Application of the topological gradient method to tomography

D. Auroux* — L. Jaafar-Belaid** — B. Rjaibi**

* Institut de Mathématiques de Toulouse
Université Paul Sabatier
31062 TOULOUSE cedex 9
FRANCE
auroux@math.univ-toulouse.fr

** ENIT-LAMSIN, Campus Universitaire
BP 37, 1002 Le Belvédère, Tunis
TUNISIE
lamia.belaid@esstt.rnu.tn, badreddine.rjaibi@lamsin.rnu.tn

RÉSUMÉ. Une nouvelle méthode de reconstruction pour la tomographie par faisceaux parallèles est proposée. Cette méthode est basée sur l'approche du gradient topologique. Des résultats expérimentaux obtenus sur des données bruitées illustrent les possibilités de cette approche prometteuse dans le domaine de traitement d'images IRM.

ABSTRACT. A new method for parallel beam tomography is proposed. This method is based on the topological gradient approach. Experimental results obtained on noisy data illustrate the efficiency of this promising approach in the case of Magnetic Resonance Imaging.

MOTS-CLÉS : Développement asymptotique topologique, gradient topologique, reconstruction tomographique, tomographie, transformation de Radon.

KEYWORDS : Topological asymptotic expansion, topological gradient, tomographic reconstruction, tomography, Radon transform.

1. Introduction

The tomographic problem can be seen as the reconstruction of an object from measurements which are line integrals of the processed object at some given orientations (or view angles). This problem has many applications including medical imaging, electron microscopy, radio astronomy, ...[8]. This reconstruction highly depends on the amount of available line integrals. Classically, when line integrals are available from many directions and when the measurements are nearly noise free, the filtered backprojection reconstruction gives good results. Unfortunately, these conditions are not possible in real life applications, and the interpretation of results will be degraded. Another drawback is the computational cost required by traditional methods for tomography. Currently, the fastest algorithms require $O(n^2 \log n)$ operations for reconstructing n^2 pixels.

The goal of this paper is to present a new method for the tomographic problem. This method is based on the topological gradient approach, that has been introduced for topological optimization purpose [1, 9, 10, 14]. The idea of topological optimization is to create a partition of a given domain (in our case, an image) into an optimal design and its complementary. A common way to consider the restoration problem is to identify the edges of the image, in order to preserve them in the restoration process. Then, the image is smoothed elsewhere. This technique was successfully used for several problems in image processing [3, 4, 5, 11]. The computational time was very interesting too. On the other hand, it has been shown in [11] that a classical way to restore a given image f is exactly the Tikhonov regularisation applied to the inversion of a compact operator. For all these reasons, the topological gradient approach seems appropriate for the tomographic problem, which is well known to be an ill-posed problem.

The Radon transform represents a set of parallel line integral projections of a $2D$ function f at different angles θ, where f is the processed image, defined on a bounded open and convex domain Ω of $\mathbb{R}^2$. These projections (or sinogram) are given by :

$$\mathcal{R}(f)(\theta, r) = \int\int f(x,y)\delta(r - x\cos\theta - y\sin\theta)dxdy, \qquad [1]$$

where r and θ are the polar coordinates and $\delta(.)$ is the Dirac delta function. We refer to [6, 8, 12, 13, 15] for the historical development of tomography and some review of all the methods proposed in the literaure for tomographic reconstruction. We compare our approach with a filtered backprojection (FBP) reconstruction method, which is a widely used technique for inverting the $2D$ Radon transform.

This paper is organized as follows : Section 2 is devoted to a filtered backprojection reconstruction method [13]. The topological gradient approach and its application to tomography is developed in section 3. In section 4, we summarize the main results, discuss the complexity of our algorithm and give some perspectives. Some numerical experiments showing the efficiency of our method are also given.

