Relating sets and types

André Hirschowitz and Marco Maggesi

Abstract

We introduce the category of (strict) *typoids* which is a variant of the category of setoids. We equip this category with a model structure where cofibrant-fibrant objects are sets. In this way, homotopy theory provides a unified approach to several categories of variants of sets.

1 Introduction

In this paper, we propose a (slightly) new discipline for the formalization of sets, based on what we call typoids, which we now describe.

The formalization of sets can be achieved in a lot of ways, and this early choice in the process of formalizing mathematics is easily recognized as a crucial one (see e.g. [?]). In [?], a systematic discussion of the formalization of sets has been undertaken and the relationship between several categories of setoids is explored. Among others, it is observed there that the categories PER and $\overline{\mathsf{PER}}$ of sets (or types) equipped with a partial equivalence relation and classes of compatible functions (resp. graphs) among them are not (naturally) equivalent. And also that the form of weak choice axiom which seems the natural solution to this problem is inconsistent. The conclusion in [?] is that the choice of PER for the formalization of mathematics should be rejected. We propose a different point of view, which gives each of the considered categories its place in a unified picture. This point of view is provided by homotopy theory, where one main problem is to relate strict and weak morphisms ([?] [?] [?]) and where the structure of model category ([?] [?]) is universally considered as the best form of answer. Indeed, we start from the category of (strict) *typoids* where objects are sets equipped with a relation and where morphisms are applications compatible with the given relations. We call equivalences those morphisms which induce bijections on quotients sets. This yields ([?]) a notion of homotopy and we check that two morphisms are homotopic iff they induce the same map on quotient sets. Then we equip our category of strict typoids with a model structure where the subcategory of cofibrant and fibrant objects is the category of sets, and the homotopy category is $\overline{\mathsf{PER}}$.

We thank Carlos Simpson for many stimulating discussions, in particular for a decisive hint toward our model structure. We also thank Sabrina Tarento for drawing our attention to [?].

2 Typoids

Definition 2.1 (Typoid). A typoid (X, r) consists of a set X equipped with an arbitrary binary relation r on X.

Definition 2.2 (Associated PER and quotient). Given a typoid $T = (X, r)$, we form the generated partial equivalence relation $\hat{r}$ and the associated PER $\hat{T} := (X, \hat{r})$; and we denote by $\overline{T}$ the associated quotient set.

Definition 2.3 (Category of strict typoids). We take as (strict) morphisms among typoids the morphisms among the associated PERs, i.e., applications among sets compatible with the relations (we do not identify equivalent applications). This yields our category STyp of strict typoids.

Definition 2.4 (Product of typoids). We check easily that the cartesian product of two sets equipped with the cartesian product of relations yields the cartesian product in the category STyp. More generally we easily construct limits and colimits in STyp.

Definition 2.5 (Category of weak typoids). We take as weak morphisms among typoids the morphisms among the associated quotient sets. This yields our category WTyp of weak typoids.

Definition 2.6 (Graph functor). By taking the graph, we build a functor Gr from STyp to WTyp.

Definition 2.7 (Equivalences in STyp). We say that a strict morphism m is an equivalence if $\mathrm{Gr}(m)$ is a bijection. Equivalences form a multiplicative system which is, by definition, inverted by Gr.

Proposition 2.8. *Two morphisms in* STyp *are homotopic (in the sense of [?]) if and only if their images by Gr are equal.*

Proposition 2.9. *The functor* Gr *is the localization of the multiplicative system of equivalences in* STyp.

For the proof, we observe that each morphism in WTyp is the graph of a right fraction, and that two morphisms in STyp with the same graph become equal when composed (on both sides) with suitable equivalences (to be understood later as fibrant and cofibrant replacements).

We now describe the model structure on STyp.

Definition 2.10 (The exotic part). We define the exotic part of a typoid to be the subset of elements which are not equivalent to themselves. This yields a functor, not to the usual category of sets, but to the category of sets with partially defined maps.

Definition 2.11 (Cofibrations). We define cofibrations in STyp to be the morphisms whose exotic part is a (total) bijection.

Definition 2.12 (Fibrations). We define a fibration in STyp to be a morphism $f : (A, r) \to (B, s)$ such that, for each non exotic x, f induces a bijection from the r-equivalence class of x to the s-equivalence class of $f(x)$.

Theorem 2.13. *Cofibrations, fibrations and equivalences as defined above make $STyp$ into a model category.*

This model category expresses a strong relationship among our categories $STyp$, $WTyp$ and the category of usual sets, which is the category of fibrant and cofibrant objects, thus equivalent to the homotopy category $WTyp$. The discipline of model categories allows to replace typoids by sets when proving homotopy-theoretic statements. We did not try yet to give a precise meaning to this principle.