Modelling (and scientific computing): Coupling, relaxation, adaptation of hyperbolic models

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Laboratoire Jacques-Louis Lions (University Pierre et Marie Curie-Paris 6): created by J.-L. Lions in 1969


LJLL covers a wide range of research topics from theoretical analysis of PDE, numerical analysis, modeling, simulation and scientific computing

Other members of LJLL involved in IFCAM:
Jean-Michel Coron and Emmanuel Trélat, Control of PDE;
with Mythily Ramaswami, Vanninathan, and Adimurthi

Philippe G. LeFloch, hyperbolic problems, relativity;
with K.T. Joseph

Nicolas Seguin
Frédéric Coquel (a member till 2010, now at Ecole Polytechnique)
collaborations

Christophe Chalons, Frédéric Coquel, Frédéric Lagoutière (now in Orsay), Pierre-Arnaud Raviart, Nicolas Seguin
Filipa Caetano, Thomas Galié, Benjamin Boutin (PhD)
more recently
Clément Cancès, Hélène Mathis (postdoc, now in Nantes)
with CEA (french nuclear agency): LRC MANON
Introduction

From the application side
Simulation of multiphase flow in nuclear energy industry (CEA): coolant circuits are formed by different components, each one with its associated specific model for the coolant flow. The coolant is a two-phase fluid.
There exists a wide variety of models (mixture, drift, homogeneous or not, two-fluid or even multi-field models).
A code dedicated to a component: need to couple them.
More or less costly to implement (number of variables, complex pressure laws,...): use a coarse (less expensive) model whenever possible.
Adaptative procedure to justify the coupling.

Example: HRM/HEM Homogeneous Relaxation/Equilibrium Model.
Relaxation provides hierarchy between the 2 systems.
Relaxation also provides tools for approximation.
Enceinte de confinement

Barres de contrôle

Cuve

Pressuriseur

Caloporteur chaud (830 °C)

Caloporteur froid (250 °C)

Pompe

Générateur de vapeur (échangeur de chaleur)

Vapeur

Liquide

Turbine

Condenseur

Pompe

Alternateur

Circuit de refroidissement

Tour de refroidissement ou rivière ou mer

Vapeur d'eau

Eau sous pression (circuit primaire)

Eau et vapeur d'eau (circuit secondaire)

Eau ou aéronéfrigérant (circuit de refroidissement)

**Figure 1** – Schéma simplifié d’un réacteur nucléaire à eau sous pression
Theoretical frame

- **Hierarchy** between systems is linked to a relaxation process. Ex: HRM/HEM. Relaxation system (*fine*)/ equilibrium (*coarse*); bifluid (*fine*) model / drift (*coarse*) model.
- Simplified models for which hierarchy can be (formally) proved.
- Need of coupling models at a fixed interface.
- Understand the **interface coupling model** on different significative examples.
- Derive numerical procedures for more general situations.
- Theoretical and numerical points of view are linked: what does a scheme compute?
- Where to put the interface? Which model coarse/fine? Study some **adaptation** principle.
- Simpler models: hyperbolic systems for which **theoretical** results can be proved.
• **Coupling** of models
  - hyperbolic framework: interfacial coupling, examples
  - two approaches: conservative / non conservative
  - fluid models in Lagrangian coordinates
  - numerical coupling through relaxation

• **Relaxation**
  - relaxation systems
  - example with Euler
  - a more realistic example: bifluid / drift

• **Adaptation** (hints)
  - aim
  - simpler example
Introduction: coupling models

Problem: fixed interface (for adaptation, at least over one time step), not necessarily physical. On each side a model given by a system of PDE (conservation laws, hyperbolicity required)

- Examples of interfaces for coupling: traffic (‘junction’), network of ‘pipelines’ (nodes), sedimentation.

‘Artificial’ interface: coupling of codes. Our analysis may help put the computational interface at a right place.

