CONFORMAL BLOCKS, FUSION RULES AND THE VERLINDE FORMULA

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Abstract. A *Rational Conformal Field Theory* (RCFT) is a functor which associates to any Riemann surface with marked points a finite-dimensional vector space, so that certain axioms are satisfied; the Verlinde formula computes the dimension of these vector spaces. For some particular RCFTs associated to a compact Lie group G (the WZW models), these spaces have a beautiful algebro-geometric interpretation as spaces of *generalized theta functions*, that is, sections of a determinant bundle (or its powers) over the moduli space of G-bundles on a Riemann surface.

In this paper we explain the formalism of the Verlinde formula: the dimension of the spaces are encoded in a finite-dimensional **Z**-algebra, the *fusion ring* of the theory; everything can be expressed in terms of the characters of this ring. We show how to compute these characters in the case of the WZW model and thus obtain an explicit formula for the dimension of the space of generalized theta functions.

Dedicated to F. Hirzebruch

Introduction.

The Verlinde formula computes the dimension of certain vector spaces, the *spaces of conformal blocks*, which are the basic objects of a particular kind of quantum field theories, the so-called Rational Conformal Field Theories (RCFT). These spaces appear as spaces of global multiform sections of some flat vector bundles on the moduli space of curves with marked points, so that their dimension is simply the rank of the corresponding vector bundles. The computation relies on the behaviour of these bundles under degeneration of the Riemann surface, often referred to as the *factorization rules.* Verlinde's derivation from the formula [V] rested on a conjecture which does not seem to be proved yet in this very general framework.

The Verlinde formula started attracting a great deal of attention from mathematicians when it was realized that for some particular RCFTs associated to a compact Lie group G (the WZW-models), the spaces of conformal blocks had a nice interpretation as spaces of *generalized theta functions*, that is, sections of a

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determinant bundle (or its tensor powers) over the moduli space of G-bundles on a Riemann surface. This interpretation has been worked out rigorously for $\mathrm{SU}(n)$ in [B-L], and for the general case in [F], while the factorization rules for these models have been established in [T-U-Y] and also in [F]. However, there seems to be some confusion among mathematicians as to whether this work implies the explicit Verlinde formula for the spaces of generalized theta functions or not – perhaps because of a few misprints and inadequate references in some of the above quoted papers.

Thus the aim of this paper is to explain how the Verlinde formula for the WZW-models (hence for the space of generalized theta functions) can be derived from the factorization rules, at least in the $\mathrm{SU}(n)$ case. As the title indicates, the paper has three parts. In the first one, which is probably the most involved technically, we fix a simple Lie algebra $\mathfrak{g}$; following [T-U-Y] we associate a vector space $\mathrm{V_C}(\vec{p},\vec{\lambda})$ to a Riemann surface C and a finite number of points of C, to each of which is attached a representation of $\mathfrak{g}$. The main novelty here is a more concrete interpretation of this space (Prop. 2.3) which gives a simple expression in the case $\mathrm{C}=\mathbf{P}^1$ – an essential ingredient of the Verlinde formula. In the second part we develop the formalism of the *fusion rings*, an elegant way of encoding the factorization rules; this gives an explicit formula for the dimension of $\mathrm{V_C}(\vec{p},\vec{\lambda})$ in terms of the characters of the fusion ring. In the third part we apply this formalism to the special case considered in part I; this leads to the fusion ring $\mathcal{R}_\ell(\mathfrak{g})$ of representations of level $\leq \ell$. We show following [F] how one can determine the characters of $\mathcal{R}_\ell(\mathfrak{g})$ when $\mathfrak{g}$ is $\mathfrak{sl}(n,\mathbf{C})$ or $\mathfrak{sp}(n,\mathbf{C})$ (Faltings handles all the classical algebras and G_2, but there seems to be no proof for the other exceptional algebras). Putting things together we obtain in these cases the Verlinde formula for the dimension of $\mathrm{V_C}(\vec{p},\vec{\lambda})$.

