Holomorphic symplectic geometry

Arnaud Beauville

Université de Nice

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I. Symplectic structure

Definition
A symplectic form on a manifold $X$ is a 2-form $\phi$ such that:

- $d\phi = 0$
- $\phi(x) \in \text{Alt}(T_x(X))$ non-degenerate $\forall x \in X$.

$\iff$ locally $\phi = dp_1 \wedge dq_1 + \ldots + dp_r \wedge dq_r$ (Darboux)

Then $(X, \phi)$ is a symplectic manifold.

(In mechanics, typically $q_i \leftrightarrow$ positions, $p_i \leftrightarrow$ velocities)

$\Rightarrow$ Unlike Riemannian geometry, symplectic geometry is locally trivial; the interesting problems are global.

$\Rightarrow$ All this makes sense with $X$ complex manifold, $\phi$ holomorphic.

$\Rightarrow$ Global $\Rightarrow X$ compact, usually projective or Kähler.

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A *symplectic form* on a manifold $X$ is a 2-form $\varphi$ such that:

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Holomorphic symplectic manifolds

Definition: holomorphic symplectic manifold $X$ compact, Kähler, simply-connected; $X$ admits a (holomorphic) symplectic form, unique up to $C^*$. 

Consequences: $\dim \mathbb{C} X = 2r$; the canonical bundle $K_X := \Omega^2 X$ is trivial, generated by $\phi \wedge \ldots \wedge \phi$ ($r$ times).

(Note: on $X$ compact Kähler, holomorphic forms are closed)

Why is it interesting?

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The Decomposition theorem

Decomposition theorem

\[ X \text{ compact Kähler with } \text{K}(X) = 0. \]

\[ \exists \tilde{X} \to X \text{ étale finite and } \tilde{X} = T \times \prod_i Y_i \times \prod_j Z_j. \]

\( T \) complex torus (= \( \mathbb{C}^g / \text{lattice} \)); \( Y_i \) holomorphic symplectic manifolds; \( Z_j \) simply-connected, projective, \( \text{dim} \geq 3 \), \( H^0(Z_j, \Omega^\ast) = \mathbb{C} \oplus \mathbb{C} \omega \), where \( \omega \) is a generator of \( K_{Z_j} \). (these are the Calabi-Yau manifolds)

Thus holomorphic symplectic manifolds (also called hyperkähler) are building blocks for manifolds with \( \text{K} \) trivial, which are themselves building blocks in the classification of projective (or compact Kähler) manifolds.

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Examples?

Many examples of Calabi-Yau manifolds, very few of holomorphic symplectic.

\[ \dim 2: X \text{ simply-connected}, \ K_X = \mathcal{O}_X \]
\[ \iff X \text{ K3 surface.} \]
(Example: \(X \subset \mathbb{P}^3\) of degree 4, etc.)

\[ \dim > 2? \text{ Idea: take } S_r \text{ for } S \text{ K3. Many symplectic forms: } \]
\[ \phi = \lambda_1 p^* \phi_S + \ldots + \lambda_r p^* \phi_S, \]
with \(\lambda_1, \ldots, \lambda_r \in \mathbb{C}^\ast\).

Try to get unicity by imposing \(\lambda_1 = \ldots = \lambda_r\), i.e. \(\phi\) invariant under \(S_r\), i.e. \(\phi\) comes from \(S_r^r := S_r / S_r = \{\text{subsets of } r \text{ points of } S\}, \text{ counted with multiplicities}\)
\(S_r^r\) is singular, but admits a natural desingularization
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Theorem

For $S K^3$, $S[r]$ is holomorphic symplectic, of dimension $2r$.

Other examples

1. Analogous construction with $S = $ complex torus (dim. 2); gives generalized Kummer manifold $K[r]$ of dimension $2r$.

2. Two isolated examples by O'Grady, of dimension 6 and 10.

All other known examples belong to one of the above families!

Example:

$V \subset P^5$ cubic fourfold.
$F(V) := \{ \text{lines contained in } V \}$ is holomorphic symplectic, deformation of $S[2]$ with $S K^3$. 

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The period map

A fundamental tool to study holomorphic symplectic manifolds is the period map, which describes the position of $\varphi$ in $H^2(X, C)$.

Proposition 1

$\exists q : H^2(X, Z) \to Z$ quadratic and $f \in Z$ such that

$$\int_X \alpha^2 r = f q(\alpha) r$$

for $\alpha \in H^2(X, Z)$.

2 For $L$ lattice, there exists a complex manifold $M_L$ parametrizing isomorphism classes of pairs $(X, \lambda)$, where $\lambda : (H^2(X, Z), q) \to -L$.

(Beware that $M_L$ is non Hausdorff in general.)
A fundamental tool to study holomorphic symplectic manifolds is the period map, which describes the position of $[\varphi]$ in $H^2(X, \mathbb{C})$. 

