Approximation of nonlinear functional equations

Laurence Grammont & Mario Ahues,
Université de Lyon, Institut Camille Jordan, UMR 5208
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Indo-French collaboration

Indian partners: Rekha Kulkarni and Balmohan Limaye (I.I.T Bombay)

- Before 2005
  - Visiting professor, Oberwofach (Mario Ahues, Alain Largillier, Balmohan Limaye)
  - IFIM (Indo-French institute of mathematics)
  - TOFNA 2005 : conference in Mumbai.
History of our Indo-French collaboration

Indo-French collaboration

- 2005-2008 ARCUS project
  (French Department of Foreign Affairs (MAE) and Rhône-Alpes region)

**Subjects:**
- Treatment of linear Fredholm equation of the second kind
- First work on nonlinear integral equation: fixed point problems.
- Numerical linear algebra.

**By product:**
- *Springs chool St-Etienne 2007.*
- *Papers.*
- *One of our Phd students visited IIT Guwahati.*
Indo-French collaboration

- 2009-2012 CEFIPRA

**Subjects:**
- Treatment of linear Fredholm equation of the second kind with nonsmooth kernels.
- Asymptotic expansions of the approximate solution, extrapolation methods.
- Approximation of the solution of a nonlinear integral equation of the second kind.
- A new aspect of a nonlinear problem: the role played by the discretization and the linearization.
- Numerical linear algebra: condition number of subspace basis.
Indo-French collaboration

- 2009-2012, CEFIPRA

  By product:
  - A French Phd student, Hamza Guebbai
  - Many Papers
  - Stay of 3 months of an Indian Phd student Akshay Rane in our team (bourse Sharpack)
History of our Indo-French collaboration

After 2012, IFCAM

Subjects:
• Treatment of linear and nonlinear Fredholm equation of the second kind with nonsmooth kernels.
• Asymptotic expansions of the approximate solutions of nonlinear equations, extrapolation methods for non smooth kernel.
• Discretization versus linearization.
• Case of the Fredholm equation of the first kind.

Collaboration with Thamban Nair (I.I.T Madras)?
• An application in Astrophysics with the “Observatoire de Lyon”.
• Numerical linear Algebra. Collaboration with Rafikul Alam (I.I.T Guwahati)?
The general problem

- $\mathcal{X}$, a complex infinite-dimensional Banach space.

- $\mathcal{O}$, an open subset of $\mathcal{X}$.

- $F : \mathcal{O} \to \mathcal{X}$, a nonlinear differentiable operator.

- $F'$, the Fréchet derivative of $K$.

- The equation:

  \[
  \text{find } \varphi \in \mathcal{O} : \quad F(\varphi) = 0.
  \]
Example: a nonlinear Fredholm equation of the second kind

- $K: \mathcal{O} \to \mathcal{X}$, a nonlinear differentiable operator.
- $T := K'$, the Fréchet derivative of $K$.
- The equation:

  Given $f \in \mathcal{X}$, find $\varphi \in \mathcal{O}$: $\varphi - K(\varphi) = f$. 
Bibliography

M. Ahues (2004),
“Newton Methods with Holder Derivative”,

K. E. Atkinson and F. A. Potra,
“Projection and iterated projection methods for nonlinear integral equations”,

D. R. Dellwo and M. B. Friedman,
“Accelerated projection and iterated projection methods with applications to nonlinear integral equations”,

G. M. Vainikko, Galerkin’s perturbation method and general theory of approximate methods for non-linear equations,
First discretize, then linearize (Option B)

The classical option: Atkinson, Potra, Dellwo, Friedman, Vainikko

- First, discretize: Replace $K$ with a finite-dimensional $K_n$,
  \[ [\varphi = K\varphi + f] \rightarrow [\psi_n = K_n\psi_n + f]. \]

- Next, linearize: Build the Newton sequence in finite dimension,
  \[ (I_n - K_n'(\psi_n^{(k)}))(\psi_n^{(k+1)} - \psi_n^{(k)}) = -\psi_n^{(k)} + K_n(\psi_n^{(k)}) + f. \]

**Theorem**

*Under suitable hypotheses, for $n$ fixed,*

\[ \|\psi_n^{(k)} - \psi_n\| \rightarrow 0 \text{ at least quadratically, as } k \rightarrow \infty. \]
First linearize, then discretize (Option A)