I have tried to make the paper as self-contained as possible, and in particular not to assume that the reader is an expert in Kac-Moody algebras; however, some familiarity with classical Lie theory will certainly help. I would like to mention the preprint [S] which contains (among other things) results related to our Parts II and III – though with a slightly different point of view.

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Part I: the spaces $\mathrm{V_C}(\vec{p},\vec{\lambda})$

1. Affine Lie algebras.

(1.1) Throughout this paper we fix a simple complex Lie algebra $\mathfrak{g}$, and a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$. I refer e.g. to [Bo] for the definition of the root system $\mathrm{R}(\mathfrak{g},\mathfrak{h})\subset\mathfrak{h}^*$, and of the coroot $\mathrm{H}_\alpha\in\mathfrak{h}$ associated to a root α. We have a decomposition $\mathfrak{g}=\mathfrak{h}\oplus\sum_{\alpha\in R(\mathfrak{g},\mathfrak{h})}\mathfrak{g}^{\alpha}$. We also fix a basis $(\alpha_1,\ldots,\alpha_r)$ of the root system, which provides us with a partition of the roots into positive and negative ones.

The *weight lattice* $P\subset\mathfrak{h}^*$ is the group of linear forms $\lambda\in\mathfrak{h}^*$ such that $\lambda(H_\alpha)\in\mathbf{Z}$ for all roots α. A weight λ is *dominant* if $\lambda(H_\alpha)\geq 0$ for all positive roots α; we denote by P_+ the set of dominant weights. To each dominant weight λ is associated a simple $\mathfrak{g}$-module V_λ, unique up to isomorphism, containing a *highest weight vector* v_λ with weight λ (this means that v_λ is annihilated by $\mathfrak{g}^\alpha$ for $\alpha>0$ and that $H\,v_\lambda=\lambda(H)v_\lambda$ for all H in $\mathfrak{h}$). The map $\lambda\mapsto[V_\lambda]$ is a bijection of P_+ onto the set of isomorphism classes of finite-dimensional simple $\mathfrak{g}$-modules.

(1.2) The *normalized Killing form* $(\ |\ )$ on $\mathfrak{g}$ is the unique $\mathfrak{g}$-invariant non-degenerate symmetric form on $\mathfrak{g}$ satisfying $(H_\beta\,|\,H_\beta)=2$ for every long root β. We'll denote by the same symbol the non-degenerate form induced on $\mathfrak{h}$ and the inverse form on $\mathfrak{h}^*$. We will use these normalized forms throughout the paper.

(1.3) Let θ be the highest root of $R(\mathfrak{g},\mathfrak{h})$, and H_θ the corresponding coroot. Following [Bo] we choose elements X_θ in $\mathfrak{g}^\theta$ and $X_{-\theta}$ in $\mathfrak{g}^{-\theta}$ satisfying

$$[H_\theta,X_\theta]=2X_\theta\quad,\quad[H_\theta,X_{-\theta}]=-2X_{-\theta}\quad,\quad[X_\theta,X_{-\theta}]=-H_\theta\ .$$

These elements span a Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, isomorphic to $\mathfrak{sl}_2$, which will play an important role in this paper.

(1.4) The affine Lie algebra $\widehat{\mathfrak{g}}$ associated to $\mathfrak{g}$ is a central extension of $\mathfrak{g}\otimes\mathbf{C}((z))$ by $\mathbf{C}$:

$$\widehat{\mathfrak{g}}=\big(\mathfrak{g}\otimes\mathbf{C}((z))\big)\oplus\mathbf{C}c\ ,$$

the bracket of two elements of $\mathfrak{g}\otimes\mathbf{C}((z))$ being given by

$$[X\otimes f,Y\otimes g]=[X,Y]\otimes fg\ +\ c\cdot(X\,|\,Y)\ \mathrm{Res}(g\,df)\ .$$

