Fano threefolds and K3 surfaces

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Introduction

A smooth anticanonical divisor in a Fano threefold is a K3 surface, endowed with a natural polarization (the restriction of the anticanonical bundle). The question we address in this note is: which K3 surfaces do we get in this way? The answer turns out to be very easy, but it does not seem to be well-known, so the Fano Conference might be a good opportunity to write it down.

To explain the result, let us consider a component $\mathcal{F}_g$ of the moduli stack[1] of pairs (V,S), where V is a Fano threefold of genus g and S a smooth surface in the linear system $|\mathrm{K}_\mathrm{V}^{-1}|$. Let $\mathcal{K}_g$ be the moduli stack of polarized K3 surfaces of degree $2g-2$. By associating to (V,S) the surface S we get a morphism of stacks

$$s_g : \mathcal{F}_g \longrightarrow \mathcal{K}_g \ .$$

We cannot expect s_g to be generically surjective, at least if our Fano threefolds have $b_2 > 1$: indeed for each (V,S) in $\mathcal{F}_g$ the restriction map $\mathrm{Pic}(\mathrm{V}) \rightarrow \mathrm{Pic}(\mathrm{S})$ is injective by the weak Lefschetz theorem, and this is a constraint on the K3 surface S. This map is actually a lattice embedding when we equip $\mathrm{Pic}(\mathrm{V})$ with the scalar product $(\mathrm{L},\mathrm{M}) \mapsto (\mathrm{L}\cdot\mathrm{M}\cdot\mathrm{K}_\mathrm{V}^{-1})$; it maps the element $\mathrm{K}_\mathrm{V}^{-1}$ of $\mathrm{Pic}(\mathrm{V})$ to the polarization of S.

To take this into account, we fix a lattice R with a distinguished element ρ of square $2g-2$, and we consider the moduli stack $\mathcal{F}_g^\mathrm{R}$ parametrizing pairs (V,S) with a lattice isomorphism $\mathrm{R} \xrightarrow{\sim} \mathrm{Pic}(\mathrm{V})$ mapping ρ to $\mathrm{K}_\mathrm{V}^{-1}$. Let $\mathcal{K}_g^\mathrm{R}$ be the algebraic stack parametrizing K3 surfaces S together with an embedding of R as a primitive sublattice of $\mathrm{Pic}(\mathrm{S})$, mapping ρ to an ample class. We have as before a forgetful morphism $s_g^\mathrm{R} : \mathcal{F}_g^\mathrm{R} \rightarrow \mathcal{K}_g^\mathrm{R}$.

Theorem .– *The morphism* $s_g^\mathrm{R} : \mathcal{F}_g^\mathrm{R} \rightarrow \mathcal{K}_g^\mathrm{R}$ *is smooth and generically surjective; its relative dimension at* (V,S) *is* $b_3(\mathrm{V})/2$.

As a corollary, a general K3 surface with given Picard lattice R and polarization class $\rho \in \mathrm{R}$ is an anticanonical divisor in a Fano threefold if and only if $(\mathrm{R},\rho) \cong (\mathrm{Pic}(\mathrm{V}),\mathrm{K}_\mathrm{V}^{-1})$ for some Fano threefold V.

The proof of the Theorem is given in § 3, after some preliminaries on deformation theory (§ 1) and construction of the moduli stacks (§ 2). We give some comments in § 4, and in § 5 we discuss the analogous question for curve sections of K3 surfaces.

[1] The frightened reader may replace "stack" by "orbifold" or even "space"; in the latter case the word "smooth" in the Theorem below has to be taken with a grain of salt.

We will work for simplicity over $\mathbf{C}$, though part of the results remain valid over an arbitrary algebraically closed field.

1. A reminder on deformation theory

In this section we will quickly review two well-known results on deformation theory that are needed for the proof. The experts are encouraged to skip this part.

Let X be a smooth variety, Y a closed, smooth subvariety of X. We denote by $\mathrm{T_X}\langle \mathrm{Y}\rangle \subset \mathrm{T_X}$ the subsheaf of vector fields which are tangent to Y, and by $r : \mathrm{T_X}\langle \mathrm{Y}\rangle \to \mathrm{T_Y}$ the restriction map.

Proposition 1.1 .– *The infinitesimal deformations of* (X,Y) *are controlled by the sheaf* $\mathrm{T_X}\langle \mathrm{Y}\rangle$ (that is, obstructions lie in $\mathrm{H}^2(\mathrm{X},\mathrm{T_X}\langle \mathrm{Y}\rangle)$, first order deformations are parametrized by H^1 and infinitesimal automorphisms by H^0). *The map which associates to a first order deformation of* (X,Y) *the corresponding deformation of* Y *is the induced map* $\mathrm{H}^1(r) : \mathrm{H}^1(\mathrm{X},\mathrm{T_X}\langle \mathrm{Y}\rangle) \to \mathrm{H}^1(\mathrm{Y},\mathrm{T_Y})$.

