Symmetry-preserving observers for water tank problems: theory and application to an oceanography data assimilation example

Silvère Bonnabel
Co-workers: Didier Auroux. Pierre Rouchon

Mines ParisTech
Centre de Robotique
Mathématiques et Systèmes
silvere.bonnabel@mines-paristech.fr

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Outline

Introduction and motivations

Water tank systems, symmetries, and observer design

Application to an oceanography example

Numerical simulations

Conclusion
Symmetries have been used in control for feedback design for non-linear systems but much less for the design of non-linear observers to our knowledge.

In control theory, a state observer is a system that uses

- a model of the real system
- noisy measurements of the input and output of the real system
- Goal: provide a real-time estimate of the internal state

It is typically a computer-implemented mathematical model (preferably low cost of computation). Exemple: Kalman filter
Linear observers

Consider the linear system

\[
\frac{d}{dt} x = Ax + Bu, \quad y = Cx
\]

where \( x \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R}^m \) the input, and \( y \in \mathbb{R}^p \) the output; \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, \) and \( C \in \mathbb{R}^{p \times n} \).

Luenberger Observer or Kalman filter

\[
\frac{d}{dt} \hat{x} = A\hat{x} + Bu + L(C\hat{x} - y)
\]

Copy + correction term \( L(C\hat{x} - y) \) equal to 0 when \( \hat{x} = x \).

L is the gain matrix
Non-linear observers

\[ \frac{d}{dt} x = f(x, u), \quad y = h(x, u) \quad y, u \text{ known signals} \]

Estimator, observer, filter, etc:

\[ \frac{d}{dt} \hat{x} = f(\hat{x}, u(t)) - L(\hat{x}, y(t)) \cdot (h(\hat{x}, u(t)) - y(t)) \]

- Luenberger observer, gain scheduling, high gains, ...
- Extended Kalman Filter (\(M, N\) "tuning" matrices)

\[ A = \frac{\partial f}{\partial x}(\hat{x}, u) \quad L = -PC^TN \]

\[ C = \frac{\partial h}{\partial x}(\hat{x}, u) \quad \frac{d}{dt} P = AP + PA^T + M^{-1} - PC^TNCP \]

- Tuning? Domain of convergence? Computational cost?
Non-linear symmetry-preserving observers

What can we do when the model

$$\frac{d}{dt}x = f(x, u), \quad y = h(x, u)$$

admits symmetries? ¹

- The linear system $\frac{d}{dt}x = Ax + Bu$ is invariant by scaling

  $$\forall \lambda > 0 \quad \frac{d}{dt}(\lambda x) = A(\lambda x) + B(\lambda u)$$

- so is the correction term in $\frac{d}{dt}\hat{x} = A\hat{x} + Bu + L(C\hat{x} - y)$

When $f(x, u)$ is not linear why should the correction term be linear??

- Redemption by geometry $\rightarrow$ make

  $$L(\hat{x}, y(t)) \cdot (h(\hat{x}, u(t)) - y(t))$$

respect the symmetries.

System considered and motivations of the talk

The amount of available data has drastically increased in the last years.

- **Measurement**: height of the ocean
- **Goal**: Estimate the marine streams.

Symmetries?

- Invariance by \( SE(2) \) (2D rotations and translations).
Water tank systems, symmetries, and observer design
The Saint-Venant equations write on the rectangular domain

\[ \frac{\partial}{\partial t} h = -\nabla (hv), \quad \frac{\partial}{\partial t} v = -v\nabla v - g\nabla h \]

where \( hv = h(v_x \mathbf{i} + v_y \mathbf{j}) \) is the horizontal transport.

There has been theoretical work on motion planning and feedback for this system but much less on observers.\(^2\)

The Saint-Venant equations write on the rectangular domain

\[ \frac{\partial}{\partial t} h = - \nabla (hv), \quad \frac{\partial}{\partial t} v = -v \nabla v - g \nabla h \]

where \( hv = h(v_x i + v_y j) \) is the horizontal transport.