We denote by $\widehat{\mathfrak{g}}_+$ and $\widehat{\mathfrak{g}}_-$ the subspaces $\mathfrak{g}\otimes z\mathbf{C}[[z]]$ and $\mathfrak{g}\otimes z^{-1}\mathbf{C}[z^{-1}]$ of $\widehat{\mathfrak{g}}$, so that we have a decomposition

$$\widehat{\mathfrak{g}}=\widehat{\mathfrak{g}}_-\oplus\mathfrak{g}\oplus\mathbf{C}c\oplus\widehat{\mathfrak{g}}_+\ .$$

By the formula for the Lie bracket, each summand is actually a Lie subalgebra of $\widehat{\mathfrak{g}}$.

(1.5) We fix an integer $\ell>0$ (the level); we are interested in the irreducible representations of $\widehat{\mathfrak{g}}$ which are *of level* ℓ, i.e. such that the central element c of $\widehat{\mathfrak{g}}$ acts as multiplication by ℓ. Let P_ℓ be the set of dominant weights λ of $\mathfrak{g}$ such that $\lambda(H_\theta)\leq\ell$. The fundamental result of the representation theory of $\widehat{\mathfrak{g}}$ (see e.g. [K]) asserts that the reasonable representations of level ℓ are classified by P_ℓ. More precisely, for each $\lambda \in \mathrm{P}_\ell$, there exists a simple $\widehat{\mathfrak{g}}$-module $\mathcal{H}_\lambda$ of level ℓ, characterized up to isomorphism by the following property:

The subspace of $\mathcal{H}_\lambda$ *annihilated by* $\widehat{\mathfrak{g}}_+$ *is isomorphic as a* $\mathfrak{g}$*-module to* V_λ.
In the sequel we will identify V_λ with the subspace of $\mathcal{H}_\lambda$ annihilated by $\widehat{\mathfrak{g}}_+$.

(1.6) We will need a few more technical details about the $\widehat{\mathfrak{g}}$-module $\mathcal{H}_\lambda$. Let us first recall its construction. Let $\mathfrak{p}$ be the Lie subalgebra $\mathfrak{g} \oplus \mathbf{C}c \oplus \widehat{\mathfrak{g}}_+$ of $\widehat{\mathfrak{g}}$. We extend the representation of $\mathfrak{g}$ on V_λ by letting $\widehat{\mathfrak{g}}_+$ act trivially and c as $\ell\,\mathrm{Id}_{\mathrm{V}_\lambda}$; we denote by $\mathcal{V}_\lambda$ the induced $\widehat{\mathfrak{g}}$-module $\mathrm{U}(\widehat{\mathfrak{g}}) \otimes_{\mathrm{U}(\mathfrak{p})} \mathrm{V}_\lambda$. It contains a unique maximal $\widehat{\mathfrak{g}}$-submodule $\mathcal{Z}_\lambda$; then $\mathcal{H}_\lambda$ is the quotient $\mathcal{V}_\lambda/\mathcal{Z}_\lambda$.

Since $\mathrm{U}(\widehat{\mathfrak{g}})$ is isomorphic as a $\mathrm{U}(\widehat{\mathfrak{g}}_-)$-module to $\mathrm{U}(\widehat{\mathfrak{g}}_-) \otimes_{\mathbf{C}} \mathrm{U}(\mathfrak{p})$, we see that *the natural map* $\mathrm{U}(\widehat{\mathfrak{g}}_-) \otimes_{\mathbf{C}} \mathrm{V}_\lambda \longrightarrow \mathcal{V}_\lambda$ *is an isomorphism of* $\widehat{\mathfrak{g}}_-$*-modules.*

Let us identify V_λ with the submodule $1 \otimes \mathrm{V}_\lambda$ of $\mathcal{V}_\lambda$. With the notation of (1.3), the submodule $\mathcal{Z}_\lambda$ is generated by the element $(\mathrm{X}_\theta \otimes z^{-1})^{\ell-\lambda(\mathrm{H}_\theta)+1}\, v_\lambda$ (*cf.* [K, exerc. 12.12]); this element is annihilated by $\widehat{\mathfrak{g}}_+$ (see remark (3.6) below).

