Holomorphic symplectic geometry

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I. Symplectic structure

**Definition**

A **symplectic form** on a manifold $X$ is a 2-form $\varphi$ such that:

- $d\varphi = 0$ and $\varphi(x) \in \text{Alt}(T_x(X))$ non-degenerate $\forall x \in X$.

- $\iff$ locally $\varphi = dp_1 \wedge dq_1 + \ldots + dp_r \wedge dq_r$ (Darboux)

Then $(X, \varphi)$ is a **symplectic manifold**.

(In mechanics, typically $q_i \leftrightarrow$ positions, $p_i \leftrightarrow$ velocities)

\[\implies\] Unlike Riemannian geometry, symplectic geometry is locally trivial; the interesting problems are **global**.

All this makes sense with $X$ complex manifold, $\varphi$ holomorphic. global \[\implies\] $X$ compact, usually projective or Kähler.
Holomorphic symplectic manifolds

Definition: holomorphic symplectic manifold

- $X$ compact, Kähler, simply-connected;
- $X$ admits a (holomorphic) symplectic form, unique up to $\mathbb{C}^\ast$.

Consequences: $\dim_{\mathbb{C}} X = 2r$; the canonical bundle $K_X := \Omega^2_X$ is trivial, generated by $\varphi \wedge \ldots \wedge \varphi$ ($r$ times).

(Note: on $X$ compact Kähler, holomorphic forms are closed)

Why is it interesting?
The Decomposition theorem

**Decomposition theorem**

Let $X$ be a compact Kähler manifold with $K_X = O_X$. There exists a finite étale cover $\tilde{X} \to X$, such that

$$\tilde{X} = T \times \prod_i Y_i \times \prod_j Z_j$$

- $T$ is a complex torus (equal to $\mathbb{C}^g / \text{lattice}$);
- $Y_i$ are holomorphic symplectic manifolds;
- $Z_j$ are simply-connected, projective, with $\dim \geq 3$,

$$H^0(Z_j, \Omega^*) = \mathbb{C} \oplus \mathbb{C} \omega,$$

where $\omega$ is a generator of $K_{Z_j}$.

(These are the Calabi-Yau manifolds)

Thus holomorphic symplectic manifolds (also called hyperkähler) are building blocks for manifolds with $K$ trivial, which are themselves building blocks in the classification of projective (or compact Kähler) manifolds.
Many examples of Calabi-Yau manifolds, very few of holomorphic symplectic.

- \( \dim 2: X \) simply-connected, \( K_X = \mathcal{O}_X \iff X \) K3 surface. (Example: \( X \subset \mathbb{P}^3 \) of degree 4, etc.)

- \( \dim > 2? \) Idea: take \( S^r \) for \( S \) K3. Many symplectic forms:

\[
\varphi = \lambda_1 p_1^* \varphi_S + \ldots + \lambda_r p_r^* \varphi_S , \quad \text{with} \quad \lambda_1, \ldots, \lambda_r \in \mathbb{C}^* .
\]

Try to get unicity by imposing \( \lambda_1 = \ldots = \lambda_r \), i.e.

\( \varphi \) invariant under \( S_r \), i.e. \( \varphi \) comes from \( S^{(r)} := S^r / S_r = \{ \text{subsets of } r \text{ points of } S, \text{ counted with multiplicities} \} \)

- \( S^{(r)} \) is singular, but admits a natural desingularization \( S^{[r]} := \{ \text{finite analytic subspaces of } S \text{ of length } r \} \) (Hilbert scheme)
### Examples

#### Theorem

*For $S$ K3, $S^{[r]}$ is holomorphic symplectic, of dimension $2r$.***

#### Other examples

1. Analogous construction with $S = $ complex torus (dim. 2); gives **generalized Kummer manifold** $K_r$ of dimension $2r$.

2. Two isolated examples by O’Grady, of dimension 6 and 10.

All other known examples belong to one of the above families!

**Example:** $V \subset \mathbb{P}^5$ cubic fourfold. $F(V) := \{\text{lines contained in } V\}$ is holomorphic symplectic, deformation of $S^{[2]}$ with $S$ K3.
A fundamental tool to study holomorphic symplectic manifolds is the period map, which describes the position of $[\phi]$ in $H^2(X, \mathbb{C})$.

**Proposition**

1. \( \exists \ q : H^2(X, \mathbb{Z}) \rightarrow \mathbb{Z} \) quadratic and \( f \in \mathbb{Z} \) such that
   \[
   \int_X \alpha^{2r} = f \ q(\alpha)^r \quad \text{for} \quad \alpha \in H^2(X, \mathbb{Z}).
   \]

2. For \( L \) lattice, there exists a complex manifold \( \mathcal{M}_L \) parametrizing isomorphism classes of pairs \((X, \lambda)\), where \( \lambda : (H^2(X, \mathbb{Z}), q) \overset{\sim}{\rightarrow} L \).