- **Assumption**: \( h(x, y, t) \) is measured (with noise) for all \( x, y, t \).
- **Goal**: Estimate all the state variables \( v(x, y) \) and \( h(x, y) \) at any point \( (x, y) \in [0, L]^2 \) of the domain.
Model symmetries : SE(2)-invariance

Take $R_\theta$ rotation of angle $\theta$, then the transformations

$$ (X, Y) = R_\theta(x, y) + (x_0, y_0) $$

$$ H(X, Y) = h(x, y) $$

$$ V(X, Y) = R_\theta v(x, y) $$

leave the system unchanged

$$ \frac{\partial}{\partial t} V = - V \nabla V - g \nabla H $$

$$ \frac{\partial}{\partial t} H = - \nabla \cdot (HV) $$

This results from : $\nabla h(x, y) = R_\theta \nabla H(X, Y)$. 

Note that the domain becomes $\left( R_\theta D + (x_0, y_0) \right) \subset \mathbb{R}^2$. 
Observer design

We consider asymptotic observer of the form

\[
\frac{\partial}{\partial t} \hat{v} = -\hat{v} \nabla \hat{v} - g \nabla \hat{h} + N_v(h, \hat{v}, \hat{h})
\]

\[
\frac{\partial \hat{h}}{\partial t} = -\nabla \cdot (\hat{h} \hat{v}) + N_h(h, \hat{v}, \hat{h})
\]

where \( N_h \) and \( N_v \) are operators (versus space variables) such that

\[
N_v(h, \hat{v}, h) = 0, \quad N_h(h, \hat{v}, h) = 0
\]

\( N_v(h, \hat{v}, \hat{h}) \) is a vector and \( N_h(h, \hat{v}, \hat{h}) \) a scalar.

How to preserve \( SE(2) \) invariance in the choice of \( N_h \) and \( N_v \) ?
Symmetry-preserving observer: scalar differential terms

- Classical result: any differential scalar operator, $SE(2)$ invariant, is polynomial in $\Delta$.
- Scalar invariant correction $N_h$:

$$N_h = Q_1(\Delta, h, \hat{v}^2, \hat{h} - h) + \nabla \left( Q_2(\Delta, h, \hat{v}^2, \hat{h} - h) \right) \cdot \hat{v}$$

where $Q_1$ and $Q_2$ of the form

$$Q_i(\Delta, h, \hat{v}^2, \hat{h} - h) = \sum_{k=0}^{N} a^i_k(h, \hat{v}^2, \hat{h} - h) \Delta^k \left( b^i_k(h, \hat{v}^2, \hat{h} - h) \right)$$

the functions $a^i_k$ and $b^i_k$ being smooth functions of their arguments such that

$$a^i_k(h, \hat{v}^2, 0) = b^i_k(h, \hat{v}^2, 0) = 0.$$
Symmetry-preserving observer: vectorial differential terms

- Vector invariant correction terms $N_v$:

$$N_v = P_1(\Delta, h, \hat{v}^2, \hat{h} - h)\hat{v} + \nabla P_2(\Delta, h, \hat{v}^2, \hat{h} - h)$$

where $P_1$ and $P_2$ are similar to the $Q_i$ used for $N_h$:

$$P_i(\Delta, h, \hat{v}^2, \hat{h} - h) = \sum_{k=0}^{N} c_k^i(h, \hat{v}^2, \hat{h} - h) \Delta^k \left( d_k^i(h, \hat{v}^2, \hat{h} - h) \right)$$
symmetry-preserving observer: integral terms

\[ N_v(x, y, t) = \int \int \left[ R_1(\Delta, h, \hat{v}^2, \hat{h} - h)\hat{v} + \nabla R_2(\Delta, h, \hat{v}^2, \hat{h} - h) \right] (x - \xi, y - \zeta, t) \phi_v(\xi^2 + \zeta^2) \, d\xi \, d\zeta \]