(1.7) An important observation (which plays a crucial role in conformal field theory) is that the representation theory of $\widehat{\mathfrak{g}}$ is essentially independent of the choice of the local coordinate z. Let $u = u(z)$ be an element of $\mathbf{C}[[z]]$ with $u(0) = 0$, $u'(0) \neq 0$. The automorphism $f \mapsto f \circ u$ of $\mathbf{C}((z))$ induces an automorphism of $\mathfrak{g} \otimes \mathbf{C}((z))$, which extends to an automorphism γ_u of $\widehat{\mathfrak{g}}$ (given by $\gamma_u(\mathrm{X} \otimes f) = \mathrm{X} \otimes f \circ u$). Let $\lambda \in \mathrm{P}_\ell$; since γ_u preserves $\widehat{\mathfrak{g}}_+$ and is the identity on $\mathfrak{g}$, the representation $\pi_\lambda \circ \gamma_u$ is irreducible, and the subspace annihilated by $\widehat{\mathfrak{g}}_+$ is exactly V_λ. Therefore the representation $\pi_\lambda \circ \gamma_u$ is isomorphic to π_λ. In other words, there is a canonical linear automorphism Γ_u of $\mathcal{H}_\lambda$ such that $\Gamma_u\big((\mathrm{X} \otimes f)v\big) = (\mathrm{X} \otimes f \circ u)\, \Gamma_u(v)$ for $v \in \mathcal{H}_\lambda$, $\mathrm{X} \otimes f \in \widehat{\mathfrak{g}}$ and $\Gamma_u(v) = v$ for $v \in \mathrm{V}_\lambda$.

(1.8) Let $\mathfrak{a}$ be a Lie algebra, V an $\mathfrak{a}$-module. The *space of coinvariants* of V, denoted by $[\mathrm{V}]_{\mathfrak{a}}$, is the largest quotient of V on which $\mathfrak{a}$ acts trivially, that is, the quotient of V by the subspace spanned by the vectors $\mathrm{X}v$ for $\mathrm{X} \in \mathfrak{a}$, $v \in \mathrm{V}$. This is also $\mathrm{V}/\mathrm{U}^+(\mathfrak{a})\mathrm{V}$, where $\mathrm{U}^+(\mathfrak{a})$ is the augmentation ideal of $\mathrm{U}(\mathfrak{a})$.

Let V and W two $\mathfrak{a}$-modules. Using the canonical anti-involution σ of $\mathrm{U}(\mathfrak{a})$ (characterized by $\sigma(\mathrm{X}) = -\mathrm{X}$ for any X in $\mathfrak{a}$) we can consider V as a right $\mathrm{U}(\mathfrak{a})$-module. *Then the space of coinvariants* $[\mathrm{V} \otimes \mathrm{W}]_{\mathfrak{a}}$ *is the tensor product* $\mathrm{V} \otimes_{\mathrm{U}(\mathfrak{a})} \mathrm{W}$: they are both equal to the quotient of $\mathrm{V} \otimes \mathrm{W}$ by the subspace spanned by the elements $\mathrm{X}v \otimes w + v \otimes \mathrm{X}w$ ($\mathrm{X} \in \mathfrak{a}$, $v \in \mathrm{V}$, $w \in \mathrm{W}$).

2. The spaces $\mathrm{V}_{\mathrm{C}}(\vec{p}, \vec{\lambda})$.

(2.1) Let C be a smooth, connected, projective curve over $\mathbf{C}$. For each affine open set $\mathrm{U}\subset\mathrm{X}$, we denote by $\mathcal{O}(\mathrm{U})$ the ring of algebraic functions on U, and by $\mathfrak{g}(\mathrm{U})$ the Lie algebra $\mathfrak{g}\otimes\mathcal{O}(\mathrm{U})$.