Authors’ suggestion

- First, linearize: Build the Newton sequence,

\[
[I - T(\varphi^{(k)})](\varphi^{(k+1)} - \varphi^{(k)}) = -\varphi^{(k)} + K(\varphi^{(k)}) + f, \varphi^{(0)} \in O.
\]

- Next, discretize: Replace \( T \) with a finite-dimensional \( T_n \),

\[
[I - T_n(\varphi_n^{(k)})](\varphi_n^{(k+1)} - \varphi_n^{(k)}) = -\varphi_n^{(k)} + K(\varphi_n^{(k)}) + f.
\]

**Theorem**

*Under suitable hypotheses, for \( n \) large enough but fixed,*

\[
\varphi_n^{(k)} \rightarrow \varphi \text{ at least linearly, as } k \rightarrow \infty.
\]
Discretization scheme: Galerkin or collocation method

\[ \text{[OptionA]} \approx \text{[OptionB]} \]

**Theorem**

The assumptions of the previous theorem are not satisfied. We have

\[ \varphi_n^{(k)} \text{ does not tend to } \varphi \text{ as } k \to \infty, \text{ but } \]

\[ \varphi_n^{(k)} \to \psi_n \text{ as } k \to \infty. \]
Option A

- **Linearization**

\[
[F(\varphi) = 0] \\
\downarrow \\
[F'(\varphi(k))(\varphi^{(k+1)} - \varphi^{(k)}) = -F(\varphi^{(k)}), \quad \varphi^{(0)} \in \mathcal{O}.] \\
\uparrow \\
[\varphi^{(k+1)} = \varphi^{(k)} - F'(\varphi^{(k)})^{-1}F(\varphi^{(k)}).]
\]

- **Discretization**

\[
[\varphi^{(k+1)} = \varphi^{(k)} - F'(\varphi^{(k)})^{-1}F(\varphi^{(k)})] \\
\downarrow \\
[\varphi_n^{(k+1)} = \varphi_n^{(k)} - \sum_n(\varphi_n^{(k)})F(\varphi_n^{(k)})]
\]
**Theorem**

Suppose that $F, \mathcal{O}, \phi \in \mathcal{O}, \mu > 0, R > 0, \ell > 0$ and $\alpha \in ]0, 1]$ are such that:

1. $F(\phi) = 0$, $F'(\phi)$ is invertible and $\|F'(\phi)^{-1}\| \leq \mu$,
2. $B_R(\phi) \subset \mathcal{O}$ and $F' : \mathcal{O} \rightarrow \mathcal{L}(\mathcal{X})$ is $(\ell, \alpha)$–Hölder continuous on $B_R(\phi)$,
3. For $r := \min \left\{ R, \frac{1}{(2\mu\ell)^{1/\alpha}} \right\}$, there exists $\gamma_n \in ]0, 1[$ such that
   \[
   \sup_{x \in B_r(\phi)} \|I - \sum_n(x)F'(x)\| \leq \gamma_n,
   \]
4. $\phi_n^{(0)} \in B_{\rho_n}(\phi)$, where $\rho_n := \min \left\{ r, \left( \frac{1-\gamma_n}{4\ell\mu(1+\gamma_n)} \right)^{1/\alpha} \right\}$.
Theorem

Then, for all \( k \), \( \varphi_n^{(k)} \in B_{\rho_n}(\varphi) \), and

\[
\| \varphi_n^{(k)} - \varphi \| \leq \rho_n \left( \frac{1 + \gamma_n}{2} \right)^k \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.
\]
Example 1: Nyström

Option A Linearization then discretization

Find $\varphi \in \mathcal{O} : \varphi - K(\varphi) - f = 0.$

$$K(x)(s) := \int_{0}^{1} \kappa(s, t, x(t)) \, dt, \quad x \in \mathcal{O}, \, s \in [0, 1],$$

- Linearization: Newton

$$\left(I - T(\varphi^{(k)})\right)(\varphi^{(k+1)} - \varphi^{(k)}) = -\varphi^{(k)} + K(\varphi^{(k)}) + f, \quad \varphi^{(0)} \in \mathcal{O}.$$

- Discretization: Nyström

$$[T_n(x)h](s) := \sum_{j=1}^{n} \omega_{n,j} \frac{\partial \kappa}{\partial u}(s, t_{n,j}, x(t_{n,j})) h(t_{n,j}), \quad h \in \mathcal{X}, \, s \in [0, 1].$$
Example 1: Nyström