This can be extracted, for instance, from [R], but in such a simple situation it is more direct to apply Grothendieck's theory, as explained in [Gi], VII.1.2. Let us sketch briefly how this works. Put $\mathrm{X}_\varepsilon = \mathrm{X}\otimes_{\mathbf{C}} \mathbf{C}[\varepsilon]$ and $\mathrm{Y}_\varepsilon = \mathrm{Y}\otimes_{\mathbf{C}} \mathbf{C}[\varepsilon]$, with $\varepsilon^2=0$; let $\mathcal{A}_{\mathrm{X,Y}}$ (resp. $\mathcal{A}_{\mathrm{Y}}$) be the sheaf of local automorphisms of $\mathrm{Y}_\varepsilon \subset \mathrm{X}_\varepsilon$ (resp. Y_ε) which induce the identity modulo ε. According to (*loc. cit.*), since the deformations of $\mathrm{Y}\subset \mathrm{X}$ (resp. Y) are locally trivial, they are controlled by the sheaf $\mathcal{A}_{\mathrm{X,Y}}$ (resp. $\mathcal{A}_{\mathrm{Y}}$) (technically, these deformations form a gerbe, and the sheaf $\mathcal{A}$ is a band for this gerbe). So we just have to identify these sheaves. For $\mathcal{A}_{\mathrm{Y}}$ this is classical: a section of $\mathcal{A}_{\mathrm{Y}}$ over an open subset U of Y is given by an algebra automorphism

$$\mathcal{O}_{\mathrm{U}}[\varepsilon] \longrightarrow \mathcal{O}_{\mathrm{U}}[\varepsilon]$$

which must be of the form $\mathrm{I}+\varepsilon\,\delta$, where δ is a derivation of $\mathcal{O}_{\mathrm{U}}$; this gives a group isomorphism $\mathcal{A}_{\mathrm{Y}} \cong \mathrm{T_Y}$. Similarly a local automorphism of (X,Y) is given by a diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathrm{X}}[\varepsilon] & \xrightarrow{\mathrm{I}+\varepsilon \mathrm{D}} & \mathcal{O}_{\mathrm{X}}[\varepsilon] \\ \big\downarrow & & \big\downarrow \\ \mathcal{O}_{\mathrm{Y}}[\varepsilon] & \xrightarrow{\mathrm{I}+\varepsilon\,\delta} & \mathcal{O}_{\mathrm{Y}}[\varepsilon] \end{array} \quad ,$$

where D and δ are local derivations of $\mathcal{O}_{\mathrm{X}}$ and $\mathcal{O}_{\mathrm{Y}}$. The commutativity of the diagram means that D, viewed as a vector field, is tangent to Y, and induces the vector field δ on Y. This gives an isomorphism $\mathcal{A}_{\mathrm{X,Y}} \cong \mathrm{T_X}\langle \mathrm{Y}\rangle$; the forgetful map $\mathcal{A}_{\mathrm{X,Y}} \to \mathcal{A}_{\mathrm{Y}}$ maps (D,δ) onto δ, thus coincides with $r : \mathrm{T_X}\langle \mathrm{Y}\rangle \to \mathrm{T_Y}$. ■

(1.2) Let now X be a smooth variety and R a free, finitely generated submodule of $\mathrm{Pic}(X)$; we consider the deformation problem for (X, R). Choosing a basis for R this amounts to deform X together with line bundles $L_1, \ldots, L_p$. As above the deformations of a pair (X, L) are controlled by the sheaf of local automorphisms of $(X \otimes_{\mathbf{C}} \mathbf{C}[\varepsilon], L \otimes_{\mathbf{C}} \mathbf{C}[\varepsilon])$ inducing the identity modulo ε; this is readily identified with the sheaf $\mathcal{D}^1(L)$ of first order differential operators of L, the map $(X, L) \mapsto [X]$ corresponding to the symbol map $\sigma : \mathcal{D}^1(L) \to T_X$ (this is of course classical). Therefore deformations of $(X, L_1, \ldots, L_p)$ are controlled by the sheaf $\mathcal{D}^1(R) := \mathcal{D}^1(L_1) \times_{T_X} \ldots \times_{T_X} \mathcal{D}^1(L_p)$, which appears as an extension

$$0 \to \mathcal{O}_X^p \longrightarrow \mathcal{D}^1(R) \longrightarrow T_X \to 0 .$$