Proposition 1: There exists a quadratic form $q: H^2(X, \mathbb{Z}) \to \mathbb{Z}$ such that

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The period package

\[ (X, \lambda) \in M_L, \lambda C : H^2(X, C) \sim \rightarrow L_C; \text{ put } \wp(X, \lambda) := \lambda C(C \phi). \]

\( \wp : M_L \rightarrow P(L_C) \) is the period map.

**Theorem**

Let \( \Omega := \{ x \in P(L_C) | q(x) = 0, q(x, \bar{x}) > 0 \} \).

1. \( \wp \) is a local isomorphism \( M_L \rightarrow \Omega \).
2. (Huybrechts) \( \wp \) is surjective.
3. (Verbitsky) The restriction of \( \wp \) to any connected component of \( M_L \) is generically injective.

Gives very precise information on the structure of \( M_L \) and the geometry of \( X \).

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The period package

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**Theorem**

Let \( \Omega := \{ x \in \mathbb{P}(L_{\mathbb{C}}) \mid q(x) = 0 , q(x, \bar{x}) > 0 \} \).
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1. \((AB)\) \(\wp\) is a local isomorphism \(\mathcal{M}_L \rightarrow \Omega.\)
(X, λ) ∈ M_L, λ_C : H^2(X, ℂ) → L_ℂ; put ϕ(X, λ) := λ_C(ℂϕ).

ϕ : M_L → ℙ(L_ℂ) is the period map.

Theorem

Let Ω := \{x ∈ ℙ(L_ℂ) | q(x) = 0 , q(x, ¯x) > 0\}.

1. (AB) ϕ is a local isomorphism M_L → Ω.

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Gives very precise information on the structure of \(\mathcal{M}_L\) and the geometry of \(X.\)
Completely integrable systems

Symplectic geometry provides a set-up for the differential equations of classical mechanics: $M$ real symplectic manifold; $\phi$ defines $\phi^\sharp: T^* (M) \xrightarrow{\sim} T (M)$.

For $h$ function on $M$, $X_h := \phi^\sharp (dh)$: hamiltonian vector field of $h$.

$X_h \cdot h = 0$, i.e. $h$ constant along trajectories of $X_h$ ("integral of motion").

$\dim (M) = 2^r$. $h: M \rightarrow \mathbb{R}^r$, $h = (h_1, \ldots, h_r)$. Suppose:

$h^{-1}(s)$ connected, smooth, compact, Lagrangian ($\phi|_{h^{-1}(s)} = 0$).

Arnold-Liouville theorem $h^{-1}(s) \sim \mathbb{R}^r / \text{lattice}$; $X_{hi}$ tangent to $h^{-1}(s)$, constant on $h^{-1}(s)$.

$\Rightarrow$ explicit solutions of the ODE $X_{hi}$ (e.g. in terms of $\theta$ functions): "algebraically completely integrable system".

Classical examples: geodesics of the ellipsoid, Lagrange and Kovalevskaya tops, etc.

Arnaud Beauville

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Holomorphic set-up

No global functions \( \Rightarrow \) replace \( R \) by \( P \).

**Definition**

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**Lagrangian fibration:**

\( h: X \to P^r \), general fiber connected Lagrangian.

\( \Rightarrow \) on \( h^{-1}(C^r) \to C^r \), Arnold-Liouville situation.

**Theorem**

\( f: X \to B \) surjective with connected fibers

1. \( h \) is a Lagrangian fibration (Matsushita);
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Is there a simple characterization of Lagrangian fibration?

**Conjecture**

\( \exists X \) such that \( P^r \) Lagrangian \( \iff \exists L \) on \( X \), \( q(c_1(L)) = 0 \).

Arnaud Beauville

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An example

Many examples of such systems. Here is one:

\[ S \subset P^5 \] given by

\[ P = Q = R = 0, \]

\( P, Q, R \) quadratic \( \Rightarrow S \subset K^3. \)

\[ \Pi = \{ \text{quadrics} \supset S \} = \{ \lambda P + \mu Q + \nu R \} \sim P^2. \]

\( \Pi^* = \text{dual projective plane} = \{ \text{pencils of quadrics} \supset S \}. \)

\[ h: S^2 \to \Pi^*: h((x, y)) = \{ \text{quadrics of } \Pi \supset \langle x, y \rangle \}. \]

By the theorem, \( h \) is a Lagrangian fibration \( \Rightarrow h^{-1}(\langle P, Q \rangle) = \{ \text{lines} \subset \{ P = Q = 0 \} \subset P^5 \} \sim 2\text{-dim'l complex torus}, \) a classical result of Kummer.
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II. Contact geometry

What about odd dimensions?

Definition

A contact form on a manifold $X$ is a 1-form $\eta$ such that:

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\ker \eta(x) = H_x \lhd T_x (X) \quad \text{and} \quad d\eta|_{H_x} \text{non-degenerate} \quad \forall x \in X;
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$\iff$ locally $\eta = dt + p_1 dq_1 + \ldots + p_r dq_r$.

A contact structure on $X$ is a family $H_x \lhd T_x (X) \quad \forall x \in X$, defined locally by a contact form.