- Assume a plane interface, and normal flux at interface: the study of 1D problems is relevant.
Notations: \( u, f \), but dimensions may be different, if so, systems are assumed compatible and one uses lift or projection operators.
Conservative interface model

One discontinuous flux: \( f(u; x) = f_L(u)1_{x<0} + f_R(u)1_{x>0} \)

\[
\partial_t u + \partial_x f(u; x) = 0, \quad x \in \mathbb{R}
\]

conservative form \(\Rightarrow\) Rankine-Hugoniot \([f(u; 0)] = 0\)

flux coupling

Scalar case. A notion of entropy is possible (Kružkov type):
usual condition outside interface and entropy condition at interface.

Several approaches, unifying framework in Andreianov, Karlsen, Risebro (ARMA 2011) with a notion of dissipative solver and admissible germ: well posedness and uniqueness results are proved.

\[
\begin{cases}
    u_\ell, & x < 0 \\
    u_r, & x > 0
\end{cases}
\]  \hspace{1in} (1)

Admissible interface model $\leftrightarrow$ define a Riemann solver at interface. Take for $u_\ell, r$ traces $u(0\pm)$ of solutions of CRP $\rightarrow$ notion of *germs* or elementary solutions: stationary solutions (1). In a conservative approach

\[
f_\ell(u_\ell) = f_r(u_r)
\]  \hspace{1in} (2)

Theory in the scalar case $f(u; x) = f_\ell(u)1_{x<0} + f_r(u)1_{x>0}$ elementary solutions (admissibility germs) play the role of constants in Kružkov entropies $|u - k|$; If admissibility germ satisfies some Kružkov type entropy inequality, it is called $L^1$ dissipative.
Example. Interface entropy condition (Audusse-Perthame 2005; Baiti-Jensen 1997): *adapted* Kružkov entropy $|u - k_\alpha(x)|$ where $k_\alpha$ satisfies

$$f(k_\alpha^\pm(x), x) = \alpha, \forall x \in \mathbb{R} \tag{3}$$

$k_\alpha$ exists under some (strong) monotonicity hypothesis on $f_{\ell,r}$

Corresponding entropy inequalities, and uniqueness results follow.

A-P germ: $(u_\ell, u_r)$ such that (2): $f_\ell(u_\ell) = f_r(u_r)$ and $\text{sgn}(u_\ell - \bar{u}_\ell) = \text{sign}(u_r - \bar{u}_r)$ where $\bar{u}_\ell, r$ point where $f_{\ell,r}$ are minimum and vanish (obvious link with (3))

existence and uniqueness result.
**coupling Non conservative approach**

- 2 IBVP boundary value problems, $L$ in $x < 0$ / $R$ in $x > 0$
- traces $u(0-, t)$ (resp. $u(0+, t)$) given as BC for $R$ (resp. $L$)
- State coupling $[u] \cong 0$, $u(0-, t) \cong u(0+, t)$ where $\cong$ means in a weak sense: $u(0\mp, t) \in O_{L/R}(u(0\pm, t))$. Then

$$\partial_t u + \partial_x f(u; x) = M, \ x \in \mathbb{R}$$

$M$ Dirac measure $\delta_0$ with weight $[f] = f_R(u(0+, t)) - f_L(u(0-, t))$
- Possibility of transmitting non conservative (primitive) variables if $u \mapsto v(u)$ admissible, $[v] \cong 0$. Many choices for $v$.
  - non uniqueness
  - give different results
  - need of some ‘physical’ criteria (preserve some solutions)
  - for some cases, resumes to a conservative coupling (characteristic interface, material CD)
**coupling Material discontinuity**

Euler system in Lagrangian coordinates, one fluid with **two** components: \( p_L, p_R \) **two** pressure laws
conservative variable \( U = (\tau, u, e) \), flux \( F(U) = (-u, p, pu) \),
\( U \rightarrow F(U) \) is not admissible; for each model \( \lambda = 0 \) eigenvalue, *characteristic* interface.