The extension class lies in $H^1(\Omega^1_X)^p$, its i-th component being the Atiyah class $c_1(L_i) \in H^1(X, \Omega^1_X)$. In a more intrinsic way this can be written as an extension

$$0 \to R^* \otimes_{\mathbf{Z}} \mathcal{O}_X \longrightarrow \mathcal{D}^1(R) \longrightarrow T_X \to 0 \qquad (1.3)$$

whose class in $H^1(X, \Omega^1_X) \otimes_{\mathbf{Z}} R^*$ is deduced from the map $c_1 : R \to H^1(X, \Omega^1_X)$.

Assume now that X is a K3 surface. We have $H^1(X, \mathcal{O}_X) = H^2(X, T_X) = 0$, and choosing a non-zero holomorphic 2-form on X defines an isomorphism $H^2(X, \mathcal{O}_X) \xrightarrow{\sim} \mathbf{C}$. The extension (1.3) gives rise to an exact sequence

$$0 \to H^1(X, \mathcal{D}^1(R)) \longrightarrow H^1(X, T_X) \xrightarrow{\partial} R^* \otimes_{\mathbf{Z}} \mathbf{C} \longrightarrow H^2(X, \mathcal{D}^1(R)) \to 0$$

where ∂ is the cup-product with the extension class; that is, for $\xi \in H^1(X, T_X)$ and L a line bundle in R, we have $\langle \partial(\xi), L \rangle = \xi \cup c_1(L)$. In other words, using Serre duality, ∂ is the transpose of the natural map $c_1 : R \otimes_{\mathbf{Z}} \mathbf{C} \to H^1(X, \Omega^1_X)$. Since c_1 is injective, ∂ is surjective, hence $H^2(X, \mathcal{D}^1(R)) = 0$ and $H^1(X, \mathcal{D}^1(R)) = \mathrm{Ker}\, \partial$. Therefore:

Proposition 1.4.– *Let* X *be a* K3 *surface and* R *a subgroup of* $\mathrm{Pic}(X)$. *The infinitesimal deformations of* (X, R) *are unobstructed. The first order deformations are parametrized by the orthogonal of* $c_1(R) \subset H^1(X, \Omega^1_X)$ *in* $H^1(X, T_X)$. ■

2. The stacks $\mathcal{K}_g^R$ and $\mathcal{F}_g^R$

(2.1) Let V be a smooth Fano threefold. Recall that the *genus* g of V is defined by the formula $2g - 2 = (K_V^{-1})^3$. If S is a smooth K3 surface in the linear system $|K_V^{-1}|$, the induced polarization $L := K_V^{-1}{}_{|S}$ satisfies $L^2 = 2g - 2$, so that the curves of $|L|$ have genus g.

As explained in the introduction, we will consider $\mathrm{Pic}(V)$ as a lattice with the product $(L, M) \mapsto (L \cdot M \cdot K_V^{-1})$.

(2.2) The definition of the moduli stack $\mathcal{F}$ of pairs (V,S) is straightforward: we start from the moduli stack $\mathcal{T}$ of Fano threefolds. Let $f:\mathcal{V}\to\mathcal{T}$ be the universal family; the projective bundle $\mathbf{P}((f_*\mathrm{K}_{\mathcal{V}/\mathcal{T}})^*)$ parametrizes pairs (V,S) with $\mathrm{S}\in|\mathrm{K}_\mathrm{V}^{-1}|$, and we take for $\mathcal{F}$ the open substack defined by the condition that S is smooth. We add the subscript g when we restrict to pairs (V,S) of genus g.

(2.3) The definition of the moduli stacks $\mathcal{K}_g^\mathrm{R}$ and $\mathcal{F}_g^\mathrm{R}$ is slightly more involved. Let $f:\mathrm{X}\to\mathrm{B}$ be a smooth, projective morphism of noetherian schemes. Following [G], we denote by $\underline{\mathrm{Pic}}_{\mathrm{X}/\mathrm{B}}$ the sheaf on B (for the faithfully flat topology) associated to the presheaf $(\mathrm{B}'\to\mathrm{B})\mapsto\mathrm{Pic}(\mathrm{X}\times_\mathrm{B}\mathrm{B}')$. According to *loc. cit.*, this sheaf is representable by a group scheme over B, for which we will use the same notation. If f has relative dimension 2, the intersection product defines a bilinear form $\underline{\mathrm{Pic}}_{\mathrm{X}/\mathrm{B}}\times\underline{\mathrm{Pic}}_{\mathrm{X}/\mathrm{B}}\to\mathbf{Z}_\mathrm{B}$; the same holds in (relative) dimension 3 by taking the intersection product with $\mathrm{K}_{\mathrm{X}/\mathrm{B}}^{-1}$.