Again the definition makes sense in the holomorphic set-up $\Rightarrow$ holomorphic contact manifold. We will be looking for projective contact manifolds.
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Examples

Examples of contact projective manifolds $P^* (M)$ for every projective manifold $M$. "contact elements".

g a simple Lie algebra; $O_{min} \subset P(g)$ unique closed adjoint orbit. (example: rank 1 matrices in $P(slr)$).

Conjecture These are the only contact projective manifolds. (⇒ classical conjecture in Riemannian geometry: classification of compact quaternion-Kähler manifolds (LeBrun, Salamon)).
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Examples of contact projective manifolds

1. $\mathbb{P} T^*(M)$ for every projective manifold $M$ ($\{ (m, H) | H \subset T_m(M) \}$: "contact elements")

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2. \( g \) simple Lie algebra; \( O_{min} \subset \mathbb{P}(g) \) unique closed adjoint orbit.

(example: rank 1 matrices in \( \mathbb{P}(\mathfrak{sI}_r) \).)

Conjecture

These are the only contact projective manifolds.

(⇒ classical conjecture in Riemannian geometry: classification of compact quaternion-Kähler manifolds (LeBrun, Salamon).)

Arnaud Beauville  Holomorphic symplectic geometry
Examples of contact projective manifolds

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Partial results

Definition: A projective manifold $X$ is Fano if $K_X$ is negative, i.e., $K_X^N$ has "enough sections" for $N \gg 0$.

$X$ is a contact manifold; let $L := T(X) / H$ be a line bundle; then $K_X \sim L - k$ with $k = \frac{1}{2} (\dim(X) + 1)$.

Thus $X$ is Fano $\iff L^N$ has enough sections for $N \gg 0$.

Theorem 1: If $X$ is not Fano, then $X \sim \mathbb{P} T^* (M)$ (Kebekus, Peternell, Sommese, Wiśniewski + Demailly).

2. $X$ is Fano and $L^N$ has "enough sections" $\implies Z \sim \mathbb{O}_{\text{min}} \subset \mathbb{P}(g)$ (AB).

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**Theorem 1**

- If $X$ is not Fano, $X \sim \mathbb{P}^{T^* M}$ (Kebekus, Peternell, Sommese, Wiśniewski + Demailly)
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If $X$ is not Fano, $X \sim P \mathbb{T}^\ast(M)$ (Kebekus, Peternell, Sommese, Wiśniewski + Demailly)

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III. Poisson manifolds

Few symplectic or contact manifolds ⇝ look for weaker structure.

ϕ symplectic ⇝ ϕ♯:
T(X) → T∗(X) ⇝ τ ∈ ∧²T(X) ⇝ (f, g) ↦ → {f, g} := ⟨τ, df ∧ dg⟩ for f, g functions on U ⊂ X.

Fact: dϕ = 0 ⇐⇒ Lie algebra structure (Jacobi identity).

Definition Poisson structure on X: bivector field τ: x ↦ → τ(x) ∈ ∧²T_x(X), such that (f, g) ↦ → {f, g} Lie algebra structure. Again this makes sense for X complex manifold, τ holomorphic.

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Examples

1. $\dim(X) = 2$: any global section of $\wedge^2 T(X) = K^{-1}X$ is Poisson.

2. $\dim(X) = 3$: wedge product $\wedge^2 T(X) \otimes T(X) \rightarrow K^{-1}X$ gives $\wedge^2 T(X) \sim \rightarrow \Omega^1 X \otimes K^{-1}X$. Then $\alpha \in H^0(\Omega^1 X \otimes K^{-1}X)$ is Poisson $\iff \alpha \wedge d\alpha = 0 \iff$ locally $\alpha = \text{fdg}$.

3. On $P^3$, $P, Q$ quadratic $\Rightarrow \alpha = PdQ - QdP \in \Omega^1 P^3(4) = \Omega^1 P^3 \otimes K^{-1}P^3$ Poisson.

4. A holomorphic symplectic manifold is Poisson.

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The Bondal conjecture

$\tau$ Poisson, $x \in X$. $\tau_x : T^* x (X) \to T x (X)$ skew-symmetric, rk even.

$X_r := \{ x \in X | \text{rk}(\tau_x) = r \}$ (r even)

$X = \bigsqcup X_r$

**Proposition**

If $X_r \neq \emptyset$, $\dim X_r \geq r$.

**Proof**:

$X_r$ is a Poisson submanifold, i.e. at a smooth $x \in X_r$

$\tau_x \in \wedge^2 T x (X_r) \subset \wedge^2 T x (X) = \Rightarrow \text{rk}(\tau_x) \leq \dim X_r$.

**Conjecture (Bondal)**

$X$ compact Poisson manifold, $X_r \neq \emptyset \Rightarrow \dim X_r > r$.

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Some evidence

The Bondal conjecture 2

Some evidence

Proposition (Polishchuk) τ Poisson on $\mathbb{P}^3$, vanishes along smooth curve $C$. Then $C$ elliptic, $\deg(C) = 3$ or $4$; if $= 4$, $\tau = PdQ - QdP$ and $C$: $P = Q = 0$.
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