\[ N_h(x, y, t) = \int \int \left[ S_1(\Delta, h, \hat{v}^2, \hat{h} - h) + \nabla S_2(\Delta, h, \hat{v}^2, \hat{h} - h) \cdot \hat{v} \right] (x - \xi, y - \zeta, t) \phi_h(\xi^2 + \zeta^2) \, d\xi \, d\zeta \]

where \( \phi_v \) and \( \phi_h \) are convolution kernels and the \( R_i \)'s and \( S_i \)'s are polynomials versus \( \Delta \).
Chosen observer

\[ \frac{\partial}{\partial t} h = -\nabla (hv), \quad \frac{\partial}{\partial t} \hat{v} = -\hat{v} \nabla \hat{v} - g \nabla h \]

Simplest symmetry-preserving observer (with integral corrections):

\[ \frac{\partial}{dt} \hat{h} = -\nabla (\hat{h} \hat{v}) + \int \int \phi_h(\xi^2 + \zeta^2)(h - \hat{h})(x-\xi, y-\zeta, t) \, d\xi d\zeta \]
\[ = -\nabla (\hat{h} \hat{v}) + \varphi_h \ast (h - \hat{h}) \]

\[ \frac{\partial}{dt} \hat{\hat{v}} = -\hat{v} \nabla \hat{\hat{v}} - g \nabla \hat{h} + \int \int \phi_v(\xi^2 + \zeta^2) \nabla (h - \hat{h})(x-\xi, y-\zeta, t) \, d\xi d\zeta \]
\[ = -\hat{v} \nabla \hat{\hat{v}} - g \nabla \hat{h} + \varphi_v \ast \nabla (h - \hat{h}) \]

where \( \phi_h \) and \( \phi_v \) have to be designed to ensure convergence
Comparison with Nudging

Here the 2D image \((h - \hat{h})\) is filtered with an isotropic smooth kernel (heat equation filtering) before being fed in the observer.

\[
\begin{align*}
\frac{\partial}{\partial t} \hat{h} &= -\nabla (\hat{h} \hat{v}) + \varphi_h * (h - \hat{h}) \\
\frac{\partial}{\partial t} \hat{v} &= -\hat{v} \nabla \hat{v} - g \nabla \hat{h} + \varphi_v * \nabla (h - \hat{h})
\end{align*}
\]
Let us linearize the system around the steady-state 
\((h, \nu) = (\bar{h}, 0)\), where the equilibrium height \(\bar{h}\) is constant.

It means we only consider small velocities \(\delta \nu = \nu - \bar{\nu} \ll \sqrt{g\bar{h}}\) and heights \(\delta h = h - \bar{h} \ll \bar{h}\).
Design of $\phi_h$ and $\phi_v$

The estimation errors, $\tilde{h} = \delta\hat{h} - \delta h$ and $\tilde{v} = \delta\hat{v} - \delta v$, obey the following linearized equations:

$$\frac{\partial}{\partial t} \tilde{h} = -\bar{h}\nabla\tilde{v} - \varphi_h * \tilde{h}, \quad \frac{\partial}{\partial t} \tilde{v} = -g\nabla\tilde{h} - \varphi_v * \nabla\tilde{h}.$$ 

Eliminating $\tilde{v}$ leads to the following modified damped wave equation with external viscous damping

$$\frac{\partial^2}{\partial t^2} \tilde{h} = gh\Delta\tilde{h} + \varphi_v * \Delta\tilde{h} - \varphi_h * \frac{\partial}{\partial t} \tilde{h}$$

since $\nabla(\varphi_v * \nabla h) = \varphi_v * \Delta h$

where $\tilde{h} = \hat{h} - h$ and $\tilde{v} = \hat{v} - v$ are the estimation errors.
Theorem