‘Natural’ choice: transmission of \( V = (\tau, u, p) = \) continuity of \( u, p \).

Link with:

- **BC**: one cannot impose \( u, p \)
- linearization at a CD
- *ghost cell* (Abgrall-Karni approach)

Extension to **fluid systems**: they have a strong algebraical structure.
Example of a characteristic interface

Euler system in Lagrangian coordinates

\[ \lambda_{L,1} < 0 < \lambda_{L,3}, \quad \lambda_{R,1} < 0 < \lambda_{R,3}, \quad \lambda_{L,2} = \lambda_{R,2} = 0 \]

- heuristics: transmission of 2 quantities
- justified by a linearized analysis
- well posed problem in the nonlinear case
Euler $\tau, v, p$: left transmission of $\mathbf{v} = (\tau, v, p)$, vs $\mathbf{u} = (\tau, v, e)$ right
coupling Theoretical results


- Interfacial **coupling** of 2 linear systems (EG, PAR with KCLT, *M2AN* 2005): some condition on the dimension of eigenspaces, coupling at a material discontinuity, example of a plasma model.

- Extension to **fluid models**: coupling in (EG et al., *Math of Comp* 2007) and justification of the relaxation approximation (FC, EG, NS *M3AS* 2011)
**Numerical interface coupling**

Numerical method: 2 conservative FV schemes, Left, Right. Non conservative approach: **TWO** fluxes at the interface $x = 0$

\[
g_{L,0}^n = g_L \left( u_{n-1/2}^n, u_{n+1/2}^n \right), \quad g_{R,0}^n = g_R \left( u_{n-1/2}^n, u_{n+1/2}^n \right)
\]

One may characterize the states $u_{\pm 1/2}$ by their primitive components $v$
**coupling Use of relaxation schemes**

In general two fluxes at interface, unless one has a material discontinuity and transmission of primitive variables (example \( u, p \)).

One can perform the coupling of larger but simpler relaxation systems which approximate the original systems \( f_{L,R} \rightarrow \text{numerical coupling} \).

For *fluid models*, Suliciu type Linearly Degenerate (LD) relaxation system on each side \( \rightarrow \) explicit solution of Riemann problem \( \rightarrow \) Godunov scheme \( \rightarrow \text{relaxation scheme} \), which is a conservative scheme, entropy satisfying left and right

Solve the CRP for these relaxation systems, with transmission of well chosen (primitive) variables \( \rightarrow \) leads to a conservative scheme with a unique flux at the interface
Example of Relaxation system

Example of a numerical relaxation model for the \( p \)–system (Suliciu)

\[
\begin{align*}
\frac{\partial}{\partial t}\tau - \frac{\partial}{\partial x} u &= 0 \\
\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} \Pi &= 0 \\
\frac{\partial}{\partial t} T &= \frac{1}{\epsilon} (\tau - T)
\end{align*}
\]

\( \Pi = \Pi(\tau, T) = p(T) + a^2 (T - \tau) \) linearization of the pressure

\( T = \tau + \epsilon T_1 + O(\epsilon^2) \),

\[
\begin{align*}
\frac{\partial}{\partial t}\tau - \frac{\partial}{\partial x} u &= 0 \\
\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} p(\tau) &= \epsilon \frac{\partial}{\partial x} \left( (p'(\tau) + a^2) \frac{\partial}{\partial x} u \right).
\end{align*}
\]

dissipative if \( p'(\tau) + a^2 > 0 \), stability criteria

\[-a < -\sqrt{-p'(\tau)} < 0 < \sqrt{-p'(\tau)} < a\]

System with LD fields, the Riemann problem has an explicit solution, Godunov scheme leads to an HLL type scheme for the \( p \)–system. Theoretical results (Chalons-Coulombel after Yong)
relaxation system and coupling procedure

