A Calabi–Yau threefold with non-Abelian fundamental group

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Abstract

This note, written in 1994, answers a question of Dolgachev by constructing a Calabi–Yau threefold whose fundamental group is the quaternion group $H_8$. The construction is reminiscent of Reid’s unpublished construction of a surface with $p_g = 0$, $K^2 = 2$ and $\pi_1 = H_8$; I explain below the link between the two problems.

1 The example

Let $H_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group of order 8, and $V$ its regular representation. We denote by $\hat{H}_8$ the group of characters $\chi: H_8 \to \mathbb{C}^*$, which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The group $H_8$ acts on $\mathbb{P}(V)$ and on $S^2 V$; for each $\chi \in \hat{H}_8$, we denote by $(S^2 V)_\chi$ the eigensubspace of $S^2 V$ with respect to $\chi$, that is, the space of quadratic forms $Q$ on $\mathbb{P}(V)$ such that $h \cdot Q = \chi(h)Q$ for all $h \in H_8$.

**Theorem 1.1** For each $\chi \in \hat{H}_8$, let $Q_\chi$ be a general element of $(S^2 V)_\chi$. The subvariety $\tilde{X}$ of $\mathbb{P}(V)$ defined by the 4 equations

$$Q_\chi = 0 \text{ for all } \chi \in \hat{H}_8$$

is a smooth threefold, on which the group $H_8$ acts freely. The quotient $X := \tilde{X}/H_8$ is a Calabi–Yau threefold with $\pi_1(X) = H_8$.

Let me observe first that the last assertion is an immediate consequence of the others. Indeed, since $\tilde{X}$ is a Calabi–Yau threefold, we have $h^{1,0}(\tilde{X}) = h^{2,0}(\tilde{X}) = \chi(\mathcal{O}_{\tilde{X}}) = 0$, hence $h^{1,0}(X) = h^{2,0}(X) = \chi(\mathcal{O}_X) = 0$. This implies

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*Partially supported by the European HCM project “Algebraic Geometry in Europe” (AGE).

1 I use Grothendieck’s notation, that is, $\mathbb{P}(V)$ is the space of hyperplanes in $V$. 
$h^{3,0}(X) = 1$, so there exists a nonzero holomorphic 3-form $\omega$ on $X$; since its pullback to $\tilde{X}$ is everywhere nonzero, $\omega$ has the same property, hence $X$ is a Calabi–Yau threefold. Finally $\tilde{X}$ is a complete intersection in $\mathbb{P}(V)$, hence simply connected by the Lefschetz theorem, so the fundamental group of $X$ is isomorphic to $H_8$.

So the problem is to prove that $H_8$ acts freely and $\tilde{X}$ is smooth. To do this, we will need to write down explicit elements of $(S^2 V)_X$. As an $H_8$-module, $V$ is the direct sum of the 4 one-dimensional representations of $H_8$ and twice the irreducible two-dimensional representation $\rho$. Thus there exists a system of homogeneous coordinates $(X_1, X_\alpha, X_\beta, X_\gamma; Y; Z; Y', Z')$ such that

$$g \cdot (X_1, X_\alpha, X_\beta, X_\gamma; Y, Z; Y', Z') = (X_1, \alpha(g)X_\alpha, \beta(g)X_\beta, \gamma(g)X_\gamma; \rho(g)(Y, Z); \rho(g)(Y', Z')).$$

To be more precise, I denote by $\alpha$ (respectively $\beta$, $\gamma$) the nontrivial character which is $+1$ on $i$ (respectively $j$, $k$), and I take for $\rho$ the usual representation via Pauli matrices:

$$\rho(i)(Y, Z) = (\sqrt{-1}Y, -\sqrt{-1}Z), \quad \rho(j)(Y, Z) = (-Z, Y),$$

$$\rho(k)(Y, Z) = (-\sqrt{-1}Z, -\sqrt{-1}Y).$$

Then the general element $Q_X$ of $(S^2 V)_X$ can be written

$$Q_1 = t_1^2 X_1^2 + t_2^2 X_2^2 + t_3^2 X_3^2 + t_4^2 Y^2 + t_5^2 (Y' - Y'Z),$$

$$Q_\alpha = t_6^2 X_\alpha^2 + t_7^2 X_\beta X_\gamma + t_8^2 YZ + t_9^2 Y'Z' + t_{10}^2 (Y' + Z' Y'),$$

$$Q_\beta = t_6^2 X_1 X_\beta + t_7^2 X_\alpha X_\beta + t_8^2 (Y^2 + Z^2) + t_9^2 (Y'^2 + Z'^2) + t_{10}^2 (Y' Y' + Z' Z'),$$

$$Q_\gamma = t_6^2 X_1 X_\gamma + t_7^2 X_\alpha X_\gamma + t_8^2 (Y^2 - Z^2) + t_9^2 (Y'^2 - Z'^2) + t_{10}^2 (Y' Y' - Z' Z').$$