If \( \varphi_v \) and \( \varphi_h \) are defined by

\[
\varphi_v(x, y) = \beta_v \exp(-\alpha_v(x^2 + y^2)),
\]

\[
\varphi_h(x, y) = \beta_h \exp(-\alpha_h(x^2 + y^2)),
\]

with \( \beta_v, \beta_h, \alpha_v, \alpha_h > 0 \), then the first order approximation of the error system around the equilibrium \((h, v) = (\bar{h}, 0)\) is strongly asymptotically convergent. Indeed if we consider the following Hilbert space and norm:

\[
\mathcal{H} = H^1(\Omega) \times L^2(\Omega), \quad \|(u, w)\|_\mathcal{H} = \left( \int_\Omega \|\nabla u\|^2 + |w|^2 \right)^{1/2},
\]

then, for every \( \tilde{h} \) solution of the error equation,

\[
\lim_{t \to \infty} \left\| \left( \tilde{h}(t), \frac{\partial \tilde{h}}{\partial t}(t) \right) \right\|_{\mathcal{H}} = 0.
\]
Convergence study

We have the equation in 2D (where $\psi_v := g\bar{h}\delta_0 + \varphi_v$)

$$\frac{\partial^2}{\partial t^2} u = \psi_v \ast \Delta u - \varphi_h \ast \frac{\partial}{\partial t} u$$

in $\mathbb{R}^+ \times \Omega = \mathbb{R}^+ \times [0, \pi]^2$, $u = 0$

on $\mathbb{R}^+ \times \partial\Omega$,

$u(0) = u_0$, $u_t(0) = u_1$

in $\Omega$,

(5)

where $u(t, x, y) = \tilde{h}$.

Dirichlet boundary condition $\leftrightarrow$ we set $\hat{h} = h$ on the boundary.

Let $(e_{pq})$ be the orthonormal basis of $H^1_0(\Omega)$ composed of eigenfunctions of the unbounded operator $\Delta$:

$$e_{pq} = \frac{2}{\pi} \sin(px) \sin(qy).$$

(6)
Convergence study - Gain design

For the kernels $\phi_v$ and $\phi_h$ we choose:

$$\phi_v(x, y) = (f(x) \ast f(x)) (f(y) \ast f(y)),$$
$$\phi_h(x, y) = (g(x) \ast g(x)) (g(y) \ast g(y)),$$

where $f$ and $g$ are smooth even functions.

To respect the symmetries $\phi_v(x, y)$ and $\phi_h(x, y)$ must be functions of $x^2 + y^2$.

Take for instance

$$\phi_v(x, y) = \beta_v \exp(-\alpha_v(x^2 + y^2)),$$
$$\phi_h(x, y) = \beta_h \exp(-\alpha_h(x^2 + y^2)).$$
Convergence study - Gain design

We have

\[ \varphi_v(x, y) = \psi_v(x^2 + y^2) = (f(x) \ast f(x)) (f(y) \ast f(y)) \quad (11) \]
\[ \varphi_h(x, y) = \psi_h(x^2 + y^2) = (g(x) \ast g(x)) (g(y) \ast g(y)) \quad (12) \]

Take \( f \) and \( g \) smooth even functions. The Fourier coefficients are real

\[ \hat{c}_{pq} = \hat{f}_p\hat{f}_q \quad \text{and} \quad \hat{g}_p\hat{g}_q \]

We have

\[ (\varphi_v \ast \Delta) e_{pq} = -(p^2 + q^2)\hat{f}_p\hat{f}_q e_{pq} = -(p^2 + q^2)f_{pq} e_{pq} \]