- Similar approach for full Euler system, \( p(\tau, s) \rightarrow \Pi(\tau, s, T) \), 4 equations, 4 LD fields
- Extension to fluid models: relaxation approach possible for all fluid models (B. Després), \( \partial_t s = 0 \) (in Lagrangian).
- Extension to other systems: relaxation scheme for a nozzle, for Baer-Nunziato system (T. Galie, K. Saleh PhD thesis)
- Equivalence: Eulerian \( \leftrightarrow \) Lagrangian frame
scheme in Lagrangian coordinates \( \leftrightarrow \) scheme in Eulerian coordinates
- Conservative coupling of the relaxation systems if \((\tau, u, \Pi)\) are transmitted
  one numerical flux = Godunov with CPR, with transmission of primitive variables
- Relaxation \( \Pi \rightarrow p \): yields conservative coupling for Euler with transmission \((\tau, u, p)\)
- \( \Sigma \) total energy of the relaxation system, satisfies some dissipation principle which justifies relaxation.
Numerical flux coupling for Euler

For Euler, $u = (\rho, \rho u, \rho e)$, flux $(\rho u, \rho u^2 + p, (\rho e + p)u)$, the eigenvalues may change sign, the flux is not an admissible change of variables.

Numerical flux coupling via a global relaxation coupling solver
- a larger relaxation system relaxing towards Euler as $\epsilon \to 0$
- a numerical coupling of the convective part of the relaxation systems with judicious choice of CC
- a splitting method: convection $+$ instantaneous relaxation $\epsilon = 0$

Results in a standard finite volume method: if it ‘converges’ to $u$, $u$ is solution of a coupled problem with continuous flux, and entropy solution in $x < 0$ and $x > 0$. 
conservative coupling by relaxation, CRP computed with transmission of \((\varrho, u, p)\)

\[
\begin{align*}
\varrho_L &= 1.902, ~ u_L = 1.6361, ~ p_L = 2.4598; \\
\varrho_R &= 1, ~ u_R = 2, ~ p_R = 1.
\end{align*}
\]

the exact solution has 4 waves: 1\(L\)—shock, 1 stationary wave (coupling), 1\(R\)—sonic rarefaction, 2\(R\)—CD et 3\(R\)—shock

\(\text{(computations by Thomas Galié)}\)
Relaxation system (general case)

\[ \partial_t w + \partial_x h(w) = \lambda R(w), \quad t > 0, \quad (4) \]

\( h'(w) \) \((n \times n)\) diagonalizable matrix with eigenvalues in \( \mathbb{R} \). Note \( w = (u, v) \in \mathbb{R}^d \times \mathbb{R}^{n-d} \), \( P \) the projection \( \mathbb{R}^n \rightarrow \mathbb{R}^d \),

\( h(w) = (f(w), g(w)) \), and assume \( R(w) = (0, e(u) - v)^T \)

\( R(w) = 0 \) equilibrium manifold: \( w^{eq} = (u, e(u)) = E(u) \)

\( \lambda > 0 \) relaxation parameter, \( \lambda \rightarrow \infty \), \( P(w) \) satisfies

\[ \partial_t u + \partial_x f(u, e(u)) = 0 \quad (5) \]

Justification of relaxation: entropy \( V \) for (4), and \( U = V(E(u)) \).

Chapman-Enskog expansion: \( w = E(u) + \lambda^{-1} v^\lambda_1 + .., \)

\[ \partial_t u^\lambda + \partial_x f(u^\lambda, e(u^\lambda)) = \frac{1}{\lambda} \partial_x (D(u^\lambda) \partial_x u^\lambda) + .... \]

dissipative wrt. entropy \( U \), i.e., \( U''(u)D(u) > 0 \) (pioneer work of Chen-Levermore-Liu). We are interested in cases where (4) is LD, thus simpler than (5) and s.t. the relaxation is justified (approximation and convergence results following Yong)
Relaxation systems (examples)

• Example of $p-$system (and fluid systems)
• Link with friction

$$\begin{cases} 
\partial_t \tau - \partial_x u = 0 \\
\partial_t u + \partial_x p(\tau) = -\lambda u,
\end{cases}$$

