Holomorphic symplectic manifolds

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If you look from very high at the collection of projective manifolds, you see 3 dominating mountains, according to the positivity of the canonical bundle:

- $K < 0$: the Fano manifolds;
- The manifolds with $K \sim 0$;
- $K > 0$: manifolds of general type.

The first one is relatively small: a finite number of families in each dimension. The third one is enormous. The second one, which we are going to discuss, is big but still somewhat accessible.

**Very roughly**, the minimal model program tries to relate all projective manifolds to these 3 types, e.g. by finding fibrations where the base and the general fiber are of these types.
Manifolds with $c_1 = 0$

For the intermediate case we have the following theorem:

**Decomposition theorem**

Let $M$ be a compact Kähler manifold with $c_1(M) = 0$ in $H^2(M, \mathbb{R})$.

There exists $M' \to M$ finite étale with $M' = T \times \prod_i X_i \times \prod_j Y_j$

- $T =$ complex torus;
- $X_i = X$ simply-connected projective, $\dim \geq 3$, $H^0(X, \Omega^*_X) = \mathbb{C} \oplus \mathbb{C}\omega$, where $\omega$ is a generator of $K_X$ (Calabi-Yau manifolds).
- $Y_j = Y$ compact simply-connected, $H^0(Y, \Omega^*_Y) = \mathbb{C}[\varphi]$, where $\varphi \in H^0(Y, \Omega^2_Y)$ is everywhere non-degenerate (irreducible symplectic manifolds, aka hyperkähler).

The theorem follows from Yau’s theorem on the existence of Ricci-flat metrics, plus the Berger classification of holonomy groups.
Thus a basic type of manifolds with $c_1 = 0$ is:

**Definition**

A *irreducible holomorphic symplectic* (IHS) manifold $X$ is a simply-connected, compact Kähler manifold $X$ with $H^0(X, \Omega^2_X) = \mathbb{C} \varphi$, where $\varphi: \bigwedge^2 T_X \rightarrow \mathcal{O}_X$ is everywhere non-degenerate.

**Some consequences**: $X$ has even dimension $2r$; the form $\varphi^r \in H^0(K_X)$ is everywhere $\neq 0$, hence $K_X \cong \mathcal{O}_X$.

**Example**: For $r = 1$, IHS = K3 surface (e.g. $S_4 \subset \mathbb{P}^3$).

For a long time no other example was known; Bogomolov (1978) claimed that they do not exist. But Fujiki constructed a 4-dimensional example ($\sim 1982$), then I found 2 series in each dimension.
The Hilbert scheme

**The idea:** Start from a K3 surface $S$, with $\sigma \neq 0$ in $H^0(S, \Omega^2_S)$. $H^0(S^r, \Omega^2_{S^r}) = \{\lambda_1 \operatorname{pr}_1 \sigma + \ldots + \lambda_r \operatorname{pr}_r \sigma\}$, symplectic if all $\lambda_i \neq 0$.

To get one symplectic form, impose $\lambda_1 = \ldots = \lambda_r$, i.e. invariance under $\mathcal{S}_r$: we look at the **symmetric product** $S^{(r)} := S^r / \mathcal{S}_r$. An element of $S^{(r)}$ can be viewed as a 0-cycle $x_1 + \ldots + x_r$ ($x_i \in S$).

Unfortunately $S^{(r)}$ is singular along $\Delta$ where $x_i = x_j$ for some $i \neq j$.

But there is a miraculous desingularization, the **Hilbert scheme** $S[r] := \{Z \subset S \mid \text{length}(Z) = r\}$. $\pi : S[r] \to S^{(r)}$, $Z \mapsto \sum_{p \in Z} m(p) p$.

$E := \pi^{-1}(\Delta)$ is a divisor in $S[r]$. $\pi : S[r] \setminus E \sim \to S^{(r)} \setminus \Delta$, but $\pi$ contracts $E$ to $\Delta$. For instance if $r = 2$:

$Z$ such that $\pi(Z) = 2p \leftrightarrow$ tangent direction at $p$ so that $\pi^{-1}(2p) = \mathbb{P} T_p(S)$. Key fact: $S[r]$ is smooth (Fogarty).
The standard examples

**Proposition**

*For* $S$ a K3 surface, $S^{[r]}$ is an irreducible symplectic manifold.*

**Sketch of proof:** The 2-form $\sum \text{pr}_i^* \sigma$ descends to $S^{(r)} \setminus \Delta \cong S^{[r]} \setminus E$, and spans $H^0(S^{[r]} \setminus E, \Omega^2)$. Local computation shows that it extends to a symplectic form on $S^{[r]}$. □

The same construction works starting from a complex torus $T$. $T^{[r]}$ is not simply-connected, but consider $K_r(T) := \text{fiber of}$

$$
T^{[r+1]} \xrightarrow{\pi} T^{(r+1)} \xrightarrow{s} T \text{ where } s(x_0 + \ldots + x_r) = \sum x_i.
$$

**Proposition**

*$K_r(T)$ is an IHS manifold* (“generalized Kummer manifold”).