(Beware that \( \mathcal{M}_L \) is non Hausdorff in general.)
\((X, \lambda) \in \mathcal{M}_L, \lambda_C : H^2(X, \mathbb{C}) \simarrow L_\mathbb{C}; \) put \(\wp(X, \lambda) := \lambda_C(\mathbb{C}\wp).\)

\(\wp : \mathcal{M}_L \longrightarrow \mathbb{P}(L_\mathbb{C})\) is the period map.

**Theorem**

Let \(\Omega := \{x \in \mathbb{P}(L_\mathbb{C}) \mid q(x) = 0, q(x, \bar{x}) > 0\}\).

1. (AB) \(\wp\) is a local isomorphism \(\mathcal{M}_L \rightarrow \Omega\).
2. (Huybrechts) \(\wp\) is surjective.
3. (Verbitsky) The restriction of \(\wp\) to any connected component of \(\mathcal{M}_L\) is generically injective.

Gives very precise information on the structure of \(\mathcal{M}_L\) and the geometry of \(X\).
Completely integrable systems

Symplectic geometry provides a set-up for the differential equations of classical mechanics:

$M$ real symplectic manifold; $\varphi$ defines $\varphi^\# : T^*(M) \to T(M)$.

For $h$ function on $M$, $X_h := \varphi^\#(dh)$: hamiltonian vector field of $h$.

$X_h \cdot h = 0$, i.e. $h$ constant along trajectories of $X_h$ ("integral of motion")

$\dim(M) = 2r$. $h : M \to \mathbb{R}^r$, $h = (h_1, \ldots, h_r)$. Suppose:

$h^{-1}(s)$ connected, smooth, compact, Lagrangian ($\varphi|_{h^{-1}(s)} = 0$).

Arnold-Liouville theorem

$h^{-1}(s) \cong \mathbb{R}^r$/lattice; $X_{h_i}$ tangent to $h^{-1}(s)$, constant on $h^{-1}(s)$.

$\Rightarrow$ explicit solutions of the ODE $X_{h_i}$ (e.g. in terms of $\theta$ functions): "algebraically completely integrable system". Classical examples: geodesics of the ellipsoid, Lagrange and Kovalevskaya tops, etc.
Holomorphic set-up

No global functions \( \rightsquigarrow \) replace \( \mathbb{R}^r \) by \( \mathbb{P}^r \).

**Definition**

\( X \) holomorphic symplectic, \( \dim(X) = 2r \). Lagrangian fibration:

\( h : X \rightarrow \mathbb{P}^r \), general fiber connected Lagrangian.

\( \Rightarrow \) on \( h^{-1}(\mathbb{C}^r) \rightarrow \mathbb{C}^r \), Arnold-Liouville situation.

**Theorem**

\( f : X \rightarrow B \) surjective with connected fibers \( \Rightarrow \)

1. \( h \) is a Lagrangian fibration (Matsushita);

2. If \( X \) projective, \( B \cong \mathbb{P}^r \) (Hwang).

Is there a simple characterization of Lagrangian fibration?

**Conjecture**

\( \exists X \rightarrow \mathbb{P}^r \) Lagrangian \( \iff \exists L \) on \( X \), \( q(c_1(L)) = 0 \).
Many examples of such systems. Here is one:

\[ S \subset \mathbb{P}^5 \text{ given by } P = Q = R = 0, \ P, Q, R \text{ quadratic} \Rightarrow S \text{ K3.} \]

\[ \Pi = \{ \text{quadrics } \supset S \} = \{ \lambda P + \mu Q + \nu R \} \cong \mathbb{P}^2 \]

\[ \Pi^* = \text{dual projective plane } = \{ \text{pencils of quadrics } \supset S \}. \]

\[ h : S^{[2]} \rightarrow \Pi^* : h(x, y) = \{ \text{quadrics of } \Pi \supset \langle x, y \rangle \}. \]

By the theorem, \( h \) Lagrangian fibration \( \Rightarrow \]

\[ h^{-1}(\langle P, Q \rangle) = \{ \text{lines } \subset \{ P = Q = 0 \} \subset \mathbb{P}^5 \} \cong 2\text{-dim'l complex torus}, \]

a classical result of Kummer.
II. Contact geometry

What about odd dimensions?

**Definition**

A contact form on a manifold $X$ is a 1-form $\eta$ such that:

- $\text{Ker } \eta(x) = H_x \subsetneq T_x(X)$ and $d\eta|_{H_x}$ non-degenerate $\forall x \in X$;
- $\iff$ locally $\eta = dt + p_1 dq_1 + \ldots + p_r dq_r$.

- A contact structure on $X$ is a family $H_x \subsetneq T_x(X)$ $\forall x \in X$, defined locally by a contact form.

Again the definition makes sense in the holomorphic set-up $\rightsquigarrow$ holomorphic contact manifold. We will be looking for projective contact manifolds.

