The action of $\text{SL}_2$ on abelian varieties

Arnaud Beauville

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$\text{SL}_2 \rightarrow \text{Aut}(A)$

Interest: Corr$(A)$ acts on functorial invariants of $A$: $H^*(A)$, $\text{CH}(A)$, ..., hence action of $\text{SL}_2$ on these spaces.

On $H^*(A)$: classical action of $\text{SL}_2$ $\iff$ Hard Lefschetz.

On $\text{CH}(A)$: gives a twisted version of Hard Lefschetz.

Note: Action of $\text{SL}_2$ on $\text{CH}(A)$ already known (K"unnemann, Polishchuk).

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Synopsis

$\text{SL}_2 \hookrightarrow \text{Aut}(A) \cap \text{Corr}(A)$

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On \(H^\ast(A)\): classical action of \(\text{SL}_2 \iff \) Hard Lefschetz.

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**Note**: Action of \( \text{SL}_2 \) on \( CH(A) \) already known (K"unnemann, Polishchuk).
Reminder on cycles and correspondences

\[ CH(A) := \left\{ \sum_i n_i Z_i \mid n_i \in \mathbb{Q} \right\} \text{ / rational equivalence} \]
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\[ \text{Corr}(A) := CH(A \times A), \text{ with } \mathbb{Q} \text{-algebra structure given by} \]

composition \((\alpha, \beta) \mapsto \alpha \circ \beta\) such that

\[ \Gamma_u \circ \Gamma_v = \Gamma_{u \circ v} \quad \text{for} \quad u, v \in \text{Aut}(A) . \]
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Action of \( \text{Corr}(A) \) on \( CH(A) \): for \( \alpha \in \text{Corr}(A), z \in CH(A) \):

\[ \alpha_z := q_*(p^* z \cdot \alpha) \]
Main theorem: \( \text{SL}_2 \rightarrow \text{Corr}(A)^* \rightarrow \text{Aut}_Q(CH(A)). \)
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I will concentrate on $\text{SL}_2 \rightarrow \text{Aut}_Q(CH(A))$. Slight refinement of the proof gives the map $\text{SL}_2 \rightarrow \text{Corr}(A)^*$. 
Some history: Mukai

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$K_* = D(A) \rightarrow D(A)$ is the Fourier-Mukai functor associated to $K$. 

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We assume that $A$ has a polarization, i.e. an ample line bundle $L$ (defined up to translation). For simplicity we will assume that the polarization is principal: it defines an isomorphism $A \sim \hat{A}$. 

Recall: $SL_2(\mathbb{Z})$ is generated by $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ with the relations $w^2 = (uw)^3$, $w^4 = 1$. 

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Central extension: $0 \to \mathbb{Z} \cdot z^2 \to \widetilde{SL}_2(\mathbb{Z}) \to SL_2(\mathbb{Z}) \to 1$
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($\tilde{SL}_2(\mathbb{Z}) = \text{trefoil knot group} = \text{braid group } B_3$)

We have $\tilde{SL}_2(\mathbb{Z}) \rightarrow \text{Aut}(D(A))$ with

\[
\begin{cases}
  w \mapsto \mathcal{P}_* \\
  u \mapsto \bigotimes L
\end{cases}
\]
From $D(A)$ to $CH(A)$

The Chern character provides $\text{Aut}(D(A)) \to \text{Aut}_Q(CH(A))$:
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D(A) & \xrightarrow{p_*} D(A) \\
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where $\theta = [L]$ in $CH^1(A)$. Since $z^2$ acts by an even shift:
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\begin{array}{ccc}
\tilde{SL}_2(\mathbb{Z}) & \longrightarrow & \text{Aut}(D(A)) \\
\downarrow & & \downarrow \\
SL_2(\mathbb{Z}) & \xrightarrow{\tau} & \text{Aut}(CH(A)) \\
\end{array}
$$

with $\tau(w) = \mathcal{F}$, $\tau(u) = \times e^\theta$. 
The action of $\text{SL}_2(\mathbb{Z})$ on $\text{CH}(A)$ extends to an action of $\text{SL}_2$, such that $\text{CH}(A)$ is a direct sum of finite-dimensional representations.

We have

$$
\begin{align*}
\begin{pmatrix} n & 0 \\ 0 & n^{-1} \end{pmatrix} \cdot z & = n^{-g} n_A^* z \\
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot z & = \mathcal{F}(z) \\
\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot z & = e^{a\theta} z \\
\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \cdot z & = a^g e^{\theta/a} \ast z
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Key point: description of $\text{SL}_2$ by generators and relations (Demazure, SGA 3).
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\[ T = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} \quad U = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \quad B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \]
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Lemma

$H$ algebraic group over $\mathbb{Q}$.

Given

$$\begin{align*}
\tau &: \text{SL}_2(\mathbb{Z}) \to H(\mathbb{Q}) \\
\beta &: B \to H
\end{align*}$$

which coincide on $B(\mathbb{Z})$

and

$$\tau(w)\beta(t)\tau(w)^{-1} = \beta(t^{-1}) \text{ for } t \in T,$$

$\exists$ a unique morphism $f : \text{SL}_2 \to H$ extending $\tau$ and $\beta$. 
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\text{Arnaud Beauville} The action of $\text{SL}_2$ on abelian varieties
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$\beta : T \to \text{Aut}(CH(A))$ given by graduation $CH(A) = \bigoplus_s CH_s(A)$

$$CH^p_s(A) = \{z \in CH^p(A) \mid n^*_Az = n^{2p-s}z \quad \forall n \in \mathbb{Z}\}$$

Relations in the lemma are satisfied $\implies$ action extends to $\text{SL}_2$. 

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Differentiating gives action of $\mathfrak{sl}_2(\mathbb{Q})$, for $z \in CH_p^s(A)$:

\[
X \cdot z = \theta z \quad H \cdot z = (2g - p - s)z \quad Y \cdot z = \frac{\theta^g}{g!} * z .
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$H$ diagonal, $X, Y$ nilpotent $\implies$ $CH(A) = \bigoplus V_i$, $\dim V_i < \infty$. 

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The “level” grading

\[ CH^0(A) = \mathbb{Q} \]
\[ CH^1(A) = CH^1_0(A) \oplus CH^1_1(A) \]
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Follows from the Beilinson conjectures – hopefully easier?
Applications

The well-known structure of finite-dimensional representations of $\text{SL}_2$ gives:

**Proposition ("Twisted" Hard Lefschetz)**

The multiplication map $\times \theta_{g-2p+s} : \text{CH}^p(A) \to \text{CH}^{g-2p+s}(A)$ is bijective.

What about "standard" Hard Lefschetz? Cannot expect surjectivity (see above), but:

**Proposition**

$\text{CH}^s(A) = 0$ for $s < 0 \iff \times \theta_{g-2p} : \text{CH}^p(A) \to \text{CH}^{g-2p}(A)$ injective.

Note: Right hand side makes sense for any smooth projective variety.

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