Same proof.
$X$ compact complex manifold. *Deformation* of $X$ over pointed space $(B, o): f : X \to B$ proper smooth, with $X_o \sim X$.

If $H^0(X, T_X) = 0$, there exists a **universal** local deformation, parametrized by $\text{Def}_X \subset H^1(X, T_X)$, with $T_o(\text{Def}_X) = H^1(X, T_X)$.

**Theorem (Tian-Todorov)**

If $K_X = O_X$, $\text{Def}_X$ is smooth at $o$ (equivalently, $\text{Def}_X$ is an open subset of $H^1(X, T_X)$).
Deformations of symplectic manifolds

For $X$ symplectic, $T_X \cong \Omega^1_X$, so $H^1(X, T_X) = H^{1,1} \subset H^2(X, \mathbb{C})$.

**Proposition**

\[ H^2(S^{[r]}, \mathbb{C}) = H^2(S, \mathbb{C}) \oplus \mathbb{C}[E], \quad H^2(K_r, \mathbb{C}) = H^2(T, \mathbb{C}) \oplus \mathbb{C}[E]. \]

**Corollary**

The IHS $S^{[r]}$ or $K_r(T)$ form a **hypersurface** in the deformation space.

Indeed we get $H^{1,1}(S^{[r]}) = H^{1,1}(S) \oplus \mathbb{C}[E]$, and the analog for $K_r$.

So a general deformation $X$ is **not** of the form $S^{[r]}$ – we say that $X$ is of type $K3^{[r]}$. Same for $K_r(T)$. We will see later examples of such deformations.
Other examples?

When I found the 2 series $S^{[r]}$ and $K_r(T)$ in 1983, I expected that many other examples would appear. This turned out to be surprisingly difficult. At the moment, the only examples known, up to deformations, are:

- $S^{[r]}$ and $K_r(T)$, of dimension $2r$;
- 2 examples $OG_{10}$ and $OG_6$, of dimension 10 and 6, due to O’Grady – again starting from a K3 or a complex torus.

It is an important problem to find more examples.

Note the contrast with Calabi-Yau manifolds: the physicists have constructed more than 30 000 families of Calabi-Yau threefolds! Whether the number of such families is finite is an open problem.
The period map

\[ H^2(X, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} \]

\[ \|C\varphi\| \|C\overline{\varphi}\| \]

Put \( H = H^2(X, \mathbb{C}) \). Let \( \mathcal{X} \to \text{Def}_X \) be a local universal deformation. The groups \( H^2(\mathcal{X}_b, \mathbb{C}) \) form a local system on \( \text{Def}_X \); we can assume that it is constant \( \rightsquigarrow H^2(\mathcal{X}_b, \mathbb{C}) \overset{\text{can}}{\longrightarrow} H \).

The period map \( \varphi : B \to \mathbb{P}(H) \) is defined by \( \varphi(b) = [\varphi_b] \).

**Theorem**

\( \varphi \) is a local isomorphism onto a quadric \( Q \subset \mathbb{P}(H) \).
The period map

**Sketch of proof:** 1) Write \( \varphi_b = a\varphi + \omega + b\bar{\varphi} \), with \( \omega \in H^{1,1}(X) \).

\[
0 = \varphi_b^{r+1} = (a\varphi + \omega + b\bar{\varphi})^{r+1} = (r + 1)(a\varphi)^r b\bar{\varphi} + \left( \frac{r+1}{2} \right)(a\varphi)^{r-1}\omega^2
\]

\( (\in H^{2r,2}) \) + terms in \( H^{p,q}, q \geq 3 \).

Multiply by \( \bar{\varphi}^{r-1} \) and integrate: \( 0 = ab\int_X (\varphi\bar{\varphi})^r + \frac{r}{2}\int_X \omega^2(\varphi\bar{\varphi})^{r-1} \)

= a quadratic form \( q \) on \( H \), such that \( \varphi(B) \subset Q : \{ q = 0 \} \).

2) \( T_\sigma(\varphi) : H^1(X, T_X) \to \text{Hom}(H^{2,0}, H^{1,1} + H^{0,2}) \) deduced from

\[
H^1(X, T_X) \otimes H^0(X, \Omega^2_X) \xrightarrow{\cup} H^1(X, \Omega^1_X) \quad \text{(Griffiths)}
\]

\( \Rightarrow \) \( T_\sigma(\varphi) \) isomorphism onto codimension 1 subspace of

\( T_{\varphi(o)}(\mathbb{P}(H)) \), thus = \( T_{\varphi(o)}(Q) \).  □
The quadratic form

Recall: $q(a\varphi + \omega + b\bar{\varphi}) = ab\int_X (\varphi \bar{\varphi})^r + \int_X \omega^2(\varphi \bar{\varphi})^{r-1}$.