We want to associate a vector space to the data of C, a finite subset $\vec{p}=\{p_1,\ldots,p_s\}$ of C, and an element λ_i of P_ℓ attached to each p_i. In order to do this we consider the $\widehat{\mathfrak{g}}$-module $\mathcal{H}_{\vec{\lambda}}:=\mathcal{H}_{\lambda_1}\otimes\ldots\otimes\mathcal{H}_{\lambda_s}$. We choose a local coordinate z_i at each p_i, and denote by f_{p_i} the Laurent series at p_i of an element $f\in\mathcal{O}(\mathrm{C}-\vec{p})$. This defines for each i a ring homomorphism $\mathcal{O}(\mathrm{C}-\vec{p})\longrightarrow\mathbf{C}((z))$, hence a Lie algebra homomorphism $\mathfrak{g}(\mathrm{C}-\vec{p})\longrightarrow\mathfrak{g}\otimes\mathbf{C}((z))$. We define an action of $\mathfrak{g}(\mathrm{C}-\vec{p})$ on $\mathcal{H}_{\vec{\lambda}}$ by the formula

$$(2.2)\qquad (\mathrm{X}\otimes f)\cdot(v_1\otimes\ldots\otimes v_s)=\sum_i v_1\otimes\ldots\otimes(\mathrm{X}\otimes f_{p_i})v_i\otimes\ldots\otimes v_s$$

(that this is indeed a Lie algebra action follows from the residue formula, which gives $\sum_i \operatorname{Res}_{p_i} f_{p_i}dg_{p_i}=0$). Using the notation of (1.8), we put

$$\mathrm{V}_{\mathrm{C}}(\vec{p},\vec{\lambda})=[\mathcal{H}_{\vec{\lambda}}]_{\mathfrak{g}(\mathrm{C}-\vec{p})}\qquad,\qquad \mathrm{V}^\dagger_{\mathrm{C}}(\vec{p},\vec{\lambda})=\operatorname{Hom}_{\mathfrak{g}(\mathrm{C}-\vec{p})}(\mathcal{H}_{\vec{\lambda}},\mathbf{C})\ ,$$

where $\mathbf{C}$ is considered as a trivial $\mathfrak{g}(\mathrm{C}-\vec{p})$-module. Of course $\mathrm{V}^\dagger_{\mathrm{C}}(\vec{p},\vec{\lambda})$ is the dual of $\mathrm{V}_{\mathrm{C}}(\vec{p},\vec{\lambda})$. By (1.7) these spaces do not depend – up to a canonical isomorphism – on the choice of the local coordinates $z_1,\ldots,z_s$. On the other hand it is important to keep in mind that they depend on the Lie algebra $\mathfrak{g}$ and the integer ℓ, though neither of these appear in the notation.

Though this will play no role in the sequel, I would like to mention that these spaces have a natural interpretation in the framework of algebraic geometry. Let me restrict for simplicity to the case $\mathfrak{g}=\mathfrak{sl}_r(\mathbf{C})$. Then the space $\mathrm{V}^\dagger_{\mathrm{C}}(\varnothing)$ is canonically isomorphic to $\mathrm{H}^0(\mathcal{S}\mathrm{U}_{\mathrm{C}}(r),\mathcal{L}^\ell)$, where $\mathcal{S}\mathrm{U}_{\mathrm{C}}(r)$ is the moduli space of semi-stable vector bundles on C with trivial determinant on C and $\mathcal{L}$ the determinant line bundle (see [B-L], and [F] for the case of an arbitrary simple Lie algebra). A similar interpretation for $\mathrm{V}^\dagger_{\mathrm{C}}(\vec{p},\vec{\lambda})$ has been worked out by C. Pauly in terms of moduli spaces of parabolic vector bundles.