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Examples

Examples of contact projective manifolds

1. $\mathbb{P} T^*(M)$ for every projective manifold $M$
   
   ($= \{(m, H) \mid H \subset T_m(M)\}$: “contact elements”);

2. $\mathfrak{g}$ simple Lie algebra; $\mathcal{O}_{\text{min}} \subset \mathbb{P}(\mathfrak{g})$ unique closed adjoint orbit.
   
   (example: rank 1 matrices in $\mathbb{P}(\mathfrak{sl}_r)$.)

Conjecture

These are the only contact projective manifolds.

( $\Rightarrow$ classical conjecture in Riemannian geometry: classification of compact quaternion-Kähler manifolds (LeBrun, Salamon).)
Definition: A projective manifold $X$ is Fano if $K_X$ negative, i.e. $K_X^{-N}$ has “enough sections” for $N \gg 0$.

$X$ contact manifold; $L := T(X)/H$ line bundle; then $K_X \cong L^{-k}$ with $k = \frac{1}{2}(\dim(X) + 1)$. Thus $X$ Fano $\iff L^N$ has enough sections for $N \gg 0$.

Theorem

1. If $X$ is not Fano, $X \cong \mathbb{P}T^*(M)$
   (Kebekus, Peternell, Sommese, Wiśniewski + Demailly)

2. $X$ Fano and $L$ has “enough sections” $\Rightarrow Z \cong \mathcal{O}_{\text{min}} \subset \mathbb{P}(g)$
   (AB)
Few symplectic or contact manifolds look for weaker structure.

\[ \varphi \text{ symplectic} \implies \varphi^\# : T(X) \overset{\sim}{\longrightarrow} T^*(X) \overset{\sim}{\longrightarrow} \tau \in \wedge^2 T(X) \overset{\sim}{\longrightarrow} \]

\[ (f, g) \mapsto \{f, g\} := \langle \tau, df \wedge dg \rangle \text{ for } f, g \text{ functions on } U \subset X . \]

**Fact:** \( d\varphi = 0 \iff \text{Lie algebra structure (Jacobi identity)} \).

**Definition**

Poisson structure on \( X \): bivector field \( \tau : x \mapsto \tau(x) \in \wedge^2 T_x(X) \), such that \( (f, g) \mapsto \{f, g\} \) Lie algebra structure.

Again this makes sense for \( X \) complex manifold, \( \tau \) holomorphic.
Examples

1. \( \dim(X) = 2 \): any global section of \( \wedge^2 T(X) = K_X^{-1} \) is Poisson.

2. \( \dim(X) = 3 \); wedge product \( \wedge^2 T(X) \otimes T(X) \to K_X^{-1} \) gives \( \wedge^2 T(X) \sim \Omega^1_X \otimes K_X^{-1} \). Then \( \alpha \in H^0(\Omega^1_X \otimes K_X^{-1}) \) is Poisson \( \iff \alpha \wedge d\alpha = 0 \iff \text{locally } \alpha = fdg \).

3. On \( \mathbb{P}^3 \), \( P \), \( Q \) quadratic
   \( \sim \alpha = PdQ - QdP \in \Omega^1_{\mathbb{P}^3}(4) = \Omega^1_{\mathbb{P}^3} \otimes K_{\mathbb{P}^3}^{-1} \) Poisson.

4. A holomorphic symplectic manifold is Poisson.

5. If \( X \) is Poisson, any \( X \times Y \) is Poisson.
The Bondal conjecture

\( \tau \) Poisson, \( x \in X \). \( \tau_x : T^*_x(X) \to T_x(X) \) skew-symmetric, rk even.

\[
X_r := \{ x \in X \mid \text{rk}(\tau_x) = r \} \quad (r \text{ even}) \quad X = \bigsqcup X_r
\]

**Proposition**
If \( X_r \neq \emptyset \), \( \dim X_r \geq r \).

*Proof:* \( X_r \) is a Poisson submanifold, i.e. at a smooth \( x \in X_r \)
\( \tau_x \in \wedge^2 T_x(X_r) \subset \wedge^2 T_x(X) \implies \text{rk}(\tau_x) \leq \dim X_r. \)

**Conjecture (Bondal)**
\( X \) compact Poisson manifold, \( X_r \neq \emptyset \) \( \Rightarrow \) \( \dim X_r > r. \)

Example: \( X_0 = \{ x \in X \mid \tau_x = 0 \} \) contains a curve.

(e.g.: on \( \mathbb{P}^3 \), \( PdQ - QdP \) vanishes on the curve \( P = Q = 0. \))
Some evidence

1. True for $X$ projective threefold (Druel: $X_0 = \emptyset$ or $\dim \geq 1$).

2. $\text{rk}(\tau_x) = r$ for $x$ general $\Rightarrow$ true for $X_{r-2}$ if $c_1(X)^q \neq 0$, $q = \dim X - r + 1$.

Proposition (Polishchuk)

$\tau$ Poisson on $\mathbb{P}^3$, vanishes along smooth curve $C$. Then $C$ elliptic, $\deg(C) = 3$ or $4$; if $= 4$, $\tau = PdQ - QdP$ and $C : P = Q = 0$.

THE END