$\lambda$ friction coefficient
formally $d = 2$, $n = 1$, $w = (\tau, u)^T$, $h(w) = (-u, p, 0)^T$, $R(w) = (0, -u)^T$, $w^{eq} = (\tau, 0)^T$, $e(\tau) = 0$, $\lambda \to \infty$,
Chapman-Enskog expansion

$$\begin{cases} 
\partial_t \tau = \frac{1}{\lambda} \partial_x (p'(\tau) \partial_x \tau) \\
u = 0 - \frac{1}{\lambda} \partial_x p(\tau)
\end{cases}$$

$\to$ Darcy law.

$t = \lambda s$, $v = \lambda u$, large friction and large time

$$\begin{cases} 
\partial_s \tau - \partial_{xx} (p(\tau)) \\
w = -\partial_x p(\tau)
\end{cases}$$ (6)

convergence results (Hsiao-Liu, Hsiao-Serre)
Other physical examples of relaxation systems, which do not enter the ideal case:

**Relaxation** HRM → HEM (thermodynamical equilibrium)

**Relaxation** of bifluid model → drift model:

*Comm. Math Science 2008, AA, CC, FC, TG, EG, FL, PAR, NS*

Thomas Galié thesis

**Relaxation** in velocity, pressure **AND** limit in large time

Link with asymptotic in Euler with friction (*M3AS 2010, CC, FC, EG, PAR and NS*)

**Coupling:** asymptotic preserving scheme (AP)

**Relaxation** of fluid systems (*M3AS 2012, FC, EG, NS*)
Relaxation Bifluid model

Model with two phases, two pressures (dimensionless form):

Baer-Nunziato

\[
\begin{align*}
\partial_t \alpha_1 + u_I \partial_x \alpha_1 &= \Theta p_r \\
\partial_t \alpha_1 \rho_1 + \partial_x \alpha_1 \rho_1 u_1 &= -\Gamma \\
\partial_t \alpha_2 \rho_2 + \partial_x \alpha_2 \rho_2 u_2 &= \Gamma \\
\partial_t (\alpha_1 \rho_1 u_1) + \partial_x (\alpha_1 \rho_1 u_1^2 + \alpha_1 \rho_1) - p_l \partial_x \alpha_1 &= \alpha_1 \rho_1 f_1 + \Lambda |u_r| u_r \\
\partial_t (\alpha_2 \rho_2 u_2) + \partial_x (\alpha_2 \rho_2 u_2^2 + \alpha_2 \rho_2) + p_l \partial_x \alpha_1 &= \alpha_2 \rho_2 f_2 - \Lambda |u_r| u_r \\
\end{align*}
\]

\( u_r \equiv u_2 - u_1, \ p_r \equiv p_1 - p_2, \ p_i = p_i(\rho_i), \ i = 1, 2 \)

\( u_I, p_I : \) velocity, interfacial pressure, one takes \( u_2 \) and \( p_1 \);

hyperbolic system \( u_2 \) LD, \( u_i \pm c_i \) GNL.

\( \Gamma(u) : \) mass transfert, phase 1 \( \rightarrow \) phase 2, taken \( = 0 \)

\( f_1(u), f_2(u) : \) external forces, \( = -g \) (gravity)

\( \Lambda(u) = \frac{\lambda(u)}{\epsilon^2} : \) drag force of order \( \frac{1}{\epsilon^2} \)

\( \Theta(u) = \frac{\theta(u)}{\epsilon^2} : \) pressure relaxation coefficient of order \( \frac{1}{\epsilon^2} \)

\( \epsilon \) small parameter: \( u_r \rightarrow 0, p_r \rightarrow 0 \) as \( \epsilon \rightarrow 0 \)
Relaxation Drift model