Option A Linear system to be solved and reconstruction formula

\[ x^{(k)}(i) := \varphi^{(k)}(t_{n,i}), \quad i \in [1, n]. \]

Theorem

\[ x^{(k+1)} \text{ satisfies } (I - A^{(k)}_{n})x^{(k+1)} = b^{(k)} \text{ where } \]

\[ A^{(k)}_{n}(i, j) := \omega_{n,j} \frac{\partial \kappa}{\partial u}(t_{n,i}, t_{n,j}, x^{(k)}(j)), \]

\[ b^{(k)}_{n}(i) := - \sum_{j=1}^{n} \omega_{n,j} \frac{\partial \kappa}{\partial u}(t_{n,i}, t_{n,j}, x^{(k)}(j))x^{(k)}(j) \]

\[ + \int_{0}^{1} \kappa(t_{n,i}, t, \varphi^{(k)}_{n}(t)) \, dt + f(t_{n,i}). \]
Example 1: Nyström

Option A Linear system to be solved and reconstruction formula

Theorem

1. \( x_n^{(k+1)} \) satisfies \( (I - A_n^{(k)})x_n^{(k+1)} = b_n^{(k)} \)

2. 

\[
\begin{align*}
\varphi_n^{(k+1)}(s) &= \sum_{j=1}^{n} \omega_{n,j} \frac{\partial \kappa}{\partial u}(s, t_{n,j}, x_n^{(k)}(j))(x_n^{(k+1)}(j) - x_n^{(k)}(j)) \\
&\quad + \int_{0}^{1} \kappa(s, t, \varphi_n^{(k)}(t)) \, dt + f(s).
\end{align*}
\]
Example 1: Nyström

Option B Nyström Discretization

Discretization:

\[
[K_n(x)](s) := \sum_{j=1}^{n} \omega_{n,j} \kappa(s, t_n, x(t_n)), \quad s \in [0, 1].
\]

\[
\psi_n - K_n(\psi_n) = f.
\]

\[
y_n(i) = \sum_{j=1}^{n} \omega_{n,j} \kappa(t_n, t_n, y_n(j)) + f(t_n, i),
\]

\[
y_n(i) := \psi_n(t_n, i).
\]

Linearization: Newton method in a finite dimensional space.
Example 1: Nyström

Option B Linear system to be solved and reconstruction formula

**Theorem**

1. \( y_n^{(k+1)} \) satisfies \( (I - A_n^{(k)}) y_n^{(k+1)} = b_n^{(k)} \) où

\[
A_n^{(k)}(i,j) := \omega_{n,j} \frac{\partial \kappa}{\partial u}(t_n,i, t_n,j, y_n^{(k)}(j)),
\]

\[
b_n^{(k)}(i) := \sum_{j=1}^{n} \omega_{n,j} \left[ \kappa(t_n,i, t_n,j, y_n^{(k)}(j)) - \frac{\partial \kappa}{\partial u}(t_n,i, t_n,j, y_n^{(k)}(j)) y_n^{(k)}(j) \right] + f(t_n,i).
\]

2. \( \psi_n^{(k+1)}(s) = \sum_{j=1}^{n} \omega_{n,j} \kappa(s, t_n,j, y_n^{(k+1)}(j)) + f(s), \quad s \in [0, 1]. \)
Example 1: Nyström

Option A vs Option B

Theorem

Matrices:

\[ (A) \quad CM(i,j) := \delta_{i,j} - \omega_{n,j} \frac{\partial \kappa}{\partial u}(t_n,i, t_n,j, x_n^{(k)}(j)), \]

\[ (B) \quad CM(i,j) := \delta_{i,j} - \omega_{n,j} \frac{\partial \kappa}{\partial u}(t_n,i, t_n,j, y_n^{(k)}(j)). \]
Example 1: Nyström

Option A vs Option B

Theorem

Right hand sides:

\[(A) \quad \text{RHS}(i) := - \sum_{j=1}^{n} \omega_{n,j} \frac{\partial \kappa}{\partial u}(t_{n,i}, t_{n,j}, \chi^{(k)}_{n}(j)) \chi^{(k)}_{n}(j)\]

\[\quad + \int_{0}^{1} \kappa(t_{n,i}, t, \varphi^{(k)}_{n}(t)) \, dt + f(t_{n,i}).\]

\[(B) \quad \text{RHS}(i) := - \sum_{j=1}^{n} \omega_{n,j} \frac{\partial \kappa}{\partial u}(t_{n,i}, t_{n,j}, y^{(k)}_{n}(j)) y^{(k)}_{n}(j)\]

\[\quad + \sum_{j=1}^{n} \omega_{n,j} \kappa(t_{n,i}, t_{n,j}, y^{(k)}_{n}(j)) + f(t_{n,i}).\]
Example 1: Nyström