For fixed $t := (t^i_X)$, let $\mathcal{X}_t$ be the subvariety of $\mathbb{P}(V)$ defined by the equations $Q_X = 0$. Let us check first that the action of $H_8$ on $\mathcal{X}_t$ has no fixed points for $t$ general enough. Since a point fixed by an element $h$ of $H_8$ is also fixed by $h^2$, it is sufficient to check that the element $-1 \in H_8$ acts without fixed point, that is, that $\mathcal{X}_t$ does not meet the linear subspaces $L_+$ and $L_-$ defined by $Y = Z = Y' = Z' = 0$ and $X_1 = X_\alpha = X_\beta = X_\gamma = 0$ respectively.

Let $x = (0, 0, 0, 0; Y; Z; Y', Z') \in \mathcal{X}_t \cap L_+$. One of the coordinates, say $Z$, is nonzero; since $Q_1(x) = 0$, there exists $k \in \mathbb{C}$ such that $Y' = kY$, $Z' = kZ$. Substituting in the equations $Q_\alpha(x) = Q_\beta(x) = Q_\gamma(x) = 0$ gives

$$(t_6^2 + t_7^2 k + t_8^2 k^2)YZ = (t_6^2 + t_7^2 k + t_8^2 k^2)(Y^2 + Z^2) = (t_3^2 + t_5^2 k + t_4^2 k^2)(Y^2 - Z^2) = 0$$

which has no nonzero solutions for a generic choice of $t$. 
Now let \( x = (X_1, X_\alpha, X_\beta, X_\gamma; 0, 0, 0) \in \mathcal{X}_t \cap L_+ \). As soon as the \( t^\psi_t \) are nonzero, two of the \( X \)-coordinates cannot vanish, otherwise all the coordinates would be zero. Expressing that \( Q_\beta = Q_\gamma = 0 \) has a nontrivial solution in \( (X_\beta, X_\gamma) \) gives \( X_\alpha^2 \) as a multiple of \( X_\beta^2 \), and similarly for \( X_\beta^2 \) and \( X_\gamma^2 \). But then \( Q_1(x) = 0 \) is impossible for a general choice of \( t \).

Now we want to prove that \( \mathcal{X}_t \) is smooth for \( t \) general enough. Let \( \mathcal{Q} = \bigoplus_{x \in \mathcal{B}_X}(S^2 V)_x \); then \( t := (t^\psi_t) \) is a system of coordinates on \( \mathcal{Q} \). The equations \( Q_x = 0 \) define a subvariety \( \mathcal{X} \) in \( \mathcal{Q} \times \mathbb{P}(V) \), whose fibre above a point \( t \in \mathcal{Q} \) is \( \mathcal{X}_t \). Consider the second projection \( p: \mathcal{X} \to \mathbb{P}(V) \). For \( x \in \mathbb{P}(V) \), the fibre \( p^{-1}(x) \) is the linear subspace of \( \mathcal{Q} \) defined by the vanishing of the \( Q_x \), viewed as linear forms in \( t \). These forms are clearly linearly independent as soon as they do not vanish. In other words, if we denote by \( B_\mathcal{X} \) the base locus of the quadrics in \( (S^2 V)_x \) and put \( B = \bigcup B_\mathcal{X} \), the map \( p: \mathcal{X} \to \mathbb{P}(V) \) is a vector bundle fibration above \( \mathbb{P}(V) \setminus B \); in particular \( \mathcal{X} \) is nonsingular outside \( p^{-1}(B) \). Therefore it is enough to prove that \( \mathcal{X}_t \) is smooth at the points of \( B \cap \mathcal{X}_t \).

Observe that an element \( x \) in \( B \) has two of its \( X \)-coordinates zero. Since the equations are symmetric in the \( X \)-coordinates we may assume \( X_\beta = X_\gamma = 0 \). Then the Jacobian matrix

\[
\left( \frac{\partial Q_x}{\partial X_\psi}(x) \right)
\]

takes the form

\[
\begin{pmatrix}
2t^1_1X_1 & 2t^1_2X_\alpha & 0 & 0 \\
t^2_1X_\alpha & t^2_1X_1 & 0 & 0 \\
0 & 0 & t^\alpha_1X_1 & t^\alpha_2X_\alpha \\
0 & 0 & t^\beta_2X_\alpha & t^\beta_1X_1
\end{pmatrix}
\]

For generic \( t \), this matrix is of rank 4 except when all the \( X \)-coordinates of \( x \) vanish; but we have seen that this is impossible when \( t \) is general enough. \( \square \)

## 2 Some comments

As mentioned in the introduction, the construction is inspired by Reid’s example [R] of a surface of general type with \( p_g = 0 \), \( K^2 = 2 \), \( \tau_1 = H_8 \). This is more than a coincidence. In fact, let \( \tilde{S} \) be the hyperplane section \( X_1 = 0 \) of \( \tilde{X} \). It is stable under the action of \( H_8 \) (so that \( H_8 \) acts freely on \( \tilde{S} \)), and we can prove as above that it is smooth for a generic choice of the parameters.