Let R be a lattice, with a distinguished element ρ. The moduli stacks $\mathcal{F}_g^\mathrm{R}$ and $\mathcal{K}_g^\mathrm{R}$ are defined as follows. An object of $\mathcal{F}_g^\mathrm{R}$ over a scheme B is a pair (V,S) over B, where $\mathrm{V}\to\mathrm{B}$ is a family of Fano threefolds, of genus g, and $\mathrm{S}\subset\mathrm{V}$ a family of K3 surfaces over B, together with a lattice isomorphism $\mathrm{R}_\mathrm{B}\xrightarrow{\sim}\underline{\mathrm{Pic}}_{\mathrm{V}/\mathrm{B}}$ mapping ρ onto the class of $\mathrm{K}_\mathrm{V}^{-1}$. Similarly, an object of $\mathcal{K}_g^\mathrm{R}$ over B is a family $\mathrm{S}\to\mathrm{B}$ of polarized K3 surfaces, of genus g, together with a lattice embedding $\mathrm{R}_\mathrm{B}\hookrightarrow\underline{\mathrm{Pic}}_{\mathrm{S}/\mathrm{B}}$ mapping ρ onto the polarization class.

That $\mathcal{K}_g^\mathrm{R}$ and $\mathcal{F}_g^\mathrm{R}$ are indeed algebraic stacks follows from the result of Grothendieck quoted above. Consider for instance the universal family $\mathcal{S}\to\mathcal{K}_g$ of K3 surfaces with a genus g polarization. Then $\underline{\mathrm{Pic}}_{\mathcal{S}/\mathcal{K}_g}$ is representable by an algebraic stack, which is a group scheme over $\mathcal{K}_g$. Choosing a basis $(e_0,\ldots,e_p)$ of R with $e_0=\rho$, we realize $\mathcal{K}_g^\mathrm{R}$ as an open and closed substack of $(\underline{\mathrm{Pic}}_{\mathcal{S}/\mathcal{K}_g})^p$.

Associating to a pair (V,S) over B the family $\mathrm{S}\to\mathrm{B}$ with the induced polarization and the composite map $\mathrm{R}_\mathrm{B}\xrightarrow{\sim}\underline{\mathrm{Pic}}_{\mathrm{V}/\mathrm{B}}\hookrightarrow\underline{\mathrm{Pic}}_{\mathrm{S}/\mathrm{B}}$ defines a morphism of stacks $s_g^\mathrm{R}:\mathcal{F}_g^\mathrm{R}\to\mathcal{K}_g^\mathrm{R}$.

(2.4) Let us say a few words about the lattice R. In order for our moduli stacks to be non-empty, R must be a sublattice of the Picard group of a K3 surface, containing a polarization; also it must be isomorphic to the Picard lattice of a Fano threefold. Thus:

- R is even, of signature $(1,r-1)$;
- R has rank $r\le 10$; if $r\ge 6$, it is isomorphic to the Picard lattice of $\mathrm{S}_{11-r}\times\mathbf{P}^1$, where S_d is the Del Pezzo surface of degree d.

(The latter property follows from Theorem 2 in [M-M]).

(2.5) Since R has signature $(1,r-1)$, the orthogonal of ρ is negative definite, and therefore the group of automorphisms of R fixing ρ is finite. It follows that the forgetful maps $\mathcal{F}_g^{\mathrm{R}} \to \mathcal{F}_g$ and $\mathcal{K}_g^{\mathrm{R}} \to \mathcal{K}_g$ are (representable and) finite. The former map is actually is an étale covering, because for any family $\mathrm{V} \to \mathrm{B}$ of Fano threefolds the sheaf $\underline{\mathrm{Pic}}_{\mathrm{V/B}}$ becomes trivial on an étale covering of B.

As for the stack $\mathcal{K}_g^{\mathrm{R}}$, we have

Proposition 2.6 .– *The stack* $\mathcal{K}_g^{\mathrm{R}}$ *is smooth, irreducible, of dimension* $20-r$.