Proposition 2.3. *Let $\vec{p}=\{p_1,\ldots,p_s\}$, $\vec{q}=\{q_1,\ldots,q_t\}$ be two finite nonempty subsets of C, without common point; let $\lambda_1,\ldots,\lambda_s,\mu_1,\ldots,\mu_t$ be elements of P_ℓ. We let $\mathfrak{g}(\mathrm{C}-\vec{p})$ act on V_{μ_j} through the evaluation map $\mathrm{X}\otimes f\mapsto f(q_j)\mathrm{X}$. The inclusions $\mathrm{V}_{\mu_j}\hookrightarrow\mathcal{H}_{\mu_j}$ induce an isomorphism*

$$[\mathcal{H}_{\vec{\lambda}}\otimes\mathrm{V}_{\vec{\mu}}]_{\mathfrak{g}(\mathrm{C}-\vec{p})}\xrightarrow{\sim}[\mathcal{H}_{\vec{\lambda}}\otimes\mathcal{H}_{\vec{\mu}}]_{\mathfrak{g}(\mathrm{C}-\vec{p}-\vec{q})}=\mathrm{V}_{\mathrm{C}}(\vec{p}\cup\vec{q},(\vec{\lambda},\vec{\mu}))\ .$$

The case $\vec{q}=\{q\}$, $\mu=0$ gives:

Corollary 2.4. *Let $q\in\mathrm{C}-\vec{p}$. There is a canonical isomorphism*

$$\mathrm{V}_{\mathrm{C}}(\vec{p},\vec{\lambda})\xrightarrow{\sim}\mathrm{V}_{\mathrm{C}}(\vec{p}\cup\{q\},(\vec{\lambda},0))\ .\quad\blacksquare$$

This is the “propagation of vacua”, *cf.* [T-U-Y], Prop. 2.2.3.

Corollary 2.5. *Let* $q \in \mathrm{C} - \vec{p}$ *. There is a canonical isomorphism*

$$\mathrm{V}_{\mathrm{C}}(\vec{p},\vec{\lambda}) \xrightarrow{\sim} [\mathcal{H}_0 \otimes \mathrm{V}_{\vec{\lambda}}]_{\mathfrak{g}(\mathrm{C}-q)} \ .$$

Apply Cor. 2.4, then the proposition inverting the role of $\vec{p}$ and $\vec{q}$. ∎

(2.6) If $\vec{\lambda} = (0,\ldots,0)$, Cor. 2.5 shows that $\mathrm{V}_{\mathrm{C}}(\vec{p},\vec{\lambda})$ is canonically isomorphic to $[\mathcal{H}_0]_{\mathfrak{g}(\mathrm{C}-q)}$, and in particular independent of $\vec{p}$. It follows that the space $[\mathcal{H}_0]_{\mathfrak{g}(\mathrm{C}-q)}$ is independent of q up to a canonical isomorphism; we'll denote it by $\mathrm{V}_{\mathrm{C}}(\varnothing)$. Note that with this convention Cor. 2.4 still holds in the case $\vec{p} = \varnothing$.

I believe that the expression for $\mathrm{V}_{\mathrm{C}}(\vec{p},\vec{\lambda})$ given by Cor. 2.5 is more flexible than the original definition. For instance, we are going to use it below to get a more explicit expression in the case $\mathrm{C} = \mathbf{P}^1$. Also, an easy proof of the "factorization rules" ([T-U-Y], Prop. 2.2.6) can be given in this set-up.

(2.7) Let me finish with an easy result which we will need later on. For each $\lambda \in \mathrm{P}_+$, the dual V_λ^* is a simple $\mathfrak{g}$-module; let us denote by λ^* its highest weight. The map $\lambda \mapsto \lambda^*$ is an involution of P_+ , which is actually the restriction of a $\mathbf{Z}$-linear involution of P (the experts have already recognized the automorphism $-w_0$, where w_0 is the element of longest element of the Weyl group). This involution also induces an involution of the root system which preserves the root system, its basis, and therefore the longest root θ . An important consequence is that P_ℓ *is preserved by the involution* $\lambda \mapsto \lambda^*$.