The surface \( S := \tilde{S}/H_8 \) is a Reid surface, embedded in \( X \) as an ample divisor, with \( h^0(X, \mathcal{O}_X(S)) = 1 \). In general, let us consider a Calabi–Yau threefold \( X \) which contains a rigid ample surface, that is, a smooth ample divisor \( S \) such that \( h^0(\mathcal{O}_X(S)) = 1 \). Put \( L := \mathcal{O}_X(S) \). Then \( S \) is a minimal surface of general type (because \( K_S = L|_S \) is ample); by the Lefschetz theorem, the natural map \( \tau_1(S) \to \tau_1(X) \) is an isomorphism. Because of the exact sequence

\[
0 \to \mathcal{O}_X \to L \to K_S \to 0,
\]
the geometric genus \( p_g(S) := h^0(K_S) \) is zero.

We have \( K_S^2 = L^3 \); the Riemann–Roch theorem on \( X \) yields

\[
1 = h^0(L) = \frac{L^3}{6} + \frac{L \cdot c_2}{12}.
\]

Since \( L \cdot c_2 > 0 \) as a consequence of Yau’s theorem (see for instance [B], Cor. 2), we obtain \( K_S^2 \leq 5 \).

For surfaces with \( p_g = 0 \) and \( K_S^2 = 1 \) or 2, we have a great deal of information about the algebraic fundamental group, that is the profinite completion of the fundamental group (see [B-P-V] for an overview). In the case \( K_S^2 = 1 \), the algebraic fundamental group is cyclic of order \( \leq 5 \); if \( K_S^2 = 2 \), it is of order \( \leq 9 \); moreover the dihedral group \( D_8 \) cannot occur. D. Naie [N] has recently proved that the symmetric group \( S_3 \) can also not occur; therefore the quaternion group \( H_8 \) is the only non-Abelian group which occurs in this range.

On the other hand, little is known about surfaces with \( p_g = 0 \) and \( K_S^2 = 3, 4 \) or 5. Inoue has constructed examples with \( \pi_1 = H_8 \times (\mathbb{Z}_2)^n \), with \( n = K^2 - 2 \) (loc. cit.); I do not know if they can appear as rigid ample surfaces in a Calabi–Yau threefold.

Let us denote by \( \tilde{X} \) the universal cover of \( X \), by \( \tilde{L} \) the pullback of \( L \) to \( \tilde{X} \), and by \( \rho \) the representation of \( G \) on \( H^0(\tilde{X}, \tilde{L}) \). We have \( \text{Tr} \rho(g) = 0 \) for \( g \neq 1 \) by the holomorphic Lefschetz formula, and \( \text{Tr} \rho(1) = \chi(\tilde{L}) = |G| \chi(L) = |G| \).

Therefore \( \rho \) is isomorphic to the regular representation. Looking at the list in loc. cit. we get a few examples of this situation, for instance:

- \( G = \mathbb{Z}_5 \), \( \tilde{X} \) is a quintic hypersurface in \( \mathbb{P}^4 \);
- \( G = (\mathbb{Z}_2)^3 \) or \( \mathbb{Z}_4 \times \mathbb{Z}_2 \), \( \tilde{X} \) is an intersection of 4 quadrics in \( \mathbb{P}^7 \) as above;
- \( G = \mathbb{Z}_3 \times \mathbb{Z}_3 \), \( \tilde{X} \) is a hypersurface of bidegree \((3,3)\) in \( \mathbb{P}^2 \times \mathbb{P}^2 \).

Of course, when looking for Calabi–Yau threefolds with interesting \( \pi_1 \), there is no reason to assume that it contains an ample rigid surface. Observe however that if we want to use the preceding method, in other words, to find a projective space \( \mathbb{P}(V) \) with an action of \( G \) and a smooth invariant linearly normal Calabi–Yau threefold \( \tilde{X} \subset \mathbb{P}(V) \), then the line bundle \( \mathcal{O}_{\tilde{X}}(1) \) will be the pullback of an ample line bundle \( L \) on \( X \), and by the above argument the representation of \( G \) on \( V \) will be \( h^0(L) \) times the regular representation. This leaves little hope to find an invariant Calabi–Yau threefold when the product \( h^0(L)|G| \) becomes large.
References


[R] M. Reid, *Surfaces with $p_g = 0$, $K^2 = 2$*. Unpublished manuscript and letters (1979)

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