2. A filtered backprojection reconstruction method

We describe briefly in this section a Fourier reconstruction algorithm based on the well known Fourier slice theorem [13]. More precisely, this reconstruction algorithm processes into three steps :

1) Compute the $1D$ Fourier transform of the projections in the variable r :

$$\mathcal{F}_r\mathcal{R}(f)(r,\theta).$$

2) Use the ramp filter $|\omega|$ for filtering the projections :

$$\widetilde{f}(\rho,\theta) = \mathcal{F}_r^{-1} |\omega| \, \mathcal{F}_r\mathcal{R}(f)(r,\theta),$$

where $\mathcal{F}_r^{-1}$ denotes the inverse Fourier transform in the variable r.

3) Apply the continuous backprojection operator $\mathcal{R}^{\#}$:

$$f(x,y) = \mathcal{R}^{\#}\widetilde{f}(\rho,\theta) = \int_0^{\pi} \widetilde{f}(xcos\theta + ysin\theta)d\theta.$$

The FBP reconstruction algorithm provides a reconstructed image of good quality in the case of sufficient noise-free data. However, these conditions are rarely available in real life applications inducing lower visual quality of the processed image.

3. Application of the topological gradient method to computed tomography

In this section, we use the topological gradient as a tool for the reconstruction problem in tomography. We recall that a standard approach for regularizing the ill-posed problem of tomographic imaging consists in the following optimization problem :

$$\min_f \|Af - g\|^2 + \lambda\phi(f), \qquad [2]$$

where A denotes a system matrix defining the discrete Radon tranform, g is the measured projection, ϕ is a regularization functional and λ represents a parameter which controls the tradeoff between a good fit to the data and the smoothness of the solution. Inspired by [7], in which the authors introduced the semi-norm of the total variation (TV) which is particularly efficient in recovering piecewise smooth functions without smoothing sharp discontinuities, we propose to consider the following minimization problem :

$$\min_f \int_\Omega |Af - g|^2 \, dx + c \int_\Omega |\nabla f|^\alpha \, dx, \qquad [3]$$

where c is a positive constant and α is equal to 1 on the edges and to 2 elsewhere. For a small $\rho \geq 0$, let $\Omega_\rho = \Omega \backslash \sigma_\rho$ be the perturbed domain by the insertion of a crack

$\sigma_\rho = x_0 + \rho\sigma(n)$, where $x_0 \in \Omega$, $\sigma(n)$ is a straight crack, and n is a unit vector normal to the crack. For a given regular function g, we consider the following problem :

$$\begin{cases} -div\,(c\nabla f_\rho) + A^\# A f_\rho = A^\# g \quad in \quad \Omega_\rho, \\ \partial_n f_\rho = 0 \quad on \quad \partial\Omega_\rho, \end{cases} \qquad [4]$$

where n denotes the outward unit normal to $\partial\Omega_\rho$ and $A^\#$ is the dual discrete Radon transform. To preserve edges as much as possible, we look for the leading term of

$$j(\rho) = J(f_\rho) = \int_{\Omega_\rho} \|\nabla f_\rho\|^2 dx. \qquad [5]$$

By considering the solution v to the adjoint problem

$$\begin{cases} -div(c\nabla v) + A^\# A v = -\partial_f J(f) \quad in \quad \Omega, \\ \partial_n v = 0 \quad on \quad \partial\Omega, \end{cases} \qquad [6]$$

we obtain the following topological asymptotic expansion in the case of a thin crack with boundary condition $\partial_n u = 0$ on $\partial\sigma(n)$ [1, 5, 11] :

$$j(\rho) - j(0) = \rho^2 G(x_0, n) + o(\rho^2), \qquad [7]$$

with

$$G(x_0, n) = -\pi c(\nabla f(x_0).n)(\nabla v(x_0).n) - \pi|\nabla f(x_0).n|^2. \qquad [8]$$

The topological gradient could be written as

$$G(x, n) = \langle M(x)n, n\rangle \qquad [9]$$

where M(x) is the symmetric matrix defined by

$$M(x) = -\pi c \frac{\nabla f(x)\nabla v(x)^T + \nabla v(x)\nabla f(x)^T}{2} - \pi\nabla f(x)\nabla f(x)^T. \qquad [10]$$