So the convolution products lead to a frequency-modified damped wave equation,

\( e_{pq} \) are still eigenvectors of the modified Laplacian operator.
Convergence study - back to convergence

\[ \frac{\partial^2}{\partial t^2} u = \psi_v \ast \Delta u - \varphi_h \ast \frac{\partial}{\partial t} u \]
\[ \text{in } \mathbb{R}^+ \times \Omega = \mathbb{R}^+ \times [0, \pi]^2, \]
\[ u = 0 \]
\[ \text{on } \mathbb{R}^+ \times \partial \Omega, \]
\[ u(0) = u_0, \quad u_t(0) = u_1 \]
\[ \text{in } \Omega, \] (13)

rewrites

\[ \frac{d}{dt} U = AU \]

where \( U = (u, u_t) = (u, v) \) and \( A \) is the unbounded linear operator
\[ A(u, v) = (v, \varphi_h \ast \Delta u - \varphi_v \ast v) \]

\[ E_{pq} = \begin{pmatrix} 1 \\ \lambda_{\pm pq} \end{pmatrix} e_{pq} \] (14)

form a Riesz basis of \( \mathcal{H} \) and are eigenvectors of \( A \) associated to the eigenvalues \( \lambda_{\pm pq} \), solutions of

\[ \lambda^2_{\pm pq} + g^2_{pq} \lambda_{\pm pq} + f^2_{pq} (p^2 + q^2) = 0. \] (15)
Convergence study - Form of the solution

The solution $u$ is given by the series

$$u(t, x, y) = \frac{2}{\pi} \sum_{1 \leq p, q} u_{pq}(t) \sin(px) \sin(qy),$$

with either

$$u_{pq}(t) = e^{-\frac{g_{pq}^2}{2} t} (A_{pq} \cos(\omega_{pq} t) + B_{pq} \sin(\omega_{pq} t)),\quad (16)$$

or

$$u_{pq}(t) = e^{-\frac{g_{pq}^2}{2} t} (A_{pq} \cosh(\tilde{\omega}_{pq} t) + B_{pq} \sinh(\tilde{\omega}_{pq} t)).$$

with

$$\omega_{pq} = \sqrt{4(p^2 + q^2)f_{pq}^2 - g_{pq}^4}$$

$$\tilde{\omega}_{pq} = \sqrt{g_{pq}^4 - 4(p^2 + q^2)f_{pq}^2}$$

but

- $g_{pq}^2$ are the Fourier coefficients of $\beta_h \exp(-\alpha_v (x^2 + y^2))$
- $f_{pq}^2$ coefficients of $g\bar{h}\delta_0 + \beta_v \exp(-\alpha_v (x^2 + y^2))$
Finally, the coefficients can be found using the Fourier series of
the initial condition. We have

\[
A_{pq} = \frac{4}{\pi^2} \int_{[0, \pi]^2} u(0, x, y) \sin(px) \sin(qy) dxdy,
\]

\[
B_{pq} = \frac{4}{\omega_{pq} \pi^2} \int_{[0, \pi]^2} \left( u_t(0, x, y) + \frac{g_{pq}^2}{2} u(0, x, y) \right) \sin(px) \sin(qy) dxdy.
\]
Convergence study

Let

\[ u_N(t, x, y) = \frac{2}{\pi} \sum_{p+q \leq N} e^{-g_{pq}^2 t} \left( A_{pq} \cos(\omega_{pq} t) \right. \]
\[ \left. + B_{pq} \sin(\omega_{pq} t) \right) \sin(p x) \sin(q y) \]

\[ \| u_N, \frac{d}{dt} u_N \|_H \rightarrow 0 \text{ exponentially (cf numerical simus).} \]

\[ \| u - u_N, \frac{d}{dt} (u - u_N) \|_H \] can be arbitrarily small for \( N \) large enough because of Parseval’s lemma (\( u_0 \in H^1_0(\Omega) \) and \( u_1 \in L^2(\Omega) \))