Option A vs Option B

Theorem

Right hand sides:

(A) \( \text{RHS}(i) := - \sum_{j=1}^{n} \omega_{n,j} \frac{\partial \kappa}{\partial u}(t_{n,i}, t_{n,j}, x^{(k)}(j))x^{(k)}(j) \)

\[ + \int_{0}^{1} \kappa(t_{n,i}, t, \varphi^{(k)}_{n}(t)) \, dt + f(t_{n,i}). \]

(B) \( \text{RHS}(i) := - \sum_{j=1}^{n} \omega_{n,j} \frac{\partial \kappa}{\partial u}(t_{n,i}, t_{n,j}, y^{(k)}(j))y^{(k)}(j) \)

\[ + \sum_{j=1}^{n} \omega_{n,j} \kappa(t_{n,i}, t_{n,j}, y^{(k)}(j)) + f(t_{n,i}). \]
Example 1: Nyström

Option A vs Option B

Theorem

Reconstruction formula

\[(B) \quad \psi_n^{(k+1)}(s) = \sum_{j=1}^{n} \omega_{n,j} \kappa(s, t_{n,j}, x_{n}^{(k+1)}(j)) + f(s), \]

\[(A) \quad \varphi_n^{(k+1)}(s) = \sum_{j=1}^{n} \omega_{n,j} \frac{\partial \kappa}{\partial u}(s, t_{n,j}, x_{n}^{(k)}(j))(x_{n}^{(k+1)}(j) - x_{n}^{(k)}(j))
\quad + \int_{0}^{1} \kappa(s, t, \varphi_n^{(k)}(t)) \, dt + f(s). \]
Example 1: Nyström

Numerical results

Kernel: \[ \kappa(s, t, x) := \cos(11\pi s) \sin(11\pi t) x^2, \]

Right hand side: \[ f(s) := \left(1 - \frac{2}{33\pi}\right) \cos(11\pi s), \]

Solution: \[ \varphi(s) := \cos(11\pi s). \]
Example 1: $\varphi - K(\varphi) = f$ and discretization = Nyström method

$\log_{10}$ of the distance between two successive iterates.
Example 1: \( \varphi - K(\varphi) = f \) and discretization = Nyström method

Error Option A: dots, Error Option B: continuous line.
Example 2: Kantorovich approximation

Option B $K_n := \pi_n K$

Atkinson, Potra [2], Dellwo, Friedman [3], Vainikko [4]:

Find $\varphi \in \mathcal{O} : \varphi - K(\varphi) = f$.

$$\psi_n - K_n \psi_n = f$$

Kantorovich: $K_n = \pi_n K.$

$$\|\psi_n - \varphi\| \leq \beta_n \to 0.$$
Example 2: Kantorovich approximation

Option B \( K_n := \pi_n K \)

Discretization:
\[
\psi_n := (I - \pi_n)f + e_n y_n,
\]
\[
y_n = \left( K((I - \pi_n)f + e_n y_n), e^*_n \right) + (f, e^*_n),
\]

Linearization:
\[
\psi_n^{(k)} := (I - \pi_n)f + e_n y_n^{(k)}.
\]

Theorem
\[
y_n^{(k+1)} \text{ satisfies } (I - A_n^{(k)}) y_n^{(k+1)} = b_n^{(k)}, where
\]
\[
A_n^{(k)} := \left( T(\psi_n^{(k)}) e_n, e^*_n \right),
\]
\[
b_n^{(k)} := \left( K(\psi_n^{(k)}), e^*_n \right) - \left( T(\psi_n^{(k)}) e_n, e^*_n \right) y_n^{(k)} + (f, e^*_n).
\]
Example 2: Kantorovich approximation

Option A \[ T_n := \pi_n T \]

- Linerization: Newton sequence:

\[
[I - T(\varphi^{(k)})](\varphi^{(k+1)} - \varphi^{(k)}) = -\varphi^{(k)} + K(\varphi^{(k)}) + f, \ varphi^{(0)} \in O.
\]

