
- It respects the dimension of linear systems and their expected dimension.

Automorphisms of S_r

- The action of W on $NS(S_r)$ comes from an action on S_r.
- Thus W permutes transitively (-1)-curves.

Generators of the Weyl group

- W contains permutations (of multiplicities).
- W contains the quadratic transformation q : $(d, m_1, \ldots, m_r) \mapsto (2d - m_1 - m_2 - m_3, d - m_2 - m_3, d - m_1 - m_3, d - m_1 - m_2, m_4, \cdots, m_r)$
- W is generated by permutations and quadratic transformations.

Standard classes

- a class $(d, m_1, \ldots, m_r)$ is said standard iff $d \geq m_1 + m_2 + m_3$, and $m_1 \geq \cdots \geq m_r$
- given a class C with positive degree, the orbit WC contains a standard class which can be computed
- a standard class $(d, m_1, \ldots, m_r)$ with $m_r \geq -1$ is not obstructed by (-1)-classes.
- This means that we can decide if a class is obstructed by (-1)-classes and reduce our problem to standard classes.

The conjecture II

Conjecture: standard classes $(d, m_1, \cdots, m_r)$ with $m_r \geq 0$ (or, if you prefer, $m_r \geq -1$) are non special.

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Low number of points

- for $r \leq 9$ the conjecture was proved by Castelnuovo
- for $r \leq 8$, this concerns Del Pezzo surfaces
- in the latter case, all classes with $d > 0$ and $\chi = 0$ are special
  (a big trouble for induction)

Low multiplicities and computers

- Hirschowitz 1985: $m_i \leq 2$ (no computer)
- Mignon 2000: $m_i \leq 4$ (some computer)
- Yang 2004: $m_i \leq 7$ (more computer)
- Dumnicki-Jarnicki 2006: $m_i \leq 11$ (much more computer)

Homogeneous multiplicities

- Hirschowitz 1985: $m_1 = \cdots = m_r \leq 3$
- Caporaso-Harris 1997: $m_1 = \cdots = m_r \leq 6$
- Ciliberto-Cioffi-Miranda-Orrechia 2000: $m_1 = \cdots = m_r \leq 20$
- Dumnicki 2006: $m_1 = \cdots = m_r \leq 37$

The case where r is a square

- Evain 2005
- Ciliberto-Miranda 2006

Nice examples

- $r := 10$
- $(19, 6, \cdots, 6)$ Hirschowitz 1985
- $(38, 12, \cdots, 12)$ Gimigliano 1987
- $(174, 55, \cdots, 55)$ Ciliberto-Miranda 2008

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The cone theory 0

- The cone theory takes place in
- $N_1(X) := NS(X) \otimes \mathbb{R} \qquad (X = S_r)$
- We are interested in the subcone $NE(X)$ generated by curves, and in its closure $\overline{NE}(X)$.

The cone theory I

- The intersection bilinear form yields :
- The cone Q of classes D with $D.D > 0$
- Q splits as $Q^+ \amalg Q^-$
- $Q^- := -Q^+$
- (the "positive" subcone Q^+ is the one with positive degrees)

The cone theory II

- $NE(X)$ contains Q^+ (eventually $D.D \geq D.K$)
- Negative curves ($D.D < 0$) yield "extremal" rays in $\overline{NE}(X)$.

The cone theory III

- $\overline{NE}(X)$ is generated by $\overline{Q^+}$ and (irreducible) negative curves
- (no closure needed)

The cone theory IV

- The K-negative side $(D.K < 0)$ is easy :
- the only negative classes there are (-1)-classes
- $(2g - 2 = D.D + D.K)$

The cone theory V

- The K-null hyperplane ($D.K = 0$) is under control:
- $\overline{NE}(X)_{K=0} = \overline{Q^+}_{K=0}$ (De Fernex 2006)

The remaining case

- Conjecture (Nagata 1959):
- (for $r \geq 10$)
- No curve D with $D.D < 0$ and $D.K > 0$.

Nagata's reduction

- From an inhomogeneous bad curve $(d, m_1, ..., m_r)$
- by symmetrization
- build a homogeneous bad curve
- (numbers follow for $r \geq 10$)
- This reduces the conjecture to the homogeneous case.

Nagata's conjecture: final form

- No curve in the class $(d, m, ..., m)$ for $d^2 < rm^2$
- in other words
- $(\sqrt{r}, 1, ..., 1)$ is in the boundary of $\overline{NE}(X)$.

Approaching Nagata's conjecture for $r = 10$

- (it is the hardest case)
- $\sqrt{10} = 3.162....$
- Problem: find a with $(a, 1, ..., 1)$ out of $\overline{NE}(X)$
- $a := 3.1606$ Harbourne-Roe 2005
- $a := 3.1609...$ Ciliberto-Miranda 2008
- $a := 3.1615...$ O. Dumitrescu 2009

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Specializing points

- Thanks to semi-continuity,
- you may treat the case where the points p_i are in special position rather than generic.

Horace method

- Put the right number of points on a curve,
- "substract" the curve
- and go ahead with the residual problem.

Collisions

- You may also specialize all your points at a single one
- This method has been used successfully by Evain and by Roe.

Specializing points on a surface

- Finally you may specialize the plane as the plane plus a new surface attached along a curve
- This method has been introduced and refined and refined by Ciliberto-Miranda.

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An active area

- This is an active area with several sattelite areas:
- consider more general singularities (than ordinary multiple points)
- study collisions for themselves
- study the resolutions of the relevant ideals (not only their Hilbert function)
- replace the plane with other surfaces
- replace the plane with higher dimensional projective spaces ...