Thus $q(\varphi) = q(\bar{\varphi}) = 0$, $q(\varphi, \bar{\varphi}) > 0$, $H^{1,1} \perp_q (H^{2,0} \oplus H^{0,2})$.

Theorem

1. $Q$ is defined by an integral quadratic form $q : H^2(X, \mathbb{Z}) \to \mathbb{Z}$, non-degenerate, of signature $(3, b_2 - 3)$.

2. $\varphi$ is a local isomorphism $B \to \Omega$, where $\Omega =$ open subset of $Q$ defined by $q(\varphi, \bar{\varphi}) > 0$.

3. $\exists f_X \in \mathbb{N}$ (the Fujiki constant) such that $\int_X \alpha^{2r} = f_X q(\alpha)^r$. 

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Holomorphic symplectic manifolds
Fix a lattice $L$ and an integer $r$.

**Marked** IHS manifold: $(X, \sigma)$ where $\sigma : H^2(X, \mathbb{Z}) \rightarrow L$.

$\mathcal{M}_L := \{ \text{iso. classes of } (X, \sigma), \dim X = 2r \}$ — complex manifold, but **non-Hausdorff**.

Period domain $\Omega_L = \{ [v] \in \mathbb{P}(L_{\mathbb{C}}) \mid v^2 = 0, \ v \bar{v} > 0 \}$.

Period map $\varphi : \mathcal{M}_L \rightarrow \Omega_L : \varphi(X, \sigma) = \sigma_{\mathbb{C}}(H^{2,0}) \subset L_{\mathbb{C}}$.

**Theorem**

1. *(Huybrechts)* $\varphi$ is surjective.

2. *(Verbitsky)* Let $\mathcal{M}_L^0$ be a connected component of $\mathcal{M}_L$. Then $\varphi : \mathcal{M}_L^0 \rightarrow \Omega_L$ is generically injective, and identifies $\Omega_L$ with the Hausdorff reduction of $\mathcal{M}_L^0$ (**"Torelli theorem"**).

**Remark:** Many examples where $\mathcal{M}_L$ is not connected.
Proposition

The projective IHS form a countable union of hypersurfaces, everywhere dense in $\mathcal{M}_L$. 

Sketch of proof: A IHS $X$ is projective iff $H^2(X, \mathbb{Z}) \ni \alpha$ with $q(\alpha) > 0$ and $q(\alpha, \varphi) = 0$ (Huybrechts). Thus the locus $\mathcal{M}_L^{\text{alg}}$ of projective IHS in $\mathcal{M}_L$ is $\bigcup \varphi^{-1}(\alpha^\perp)$, for $\alpha \in L$ with $q(\alpha) > 0$. One checks that $\bigcup (\alpha^\perp \cap Q)$ is everywhere dense in $Q$. 

Example: • For K3, dim $\mathcal{M}_L = 20$, dim $\mathcal{M}_L^{\text{alg}} = 19$.
• For $X$ of type $S^{[r]}$, dim $\mathcal{M}_L = 21$, dim $\mathcal{M}_L^{\text{alg}} = 20$. Thus the IHS $S^{[r]}$ with $S$ projective form only a hypersurface in $\mathcal{M}_L^{\text{alg}}$.

Problem: find complete families ($= \text{dim } 20$) in $\mathcal{M}_L^{\text{alg}}$. 

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(AB-Donagi) $V \subset \mathbb{P}^5$ smooth cubic, depends on 20 moduli. $F(V) := \text{variety of lines} \subset V$ is an IHS, of type $S^{[2]}$.

(Debarre-Voisin) $\psi$ general in $\text{Alt}^3(V_{10})$ depends on 20 moduli; $X_\psi := \{L \in G(6, V_{10}) | \psi|_L = 0\}$ is an IHS of type $S^{[2]}$.

(O’Grady) The “EPW-sextics” $V \subset \mathbb{P}^4$ depend on 20 moduli. $X$ double cover of $V$ is an IHS of type $S^{[2]}$.

(Lehn-Lehn-Sorger-Van Straten) $V \subset \mathbb{P}^5$ cubic, $H = \{\text{twisted cubic curves} \subset V\}; \exists H \to X$ with $X$ IHS of type $S^{[4]}$.

(Bayer, Lahoz, Macrì, Nuer, Perry, Stellari, 2019) $V \subset \mathbb{P}^5$ cubic. For $a, b \in \mathbb{N}$ coprime and $r = a^2 - ab + b^2 + 1$, $X_r$ moduli space of certain stable objects in $D(V)$ is an IHS of type $S^{[r]}$.

A few other examples… But no example known for type $K_r(T)$. 
THE END

Thank you!