Proposition 2.8. *Put* $\vec{\lambda}^* = (\lambda_1^*,\ldots,\lambda_s^*)$ *. There is a natural isomorphism*

$$\mathrm{V}_{\mathrm{C}}(\vec{p},\vec{\lambda}) \xrightarrow{\sim} \mathrm{V}_{\mathrm{C}}(\vec{p},\vec{\lambda}^*) \ .$$

(This isomorphism is canonical once certain choices (a "Chevalley basis") have been made for the Lie algebra $\mathfrak{g}$.)

There exists an automorphism σ of $\mathfrak{g}$ such that for each finite-dimensional representation $\rho : \mathfrak{g} \longrightarrow \mathrm{End}(\mathrm{V})$, $\rho \circ \sigma$ is isomorphic to the dual representation ([Bo, ch. VIII, §7, n°6, remarque 1]). The automorphism σ extends to an automorphism $\hat{\sigma}$ of $\widehat{\mathfrak{g}}$, which preserves the decomposition $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_- \oplus \mathfrak{g} \oplus \mathbf{C}c \oplus \widehat{\mathfrak{g}}_+$.

Let $\lambda \in \mathrm{P}_\ell$, and let $\pi_\lambda : \widehat{\mathfrak{g}} \longrightarrow \mathrm{End}(\mathcal{H}_\lambda)$ be the corresponding representation. The representation $\pi_\lambda \circ \hat{\sigma}$ is simple, the subspace of $\mathcal{H}_\lambda$ annihilated by $\widehat{\mathfrak{g}}_+$ is V_λ , on which $\mathfrak{g}$ acts by the representation $\rho_\lambda \circ \sigma$; therefore $\pi_\lambda \circ \hat{\sigma}$ is isomorphic to π_{λ^*} . In other words, there exists for each $\lambda \in \mathrm{P}_\ell$ a $\mathbf{C}$-linear isomorphism $t_\lambda : \mathcal{H}_\lambda \longrightarrow \mathcal{H}_{\lambda^*}$ such that $t_\lambda(\mathrm{X}v) = \hat{\sigma}(\mathrm{X})v$ for $\mathrm{X} \in \widehat{\mathfrak{g}}$, $v \in \mathcal{H}_\lambda$.

Now let $t_{\vec{\lambda}} : \mathcal{H}_{\vec{\lambda}} \longrightarrow \mathcal{H}_{\vec{\lambda}^*}$ be the $\mathbf{C}$-linear isomorphism $t_{\lambda_1} \otimes \ldots \otimes t_{\lambda_s}$. It follows from (2.2) and the above formula that $t_{\vec{\lambda}}\big((\mathrm{X} \otimes f)\, v\big) = (\sigma(\mathrm{X}) \otimes f)\, t_{\vec{\lambda}}(v)$ for $\mathrm{X} \in \mathfrak{g}$, $f \in \mathcal{O}(\mathrm{X} - \vec{p})$, $v \in \mathcal{H}_{\vec{\lambda}}$. Therefore $t_{\vec{\lambda}}$ induces an isomorphism of $\mathrm{V}_{\mathrm{C}}(\vec{p},\vec{\lambda})$ onto $\mathrm{V}_{\mathrm{C}}(\vec{p},\vec{\lambda}^*)$. ∎

3. Proof of Proposition 2.3.

Put $q=q_t$, $\mu=\mu_t$, $\mathrm{U}=\mathrm{C}-\vec{p}$, and $\mathcal{H}=\mathcal{H}_{\vec{\lambda}}\otimes \mathrm{V}_{\mu_1}\otimes\ldots\otimes \mathrm{V}_{\mu_{t-1}}$. Reasoning by induction on t it will be enough to prove that the inclusion $\mathrm{V}_\mu\hookrightarrow\mathcal{H}_\mu$ induces an isomorphism

$$[\mathcal{H}\otimes \mathrm{V}_\mu]_{\mathfrak{g}(\mathrm{U})}\xrightarrow{\ \sim\ }[\mathcal{H}\otimes\mathcal{H}_\mu]_{\mathfrak{g}(\mathrm{U}-q)}\ .$$