For a given x, $G(x, n)$ takes its minimal value when n is the eigenvector associated to the lowest eigenvalue λ_{min} of M. This value will be considered as the topological gradient. We have then to calculate the reconstruction solution by solving the following problem :

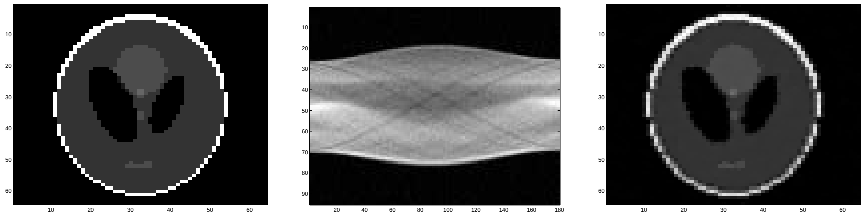
$$\begin{cases} -div\,(c_1(x_0)\nabla f) + A^\# A f = A^\# g \quad in \quad \Omega, \\ \partial_n f = 0 \quad on \quad \partial\Omega, \end{cases} \qquad [11]$$

with

$$c_1(x_0) = \begin{cases} \dfrac{1}{|\nabla f(x_0)|}, \quad x_0 \in \{x \in \Omega, \lambda_{min} < \alpha_0 < 0\}, \\ c_0 \quad \text{elsewhere}, \end{cases} \qquad [12]$$

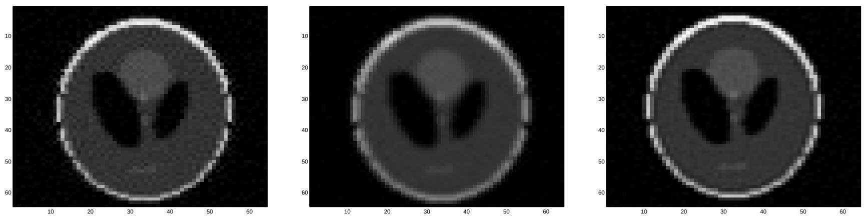
where c_0 is a given constant.

We have tested our method on the well known Shepp-Logan head phantom, for which projections data have been computed using a discrete Radon transform. Figure 1 (a) shows the original image. A Gaussian noise is added to the data with a signal to noise ratio $SNR = 22$, the sinogram is represented in figure 1(b). Figure 1(c) shows the reconstruction result ($PSNR = 26.18$). In order to visualize the performance of our approach, we



(a) (b) (c)

Figure 1. *Reconstruction of the Shepp-Logan head phantom using the topological gradient method : (a) original image, (b) noised sinogram (SNR=22), (c) reconstructed image (PSNR=26.18).*



(a) (b) (c)

Figure 2. *Reconstruction of the Shepp-Logan head phantom using the FBP and TV approaches : (a) reconstructed image using FBP (PSNR=14.59), (b) FBP reconstruction followed by a Hamming filter (PSNR=15.91), (c) reconstructed image using the TV method (PSNR=22.43).*

propose to give numerical results for both FBP and TV approaches. The first method is shown in figure 2(a), and $PSNR = 14.59$. In order to attenuate the distorsion observed in the reconstructed image, we apply then a Hamming filter on the image, but this technique widely smoothes the edges as seen in figure 2(b), and the quality of the result is then highly reduced. According to [15], a total variation based reconstruction for computed tomography has also been tested and the result is shown in figure 2(c), and $PSNR = 22.43$ in this case.

4. Conclusion

We have presented in this work a new method for the reconstruction problem from $2D$ tomographic data. To make this method relevant for real life applications, we can consider a spectral approach, based on the discrete cosine transform, and a preconditioned conjugate gradient method for solving the equations. In this case, the topological gradient algorithm requires $O(n^2 \log n)$ operations, where n is the size of the image. The numerical results show reconstructed images of good visual quality compared to the FBP and

TV methods. Our approach also produces images of higher PSNR. Several other reconstruction algorithms based on the topological gradient approach are under consideration in both $2D$ and $3D$ cases.

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