We proved the strong convergence of the linearized error system \( u = \tilde{h} \):

\[ \lim_{t \to \infty} \| \tilde{h}, \frac{d}{dt} \tilde{h} \|_H = 0 \]
Application to an oceanography exemple
Single layer model model

\[
\frac{\partial (hv)}{\partial t} + (\nabla \cdot (hv) + (hv) \cdot \nabla)v = \ldots
\]
\[
\ldots - g' h \nabla h - k \times f(hv) + (\alpha_A \mathbf{A} \nabla^2 - R)(hv) + \alpha_{\tau u} \tilde{\tau} \mathbf{i}/\rho
\]
\[
\frac{\partial h}{\partial t} = -\nabla \cdot (hv)
\]

where

- density \( \rho \), layer height \( h(x, y, t) \), fluid velocity \( v(x, y, t) \), rectangular domain \( 0 < x < L, 0 < y < L \) where \( x \) and \( y \) point east and north
- \( f \) represents Coriolis effect, \( k \) points upward, \( g' \) is the reduced gravity
- \( \tilde{\tau} \mathbf{i} \) wind term of intensity \( \tilde{\tau} \)
- \( R \) and \( A \) known damping coefficients.

The Goal: to estimate \( v(x, y, t) \) from the satellite data \( h(x, y, t) \).

SE(2) invariance

i and j point respectively towards East and North...

Take $R_\theta$ rotation of angle $\theta$, then the transformations

$$(X, Y) = R_\theta(x, y) + (x_0, y_0)$$

$H(X, Y) = h(x, y)$

$V(X, Y) = R_\theta v(x, y)$

leave the dynamics unchanged

$$\frac{\partial (HV)}{\partial t} + (\nabla \cdot (HV) + (HV) \cdot \nabla)V = -g' H \nabla H - K \times F(HV)$$

$$+ (\alpha_A A \nabla^2 - R)(HV) + \alpha_{\tau u} \tilde{r} l / \rho$$

$$\frac{\partial H}{\partial t} = -\nabla \cdot (HV)$$

This results from : $\nabla h(x, y) = R_\theta \nabla H(X, Y)$, $K = k$, $l = R_\theta i$. ...
Symmetry-preserving observers

As in the case of Saint-Venant equations, we take

\[
\frac{\partial (\hat{h}\hat{v})}{\partial t} + (\nabla \cdot (\hat{h}\hat{v}) + (\hat{h}\hat{v}) \cdot \nabla)\hat{v} = -g'\hat{h}\nabla\hat{h} - \mathbf{k} \times f(\hat{h}\hat{v})
\]

\[
+ (\alpha_A A\nabla^2 - R)(\hat{h}\hat{v}) + \alpha_{\text{tau}} \tilde{\tau}_i / \rho + \varphi_v \ast \left(\nabla(h - \hat{h})\right)
\]

\[
\frac{\partial \hat{h}}{\partial t} = -\nabla \cdot (\hat{h}\hat{v}) + \phi_h \ast (h - \hat{h})
\]

and we use a heuristic gain tuning on the linearized simplified system.
Heuristic gain tuning on the linearized simplified model

Reminding

\[ \varphi_h(x, y) = \beta_h \exp(-\alpha_h(x^2 + y^2)) \]
\[ \varphi_v(x, y) = g\bar{h}\delta_0 + \beta_v \exp(-\alpha_v(x^2 + y^2)) \]

The error system can be approximated by the following system \((\alpha = +\infty)\):

\[ \frac{\partial^2 \tilde{h}}{\partial t^2} + 2\xi_0\omega_0 \frac{\partial \tilde{h}}{\partial t} = (L_0\omega_0)^2 \Delta \tilde{h}. \]

where \(L_0^2\omega_0^2 = g\bar{h} + \bar{h}\beta_v, 2\xi_0\omega_0 = \beta_h, \)

A dimensional analysis allows to choose :

- \(\omega_0\) and \(L_0\) the characteristic pulsation and length
- \(\alpha_h^{-2} = \alpha_v^{-2}\) is the size of the region of influence.
Numerical simulations