The smoothness and dimension of $\mathcal{K}_g^{\mathrm{R}}$ follow from Proposition 1.4; its irreducibility is a consequence of the theory of the period mapping. Let us recall briefly how this works, following the exposition in [D], 4.1. Let L be an even unimodular lattice of signature $(3,19)$ (all such lattices are isomorphic). We choose an embedding of R as a primitive sublattice of L (such an embedding is unique up to an automorphism of L by Nikulin's results, see [D], thm. 1.4.8). We consider *marked* K3 *surfaces of type* R, that is, K3 surfaces S with a lattice isomorphism $\sigma : \mathrm{L} \xrightarrow{\sim} \mathrm{H}^2(\mathrm{S},\mathbf{Z})$ such that $\sigma(\mathrm{R})$ is contained in $\mathrm{Pic}(\mathrm{S}) \subset \mathrm{H}^2(\mathrm{S},\mathbf{Z})$, and $\sigma(\rho)$ is an ample class. These marked surfaces admit a fine (analytic) moduli space $\widetilde{\mathcal{K}}_g^{\mathrm{R}}$; the period map induces an isomorphism of $\widetilde{\mathcal{K}}_g^{\mathrm{R}}$ onto the period domain $\mathrm{D_R}$, which is the disjoint union of two copies of a bounded symmetric domain of type IV (*loc. cit.*). Our stack $\mathcal{K}_g^{\mathrm{R}}$ is isomorphic to the quotient of $\widetilde{\mathcal{K}}_g^{\mathrm{R}}$ by the group Γ_{R} of automorphisms of L which fix the elements of R. This group acts on $\mathrm{D_R}$ permuting its two connected components (this can be seen exactly as in [B], Cor. p. 151). Thus the quotient stack $\mathcal{K}_g^{\mathrm{R}}$ is irreducible. ∎

3. Proof of the theorem

(3.1) By Proposition 1.1 the infinitesimal behaviour of $\mathcal{F}_g$ (or $\mathcal{F}_g^{\mathrm{R}}$, since the forgetful map $\mathcal{F}_g^{\mathrm{R}} \to \mathcal{F}_g$ is étale) at a pair (V,S) is controlled by the sheaf $\mathrm{T_V}\langle \mathrm{S}\rangle$, which is defined by the exact sequence

$$0 \to \mathrm{T_V}\langle \mathrm{S}\rangle \longrightarrow \mathrm{T_V} \longrightarrow \mathrm{N_{S/V}} \to 0 \; . \tag{3.2}$$

We have $\mathrm{H}^2(\mathrm{V},\mathrm{T_V}) = \mathrm{H}^2(\mathrm{V},\Omega^2_{\mathrm{V}} \otimes \mathrm{K}_{\mathrm{V}}^{-1}) = 0$ by the Akizuki-Nakano theorem, and $\mathrm{H}^1(\mathrm{S},\mathrm{N_{S/V}}) = 0$ because $\mathrm{N_{S/V}}$ is an ample line bundle on S. Thus the exact sequence (3.2) gives $\mathrm{H}^2(\mathrm{S},\mathrm{T_V}\langle \mathrm{S}\rangle) = 0$, so that the first order deformations of (V,S) are unobstructed (in other words, the stack $\mathcal{F}_g^{\mathrm{R}}$ is smooth).

It follows from Proposition 1.1 that the tangent map to $s_g : \mathcal{F}_g \to \mathcal{K}_g$ at (V,S) is $\mathrm{H}^1(r)$, where $r : \mathrm{T_V}\langle \mathrm{S}\rangle \to \mathrm{T_S}$ is the restriction map. The map r is surjective, and its kernel is the subsheaf $\mathrm{T_V}(-\mathrm{S})$ of vector fields vanishing along S, which in our case is isomorphic to Ω^2_{V}. Thus we have an exact sequence

$$0 \to \Omega^2_{\mathrm{V}} \longrightarrow \mathrm{T_V}\langle \mathrm{S}\rangle \xrightarrow{\;r\;} \mathrm{T_S} \to 0 \; . \tag{3.3}$$

Let us consider the associated cohomology exact sequence. Since $\mathrm{H}^0(\mathrm{V},\Omega^2_{\mathrm{V}})$ and $\mathrm{H}^0(\mathrm{S},\mathrm{T_S})$ are zero, we get first of all $\mathrm{H}^0(\mathrm{V},\mathrm{T_V}\langle\mathrm{S}\rangle)=0$, so that (V,S) has no infinitesimal automorphisms (that is, $\mathcal{F}^{\mathrm{R}}_g$ is a Deligne-Mumford stack). Then we get the exact sequence

$$0\to \mathrm{H}^1(\mathrm{V},\Omega^2_{\mathrm{V}})\longrightarrow \mathrm{H}^1(\mathrm{V},\mathrm{T_V}\langle\mathrm{S}\rangle)\xrightarrow{\mathrm{H}^1(r)}\mathrm{H}^1(\mathrm{S},\mathrm{T_S})\xrightarrow{\partial}\mathrm{H}^2(\mathrm{V},\Omega^2_{\mathrm{V}})\to 0\ . \qquad (3.4)$$