Mixture model

\[
\begin{aligned}
\partial_t \rho + \partial_x \rho u &= 0 \\
\partial_t \rho Y + \partial_x (\rho Y u + \rho Y (1 - Y) u_r) &= \Gamma(v) \\
\partial_t \rho u + \partial_x (\rho u^2 + p + \rho Y (1 - Y) u_r^2) &= \rho(1 - Y) f_1(v) + \rho Y f_2(v)
\end{aligned}
\]

\(v = (\rho, Y, u)\), mean density, velocity, \(Y\) mass fraction
\(\Gamma(v)\): mass transfert, phase 1 \(\rightarrow\) phase 2, set to \(= 0\)
\(f_1(v), f_2(v)\): external forces, \(= g\) (gravity)
\(p = p(v)\): pressure law, if \(p_1, p_2\) given associated to phases 1, 2,
\(\alpha_2 \rho_2 = \rho Y\), \(\rho = (1 - \alpha_2) \rho_1 + \alpha_2 \rho_2\),

\[
\begin{aligned}
p &= p_2 \left( \frac{\rho Y}{\alpha_2} \right) \\
p_1 \left( \frac{\rho(1 - Y)}{1 - \alpha_2} \right) &= p_2 \left( \frac{\rho Y}{\alpha_2} \right)
\end{aligned}
\]

\(u_r = u_r(v)\): closure for relative velocity \(u_r = u_2 - u_1\),
\(u_k\) phase velocity \(k\), \(u = (1 - Y) u_1 + Yu_2\) mixture velocity
Starting from the bifluid model, one defines mixture quantities:
\[ \rho = \alpha_1 \rho_1 + \alpha_2 \rho_2, \ u = \alpha_1 \rho_1 u_1 + \alpha_2 \rho_2 u_2, \ \rho Y = \alpha_2 \rho_2, \ldots \]
Simply adding, we get \[ \partial_t \rho + \partial_x \rho u = 0 \]
one looks for equations for \( p_r := p_1(\rho_1) - p_2(\rho_2) \) and \( u_r := u_2 - u_1 \).
To justify the numerical coupling and adaptation, one performs an asymptotic analysis of the bifluid model \( \epsilon \to 0 \).
Solution near equilibrium defined by \( p_r = 0, \ u_r = 0 \),
Hilbert expansion
\[ p_r = 0 + \epsilon p^1_r + \mathcal{O}(\epsilon^2) \]
\[ u_r = 0 + \epsilon u^1_r + \mathcal{O}(\epsilon^2) \]
Result: the 1rst order system at equilibrium is a drift model with a Darcy type law
Relaxation \ large \ time \ limit: \ drift \ model

At equilibrium, we get the 1rst order system

\[
\begin{align*}
\partial_t \rho + \partial_x \rho u &= 0 \\
\partial_t \rho Y + \partial_x (\rho Yu + \rho Y(1 - Y)u_r) &= 0 \\
\partial_t \rho u + \partial_x (\rho u^2 + p + \rho Y(1 - Y)u_r^2) &= -\rho g
\end{align*}
\]

where \( u_r = \epsilon u^1_r \)

\[
|u_r|u_r = \epsilon^2 (\rho_2 - \rho_1) \alpha_1 \alpha_2 \frac{\partial_x p}{\rho}
\]

large time (permanent flow): \( s = \epsilon t \) gives \( \partial_x p = -\rho g \)

\( \rightarrow \) drift model with algebraic closure in the form \( u_r = \Phi(v) \)

Thanks to this asymptotic analysis, we have proved a hierarchy or compatibility between the bifluid 2 pressure model and the drift model \( |u_r|u_r = -\epsilon^2 (\rho_2 - \rho_1) \alpha_1 \alpha_2 g, \)
Relaxation end Coupling numerical illustration

Coupling bifluid / drift
1D test case: bubble column, dispersed vertical flux (rising)
Bottom at $x = -0.5$, top at $x = 0.5$ interface at $x = 0$, 100 meshes.