- Discretization: Replace \( T \) with a finite rank operator \( T_n = \pi_n T \),

\[
[I - T_n(\varphi_{n}^{(k)})](\varphi_{n}^{(k+1)} - \varphi_{n}^{(k)}) = -\varphi_{n}^{(k)} + K(\varphi_{n}^{(k)}) + f.
\]
Example 2: Kantorovich approximation

Option A: \( T_n := \pi_n T \)

**Theorem**

\[
\varphi_n^{(k+1)} = (I - \pi_n)(K(\varphi_n^{(k)}) + f) + e_n x_n^{(k+1)}
\]

\( x_n^{(k+1)} \) satisfies \( (I - A_n^{(k)})x_n^{(k+1)} = b_n^{(k)} \), où

\[
A_n^{(k)} := \langle T(\varphi_n^{(k)})e_n, e_n^* \rangle,
\]

\[
b_n^{(k)} := \langle g_n^{(k)} + T(\varphi_n^{(k)})(I - \pi_n)(K(\varphi_n^{(k)}) + f), e_n^* \rangle,
\]

\[
g_n^{(k)} := K(\varphi_n^{(k)}) - \pi_n T(\varphi_n^{(k)})\varphi_n^{(k)} + f.
\]
Example 2: Kantorovich approximation

Option A vs Option B

**Theorem**

*Matrices:*

(A) \( CM := I - \langle T(\varphi_n^{(k)})e_n, e_n^* \rangle, \)

(B) \( CM := I - \langle T(\psi_n^{(k)})e_n, e_n^* \rangle. \)
Example 2: Kantorovich approximation

Option A vs Option B

**Theorem**

**Right hand sides:**

(A) \( \text{RHS} = \pi_n K(\varphi_n^{(k)}) - \pi_n T(\varphi_n^{(k)})\varphi_n^{(k)} + \pi_n f \)

+ \( \pi_n T(\varphi_n^{(k)})(I - \pi_n)(K(\varphi_n^{(k)}) + f) \),

(B) \( \text{RHS} = \langle K(\psi_n^{(k)}), e^*_n \rangle - \langle T(\psi_n^{(k)})e_n, e^*_n \rangle y_n^{(k)} + \langle f, e^*_n \rangle. \)
Example 2: Kantorovich approximation

Option A vs Option B

Theorem

Right hand sides:

(A) \[ \text{RHS} = \pi_n K(\varphi_n^{(k)}) - \pi_n T(\varphi_n^{(k)}) \varphi_n^{(k)} + \pi_n f + \pi_n T(\varphi_n^{(k)})(I - \pi_n)(K(\varphi_n^{(k)}) + f), \]

(B) \[ \text{RHS} = \langle K(\psi_n^{(k)}), e_n^* \rangle - \langle T(\psi_n^{(k)}) e_n, e_n^* \rangle y_n^{(k)} + \langle f, e_n^* \rangle. \]
Example 2: Kantorovich approximation

Option A vs Option B

\[ K(x)(s) := \alpha(s) \int_0^1 \beta(t) x(t)^2 \, dt, \]

\[ \pi_n := \text{Continuous piecewise linear interpolation on a uniform grid on } n \text{ points.} \]
Discretization = Kantorovitch method

\[ T_n(x) := \pi_n T(x) \]

Kernel: \[ \kappa(s, t, x) := \sin(4\pi s) t u^2, \]

Second member: \[ f(s) := \frac{3}{4} \sin(4\pi s), \quad s \in [0, 1]. \]
Example 2: Kantorovich approximation

Logarithm of the errors for $n = 10$

Option A: +
Option B: ×
Example 2: Kantorovich approximation

Logarithm of the errors for $n = 1000$

Option A: +
Option B: ×
Conclusion

**Theorem**

**Option A**

*Newton then projection:*

\[ \varphi_n^{(k)} \to \varphi \text{ at least linearly, as } k \to \infty. \]

**Theorem**

**Option B**

*Projection then Newton:*

\[ \psi_n^{(k)} \to \psi_n \text{ at least quadratically, as } k \to \infty. \]
- **Nonlinear integral equations of the second kind: a new version of Nyström method**
  Laurence Grammont
  Accepted in NFAO

- **For nonlinear infinite dimensional equations, which to begin with: linearization or discretization?**
  Laurence Grammont, Mario Ahues, Filomena D. d’Almeida
  submitted to SINUM