Let $i:\mathrm{S}\hookrightarrow\mathrm{V}$ be the inclusion map. To evaluate ∂, consider the exact sequence

$$0\to\Omega^1_{\mathrm{V}}(\log\mathrm{S})(-\mathrm{S})\longrightarrow\Omega^1_{\mathrm{V}}\xrightarrow{i^*}\Omega^1_{\mathrm{S}}\to 0 \qquad (3.5)$$

deduced from (3.3) by applying the duality functor $\mathrm{R}\underline{\mathrm{Hom}}_{\mathrm{V}}(\ ,\mathrm{K_V})$ and using the canonical isomorphisms $\mathrm{R}\underline{\mathrm{Hom}}_{\mathrm{V}}(\mathrm{T_S},\mathrm{K_V})\cong\mathrm{R}\underline{\mathrm{Hom}}_{\mathrm{S}}(\mathrm{T_S},\mathrm{K_S})\cong\Omega^1_{\mathrm{S}}$. By general non-sense the cohomy exact sequence associated to (3.5) is the dual of the one associated to (3.4); in particular the transpose of ∂ is identified (through Serre duality on V and S) with the restriction map $\mathrm{H}^1(i^*):\mathrm{H}^1(\mathrm{V},\Omega^1_{\mathrm{V}})\to\mathrm{H}^1(\mathrm{S},\Omega^1_{\mathrm{S}})$ – up to a sign which is irrelevant for our purpose.

Therefore $\mathrm{Ker}\,\partial$ is the orthogonal of the image of $\mathrm{H}^1(i^*)$. Because of the commutative diagram

$$\begin{array}{ccc}
\mathrm{R} & & \\
\downarrow & \searrow & \\
\mathrm{Pic}(\mathrm{V}) & \xrightarrow{i^*} & \mathrm{Pic}(\mathrm{S}) \\
\downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle c_1} \\
\mathrm{H}^1(\mathrm{V},\Omega^1_{\mathrm{V}}) & \xrightarrow{\mathrm{H}^1(i^*)} & \mathrm{H}^1(\mathrm{S},\Omega^1_{\mathrm{S}})
\end{array}$$

it is also the orthogonal of $c_1(\mathrm{R})\subset\mathrm{H}^1(\mathrm{S},\Omega^1_{\mathrm{S}})$. By Proposition 1.4 this is exactly the tangent space to $\mathcal{K}^{\mathrm{R}}_g$ at S, so the induced map $\mathrm{T_V}\langle\mathrm{S}\rangle\to\mathrm{Ker}\,\partial$ is the tangent map to s^{R}_g at (V,S). This proves that this map is surjective, and the exact sequence (3.4) shows that its kernel is isomorphic to $\mathrm{H}^1(\mathrm{V},\Omega^2_{\mathrm{V}})$. Hence s^{R}_g is smooth, of relative dimension $b_3(\mathrm{V})/2$, and generically surjective because $\mathcal{K}^{\mathrm{R}}_g$ is irreducible (Proposition 2.6). ∎

4. Consequences and comments

Corollary 4.1 .– *Let* (S,h) *be a polarized* K3 *surface,* P *its Picard group; assume that* (S,h) *is general in* $\mathcal{K}^{\mathrm{P}}_g$. *Then* S *is an anticanonical divisor in a Fano threefold if and only if* (P,h) *is isomorphic to* $(\mathrm{Pic}(\mathrm{V}),\mathrm{K}_{\mathrm{V}}^{-1})$ *for some Fano threefold* V . ∎

We leave to the reader the enjoyable task of listing the pairs (P,h) for the 87 types of Fano threefolds with $b_2>1$ classified in [M-M]. In the case $b_2=1$ we get the generic surjectivity of $s_g : \mathcal{F}_g \to \mathcal{K}_g$; this is actually well-known, and follows for instance from the work of Mukai [M1].

(4.2) In most cases the map s_g^{R} is not surjective. Consider for instance the component of $\mathcal{F}_5$ parametrizing pairs (V,S) with $\mathrm{Pic}(\mathrm{V}) = \mathbf{Z}\cdot \mathrm{K}_{\mathrm{V}}$ and $g=5$. Each threefold V is the complete intersection of 3 quadrics in $\mathbf{P}^6$, so we get in the image of s_5 all complete intersections of 3 quadrics in $\mathbf{P}^5$, which form a proper open substack of $\mathcal{K}_5$ (it does not contain hyperelliptic and trigonal K3 surfaces).