Boundary conditions: in $\alpha_1 = 0.97$ (liquid fraction), $p_1 = p_2 = 155 \cdot 10^5 \text{Pa}$, $u_1 = 5 \text{m/s}$, $u_2 = 15 \text{m/s}$
out $p_1 = p_2 = 150 \cdot 10^5 \text{Pa}$.
One compares the stationary solution in 3 cases:
- bifluid everywhere
- drift everywhere
- coupling: bifluid left / drift right

Two values $\varepsilon: \varepsilon^2 = 10^{-1}, 10^{-3}$
Results: $\varepsilon^2 = 10^{-1}$, $T = T_{stat}$

**liquid fraction $\alpha_1$**

**relative velocity $u_r$**

$x = -0.5$: in $\rightarrow$ top = out: $x = +0.5$

**coupling** at 0: left bifluid / right drift

*(computations by Thomas Galié)*
**Results:** $\varepsilon^2 = 10^{-3}$, $T = T_{stat}$

**liquid fraction** $\alpha_1$

**relative velocity** $u_r$

*(computations by Thomas Galié)* the smaller $\varepsilon$ the better the computations look like
We assume the flow such that the drift model is hyperbolic. **Remark:** the coupling is relevant if the interface is localized in a region where the flow is nearly permanent.

In his thesis T. Galié has used a “father model”

- solve the bifluid $2p$–model in the whole domain, with a relaxation scheme (numerical LD relaxation system for each fluid and coupling by a CD)
- relaxation of the velocity and pressure in the region of the drift model

He did not use a specific scheme for the drift model. *If one has a scheme...*
Model adaptation: project

Understand and try an adaptative procedure on this case:

- fine model: system with a relaxation term
  example: bifluid 5 equation system
- coarse model: system at equilibrium
  example: 3 equation drift model

the first one relaxes to the second, whose solutions are at ‘equilibrium’.

In the case where the numerical simulation of the fine model is costly try an adaptative procedure: at each time step, one uses the coarse mode in the region where the solution is near equilibirum the fine model elsewhere.

At the interface one couples the two models.

Supposes a numerical scheme for the drift model.

Build a procedure which selects the region where the drift model is sufficient.
Two models: fine $\mathcal{M}_{fi}$ and coarse $\mathcal{M}_{co}$, associated schemes with numerical fluxes $G_{fi}$, $G_{co}$

**Numerical model adaptation**: scheme on one time step $t_n \to t_{n+1}$, starting from $u_{adap}(\cdot, t_n)$ piecewise constant

- determine $\Omega^n_{fi}$
- solve $\mathcal{M}_{fi}$ in $\Omega^n_{fi}$
- solve $\mathcal{M}_{co}$ in $\Omega^n_{co} = \Omega \setminus \Omega^n_{fi}$
- one couples at interface $\partial \Omega^n_{fi} \cap \partial \Omega^n_{co}$

Build an **adaptative procedure** which selects $\Omega^n_{fi}$: supposes some error indicator $\delta^n(x) \sim |u_{adap}(x, t_{n+1}) - u_{fi}(x, t_{n+1})|$, computed without $\mathcal{M}^{n+1}_{fi}$; given some tolerance $\theta$ if $\delta^n(x) \geq \theta$, $x \in \Omega^n_{fi}$
Adaptation example 1

fine model

\[
\begin{align*}
\partial_t u + \partial_x f(u, v) &= 0 \\
\partial_t v + \partial_x g(u, v) &= \frac{1}{\varepsilon}(e(u) - v),
\end{align*}
\]

\(\varepsilon \to 0, (u, v) \to (u_{eq}, h(u_{eq})),\) satisfies the equilibrium model:

\[
\partial_t u_{eq} + \partial_x f(u_{eq}, e(u_{eq})) = 0
\]

Coupling hyperbolic / hyperbolic

Adaptative procedure: theoretical criteria = corrector in the Chapman Enskog expansion: \(v = e(u) + \varepsilon v_1 + \mathcal{O}(\varepsilon^2),\)