1. Saint-Venant system (water tank)
2. Full non-linear shallow water model (ocean)
Model parameters and gain tuning

- Domain = square box, of dimension 2000 km \( \times \) 2000 km.
- equilibrium height \( \bar{h} = 500 \text{ m} \),
- regular spatial discretization with 81 \( \times \) 81 gridpoints \( \rightarrow \) space step of 25 km. The time step 30 mn, time periods of 1 to 4 months (1440 to 5760 time steps).
- height varies between 497.7 and 501.9 m, transversal velocity in \( \pm 0.008 \text{ m.s}^{-1} \).
- \( \alpha = 1 \text{ m}^{-2} \).
- frequency for the error system \( \omega_0 \sqrt{1 - \xi_0^2} \) chosen close to the natural frequency \( \sqrt{g\bar{h}/L_0} \) of the physical system
- Truncation of the Gaussian at 10 pixels away from center
1) Saint-Venant system

we consider the Saint-Venant system with small velocities
\( \delta v = v - \bar{v} \ll \sqrt{gh} \) and heights \( \delta h = h - \bar{h} \ll \bar{h} \)

\[ e_h = \frac{\|(\hat{h} - \bar{h}) - (h - \bar{h})\|}{\|h - \bar{h}\|}, \quad e_v = \frac{\|(\hat{v} - \bar{v}) - (v - \bar{v})\|}{\|v - \bar{v}\|} \quad (17) \]

where \( \| . \| \) is the standard \( L^2 \) norm. We observe

\[ e_h(t) = e_h(0) \exp(-c_h t), \quad e_v(t) = e_v(0) \exp(-c_v t) \quad (18) \]

With a 20% noise:
1) Comparison with the standard Nudging technique

The nudging algorithm (Luenberger observer) writes

\[
\frac{\partial \hat{h}}{\partial t} = -\nabla (\hat{h} \hat{v}) + K_h (h - \hat{h}), \tag{19}
\]

\[
\frac{\partial \hat{v}}{\partial t} = -\hat{v} \nabla \hat{v} - g \nabla \hat{h} + K_v \nabla (h - \hat{h}). \tag{20}
\]

It corresponds to \( \varphi_v = \varphi_h = \delta_0 \), i.e. \( \alpha = +\infty \).

<table>
<thead>
<tr>
<th>Size of the Gaussian kernel</th>
<th>Decrease rate ((h, v_x, v_y))</th>
<th>Estimation error at convergence ((h, v_x, v_y))</th>
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<td>( \alpha_h = \alpha_v = 0.5 )</td>
<td>(1.49 \times 10^{-6})</td>
<td>(4.43 \times 10^{-3})</td>
</tr>
<tr>
<td></td>
<td>(1.40 \times 10^{-6})</td>
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<td>( \alpha_h = \alpha_v = 10^3 )</td>
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<td>(2.48 \times 10^{-7})</td>
<td>(1.59 \times 10^{-2})</td>
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</table>
2) Full non-linear oceanographic shallow water model

\[ e_h = \frac{\|(\hat{h} - \bar{h}) - (h - \bar{h})\|}{\|h - \bar{h}\|}, \quad e_v = \frac{\|(\hat{v} - \bar{v}) - (v - \bar{v})\|}{\|v - \bar{v}\|} \]  

(21)

where \(\| . \|\) is the standard \(L^2\) norm. We observe

\[ e_h(t) = e_h(0) \exp(-c_h t), \quad e_v(t) = e_v(0) \exp(-c_v t) \]  

(22)

only at the beginning.