(4.3) Part of the argument extends to Fano manifolds of arbitrary dimension n, but the exact sequence (3.4) becomes

$$0 \to \mathrm{H}^1(\mathrm{V},\Omega_{\mathrm{V}}^{n-1}) \longrightarrow \mathrm{H}^1(\mathrm{V},\mathrm{T}_{\mathrm{V}}\langle \mathrm{S}\rangle) \longrightarrow \mathrm{H}^1(\mathrm{S},\mathrm{T}_{\mathrm{S}}) \xrightarrow{\partial} \mathrm{H}^2(\mathrm{V},\Omega_{\mathrm{V}}^{n-1}) \to 0 \ ,$$

so that the geometric meaning of $\mathrm{Ker}\,\partial$ is not so clear. When $b_{n-1}(\mathrm{V})=0$ we see that the map $(\mathrm{V},\mathrm{S})\mapsto \mathrm{S}$ is smooth.

(4.4) A glance at the list of [M-M] shows that roughly half of the families of Fano threefolds have $b_3=0$; for these the map s_g^{R} is étale, and one can ask whether it is an isomorphism onto an open substack. This is easy to prove in some cases $(\mathrm{V}=\mathbf{P}^3,\mathrm{Q}_3,\mathbf{P}^1\times\mathbf{P}^2,\ldots)$. For Fano threefolds of index 2 and genus 6, it has been proved by Mukai ([M1], Cor. 4.3). An interesting open case is the one of Fano threefolds of genus 12 with $b_2=1$.

5. K3 surfaces and canonical curves

(5.1) Let $\mathcal{KC}_g$ be the moduli stack of pairs (S,C), where S is a K3 surface with a primitive polarization of genus g, and $\mathrm{C}\subset\mathrm{S}$ a smooth curve in the polarization class; let $\mathcal{M}_g$ be the moduli stack of curves of genus g. We have as before a morphism of stacks

$$c_g : \mathcal{KC}_g \longrightarrow \mathcal{M}_g \ .$$

This morphism has been studied extensively. Let me summarize the main results. Recall first that $\dim \mathcal{KC}_g = 19+g$ is greater than $\dim\mathcal{M}_g = 3g-3$ for $g\le 10$, equal for $g=11$ and smaller for $g\ge 12$.

- c_g is generically surjective for $g\le 9$ and $g=11$ [M1].
- c_g is *not* surjective for $g=10$ [M1]; its image is the hypersurface of $\mathcal{M}_g$ where the Wahl map $\wedge^2\mathrm{H}^0(\mathrm{C},\mathrm{K}_{\mathrm{C}})\to\mathrm{H}^0(\mathrm{C},\mathrm{K}_{\mathrm{C}}^{\otimes 3})$ fails to be bijective [C-U].
- c_g is generically finite for $g=11$ and $g\ge 13$, but *not* for $g=12$ [M2].

(5.2) Let us consider the map c_g from the differential point of view that we have adopted in this note. Let $(\mathrm{S},\mathrm{C})\in\mathcal{KC}_g$; we have by Serre duality $\mathrm{H}^2(\mathrm{S},\mathrm{T}_{\mathrm{S}}\langle \mathrm{C}\rangle) = \mathrm{H}^0(\mathrm{S},\Omega^1_{\mathrm{S}}(\log \mathrm{C}))^* = 0$, hence the stack $\mathcal{KC}_g$ is smooth. By Proposition 1.1, the tangent map to c_g at (S,C) is $\mathrm{H}^1(r):\mathrm{H}^1(\mathrm{S},\mathrm{T_S}\langle\mathrm{C}\rangle)\rightarrow\mathrm{H}^1(\mathrm{C},\mathrm{T_C})$. It appears in the cohomology exact sequence analogous to (3.4)

$$0\rightarrow\mathrm{H}^1(\mathrm{S},\mathrm{T_S}(-\mathrm{C}))\longrightarrow\mathrm{H}^1(\mathrm{S},\mathrm{T_S}\langle\mathrm{C}\rangle)\xrightarrow{\mathrm{H}^1(r)}\mathrm{H}^1(\mathrm{C},\mathrm{T_C})\xrightarrow{\partial}\mathrm{H}^2(\mathrm{S},\mathrm{T_S}(-\mathrm{C}))\rightarrow 0\ .$$

Using Serre duality, we see that c_g *is smooth at* (C,S) *if and only if* $\mathrm{H}^0(\mathrm{S},\Omega^1_\mathrm{S}(\mathrm{C}))=0$, *and unramified at* (C,S) *if and only if* $\mathrm{H}^1(\mathrm{S},\Omega^1_\mathrm{S}(\mathrm{C}))=0$. Note that this condition depends only on the polarization $\mathrm{L}=\mathcal{O}_\mathrm{S}(\mathrm{C})$ and not on the particular curve C in $|\mathrm{L}|$ – a fact which is not a priori obvious.