\(f_e(u) = f(u, e(u)), U_e \equiv (u, e(u)),\)

\[
\partial_t u + \partial_x f(u, e(u)) = -\varepsilon \partial_x \left( \partial_v f(U_e) (e'(u) \partial_x f(U_e) - \partial_x g(U_e)) \right)
\]

numerical flux \(\mathcal{F}, \mathcal{G}\) (resp. \(\mathcal{F}_e\)) consistent with \(f, g\) (resp. \(f_e\)) and compatible \(\mathcal{F}_e(u) = \mathcal{F}(u_e)\) → effective indicator

\[
\nu_{1,i}^{n+1} = \lambda e'(u_i^n) \frac{\mathcal{F}_{e,i+1/2}^n - \mathcal{F}_{e,i-1/2}^n}{\Delta x} - \frac{\mathcal{G}_{i+1/2,eq}^n - \mathcal{G}_{i-1/2,e}^n}{\Delta x}
\]
Adaptation example 2 (simplified)

fine model
\[
\begin{cases}
\varepsilon \partial_t u + a \partial_x v = \frac{-2\sigma}{\varepsilon} u \\
\varepsilon \partial_t v + a \partial_x u = 0
\end{cases}
\]

telegraph equation \(\varepsilon: t \mapsto t/\varepsilon, \sigma \mapsto \sigma/\varepsilon\), large time, large friction

coarse model \(\varepsilon \rightarrow 0\)
\[
\begin{cases}
u = 0 \\
\partial_t v - \partial_{xx} \frac{a^2}{2\sigma} v = 0
\end{cases}
\]

Coupling hyperbolic / parabolic

numerical couling at interface (adapt F. Caetano results)

find an error estimate: \(u^n_i\)

AP schema

Cemracs 2011 project LRC Manon with C. Cancès and H. Mathis
K. Saleh, A.-C. Boulanger
Adaptation model 2 (more realistic)

fine model

\[
\begin{align*}
\partial_t \tau - \partial_m u &= 0, \\
\partial_t u + \partial_m p &= g - \sigma u, \\
\partial_t e + \partial_m (pu) &= gu - \sigma u^2
\end{align*}
\]

\[\sigma : t \mapsto s = \sigma t, \text{ large time } \sigma \to \infty \ (\varepsilon = 1/\sigma, \text{ large friction, large time : } u \to 0 \text{ and if } v = \sigma u, \text{ the limit model is} \]

\[
\begin{align*}
\partial_s \tau + \partial_{mm}^2 p &= 0, \\
\partial_s e - p \partial_{mm}^2 p &= 0 \\
v &= g - \partial_m p, \ p = \tilde{p}(\tau, \varepsilon).
\end{align*}
\]

we have an AP scheme

projet Cemracs 11: barotropic case (without energy)
Adaptation model 1: scalar example

fine model: 2 × 2 system (relaxation)

\[
\begin{aligned}
\partial_t u + \partial_x f(u, v) &= 0, \\
\partial_t v + \partial_x v &= \frac{1}{\varepsilon}(v_{eq}(x) - v),
\end{aligned}
\]

scalar equilibrium (\(\varepsilon \to 0\))

\[
\partial_t u + \partial_x f(u, v_{eq}(x)) = 0
\]

Theoretical results are possible: compare the solutions of the two equations with respective flux \(f(u, v_{eq}(x))\) and \(f(u, v_a(x, t))\) (Bouchut-Perthame, Chainais, Colombo-Mercier), + a posteriori error estimate (Ohlberger)
Conclusion and future work

- An original approach to coupling
  - formalize some uniqueness criteria (Andreianov, Karlsen & Risebro; Garavello, Natalini, Piccoli & Terracina, Adimurthi, Mishra, and Veerappa Gowda)
  - study resonant source term (Andreianov & Seguin)

- Interest in simple relaxation schemes

- For adaptation, a lot of work remains!
  consider at least formally the general case relaxation / equilibrium