With a 20 % noise:
2) Comparison with the standard Nudging technique

<table>
<thead>
<tr>
<th>Size of the Gaussian kernel</th>
<th>Decrease rate $(h, v_x, v_y)$</th>
<th>Estimation error at convergence $(h, v_x, v_y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_h = \alpha_v = 0.5$</td>
<td>$2.74 \times 10^{-6}$</td>
<td>$1.71 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$1.87 \times 10^{-6}$</td>
<td>$1.72 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$1.62 \times 10^{-6}$</td>
<td>$2.21 \times 10^{-1}$</td>
</tr>
<tr>
<td>$\alpha_h = \alpha_v = 1$</td>
<td>$1.36 \times 10^{-6}$</td>
<td>$1.57 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$9.65 \times 10^{-7}$</td>
<td>$1.30 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$8.38 \times 10^{-7}$</td>
<td>$1.59 \times 10^{-1}$</td>
</tr>
<tr>
<td>$\alpha_h = \alpha_v = 10^3$</td>
<td>$4.42 \times 10^{-7}$</td>
<td>$2.26 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$2.98 \times 10^{-7}$</td>
<td>$2.25 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$2.55 \times 10^{-7}$</td>
<td>$3.04 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

**Table:** Full non-linear model: decrease rate and value at convergence of the estimation error, for the three variables $h$, $v_x$ and $v_y$, in the case of noisy observations (20% noise).

In the next slide we will see that the observer allows to identify very well the main currents in a realistic setting (e.g. gulf stream).
2) Results with the chosen observer

- **Initial guess**
- **Noisy observation**
- **Identified height**
- **True height**
2) Results with the chosen observer
Conclusion

We designed an observer

- Much more economical computationally than EKF or variational methods.
- Gives better results than the nudging (Luenberger). Especially much more robust to gaussian noise.
- Gain design based on heuristic arguments on the linear first order system (easy to tune).
- Practical gain design in two steps:
  1. Convergence analysis easy around a steady state: linear and local gain design.
  2. Gain extrapolations to the nonlinear regime becomes natural via invariance.

Here we consider space-continuous and time-continuous measurements: other situations of practical interest exist: boundary measures, discrete-time and/or discrete-space measurements.
Other possibilities

\[ \frac{\partial}{\partial t} \hat{h} = -\nabla(\hat{h}\hat{v}) + \phi_h \ast (h - \hat{h}) + \phi_h^\Delta \ast \Delta(\hat{h} - h) \]

\[ \frac{\partial}{\partial t} \hat{v} = -\hat{v}\nabla\hat{v} - g\nabla\hat{h} + \phi_v \ast \nabla(h - \hat{h}) \]

Then the first variation around \( h = \bar{h} \) and \( v = 0 \), can be identified to the wave equation

\[ \frac{\partial^2}{\partial t^2} \tilde{h} = (g\bar{h} + \beta_v)\Delta \tilde{h} - \beta_h \frac{\partial}{\partial t} \tilde{h} + \beta_h^\Delta \Delta \left( \frac{\partial}{\partial t} \tilde{h} \right) \]

An additional **structural damping** changes drastically the spectrum.
Dimensional analysis design:

\[ \beta_h = \omega_0, \quad K \beta_v = \max(0, (L_0\omega_0)^2 - g\bar{h}), \quad \beta_h^\Delta = L_0^2\omega_0. \]
Back to the single layer system

Forward Nudging

\[
\frac{\partial (\hat{h}\hat{v})}{\partial t} + (\nabla \cdot (\hat{h}\hat{v}) + (\hat{h}\hat{v}) \cdot \nabla)\hat{v} = -g' \hat{h} \nabla \hat{h} - k \times f(\hat{h}\hat{v}) \\
+ (\alpha_A A \nabla^2 - R)(\hat{h}\hat{v}) + \alpha_{\text{tau}} \tilde{\tau}i/\rho + \phi_v * \left(h \nabla (h - \hat{h})\right)
\]

\[
\frac{\partial \hat{h}}{\partial t} = -\nabla \cdot (\hat{h}\hat{v}) + \phi_h * (h - \hat{h}) + \phi_h^A * \Delta(\hat{h} - h)
\]

For backward nudging: $\phi_h \mapsto -\phi_h$, $K_h^A \mapsto -\phi_h^A$ and $\phi_v$ unchanged.