The results of (5.1) are thus equivalent to:

Let (S,L) *be a general* K3 *surface with a primitive polarization of genus* g. *We have:*

- $\mathrm{H}^0(\mathrm{S},\Omega^1_\mathrm{S}\otimes\mathrm{L})=0$ *for* $g\le 9$ *and* $g=11$;
- $\dim\mathrm{H}^0(\mathrm{S},\Omega^1_\mathrm{S}\otimes\mathrm{L})=1$ *for* $g=10$;
- $\mathrm{H}^1(\mathrm{S},\Omega^1_\mathrm{S}\otimes\mathrm{L})=0$ *for* $g=11$ *and* $g\ge 13$.

A direct proof of these results would provide an alternative approach to the results of (5.1).

(5.3) Let us observe that though c_g is generically surjective for $g\le 9$ and $g=11$, it is *not* everywhere smooth. Take for instance a K3 surface S with an elliptic pencil $|\mathrm{E}|$ and a smooth curve Γ of genus $\gamma\in\{0,1\}$ with $\mathrm{E}\cdot\Gamma=2$; put $\mathrm{L}=\mathcal{O}_\mathrm{S}(k\mathrm{E}+\Gamma)$. Then L is a primitive polarization of genus $2k+\gamma$. Let $f:\mathrm{S}\rightarrow\mathbf{P}^1$ be the map defined by the pencil $|\mathrm{E}|$; since Ω^1_S contains $f^*\Omega^1_{\mathbf{P}^1}$, we get $\dim\mathrm{H}^0(\mathrm{S},\Omega^1_\mathrm{S}\otimes\mathrm{L})\ge k-1$. This gives pairs (S,C) in $\mathcal{KC}_g$, for $g\ge 4$, where c_g is not smooth.

Similarly, c_g is *not* everywhere unramified for $g=11$ or $g\ge 13$. A series of examples is provided by the following result, which is essentially due to Mukai ([M2], Prop. 6):

Proposition 5.4.– *Let* V *be a Fano threefold of index* 1 *and genus* g *such that* $\mathrm{K}_\mathrm{V}^{-1}$ *is very ample,* $\mathrm{S}\in|\mathrm{K}_\mathrm{V}^{-1}|$ *a* K3 *surface,* $\mathrm{L}:=\mathrm{K}_{\mathrm{V}\,|\mathrm{S}}^{-1}$, C *a smooth curve in the linear system* $|\mathrm{L}|$. *The fibre of* $c_g:\mathcal{KC}_g\rightarrow\mathcal{M}_g$ *at* (S,C) *is positive-dimensional. In particular, the space* $\mathrm{H}^1(\mathrm{S},\Omega^1_\mathrm{S}\otimes\mathrm{L})$ *is non-zero.*

Proof: Consider V embedded in $\mathbf{P}(\mathrm{H}^0(\mathrm{V},\mathrm{K}_\mathrm{V}^{-1}))$. A general C in $|\mathrm{L}|$ is contained in a Lefschetz pencil $(\mathrm{S}_t)_{t\in\mathbf{P}^1}$ of hyperplane sections of V : there is a finite subset Δ of $\mathbf{P}^1$ such that S_t is smooth for $t\in\mathbf{P}^1-\Delta$ and has an ordinary node for $t\in\Delta$. The corresponding map $\mathbf{P}^1-\Delta\rightarrow\mathcal{K}_g$ goes to the boundary of $\mathcal{K}_g$ (consisting of K3 surfaces with a pseudo-polarization of degree $2g-2$), and therefore cannot be constant. Thus we get a 1-dimensional family of pairs $(\mathrm{S}_t,\mathrm{C})$, for $t\in\mathbf{P}^1-\Delta$, which maps to the same point $[\mathrm{C}]$ of $\mathcal{M}_g$. This gives the result for C general in $|\mathrm{L}|$, hence for every smooth C in $|\mathrm{L}|$. ∎

In view of the list in [M-M], we get examples of positive-dimensional fibres of c_g for all $g \leq 28$ and for $g = 32$ (note that we want the polarization of S to be primitive, so V must be of index one). We know no examples in higher genus.

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