(3.1) Let me first explain the action of $\mathfrak{g}(\mathrm{U}-q)$ on $\mathcal{H}\otimes\mathcal{H}_\mu$. We choose a local coordinate z at q . As before, the map $\mathrm{X}\otimes f\mapsto \mathrm{X}\otimes f_q$ defines a Lie algebra homomorphism $\varepsilon:\mathfrak{g}(\mathrm{U}-q)\longrightarrow\mathfrak{g}\otimes\mathbf{C}((z))$. Let us denote by $\widehat{\mathfrak{g}}(\mathrm{U}-q)$ the pull-back by ε of the extension $\widehat{\mathfrak{g}}\longrightarrow\mathfrak{g}\otimes\mathbf{C}((z))$; in other words, $\widehat{\mathfrak{g}}(\mathrm{U}-q)$ is the space $\mathfrak{g}(\mathrm{U}-q)\oplus\mathbf{C}c$, the bracket of two elements $\mathrm{X}\otimes f$, $\mathrm{Y}\otimes g$ being given by

$$[\mathrm{X}\otimes f,\mathrm{Y}\otimes g]=[\mathrm{X},\mathrm{Y}]\otimes fg\ +\ c\cdot(\mathrm{X}\,|\,\mathrm{Y})\,\mathrm{Res}_q(g\,df)\ .$$

Applying again the Residue formula we see that the action of $\mathfrak{g}(\mathrm{U}-q)$ on $\mathcal{H}_{\vec{\lambda}}$ given by formula (2.2) extends to an action of $\widehat{\mathfrak{g}}(\mathrm{U}-q)$, which is *of level* $-\ell$ in the sense that the central element c acts as multiplication by $-\ell$. On the other hand ε extends by construction to a homomorphism $\widehat{\mathfrak{g}}(\mathrm{U}-q)\longrightarrow\widehat{\mathfrak{g}}$ through which $\widehat{\mathfrak{g}}(\mathrm{U}-q)$ acts on $\mathcal{H}_\mu$ with level ℓ , hence the action on $\mathcal{H}\otimes\mathcal{H}_\mu$ is of level 0 and therefore factors through $\mathfrak{g}(\mathrm{U}-q)$.

Besides the fact that it is of level $-\ell$, the only property we will use of the action of $\widehat{\mathfrak{g}}(\mathrm{U}-q)$ on $\mathcal{H}$ is the following:

(*) *The endomorphism* $\mathrm{X}_{-\theta}\otimes f$ *of* $\mathcal{H}$ *is locally nilpotent for all* $f\in\mathcal{O}(\mathrm{U})$.

(This is because $\mathrm{X}_{-\theta}$ is a nilpotent element of $\mathfrak{g}$, while every element of $\widehat{\mathfrak{g}}_+$ is locally nilpotent in the integrable modules $\mathcal{H}_{\lambda_i}$.)

(3.2) We first check that the map $\mathcal{H}\otimes\mathrm{V}_\mu\hookrightarrow\mathcal{H}\otimes\mathcal{H}_\mu$ is equivariant with respect to $\mathfrak{g}(\mathrm{U})$. This amounts to proving that the inclusion $\mathrm{V}_\mu\hookrightarrow\mathcal{H}_\mu$ is equivariant. But V_μ is the subspace of $\mathcal{H}_\mu$ annihilated by $\widehat{\mathfrak{g}}_+$ (1.5), so an element $\mathrm{X}\otimes f$ of $\mathfrak{g}(\mathrm{U})$ acts on $\mathcal{H}_\mu$ as the element $f(q)\mathrm{X}$ of $\mathfrak{g}$, hence our assertion. Therefore the inclusion induces a linear map

$$i:[\mathcal{H}\otimes\mathrm{V}_\mu]_{\mathfrak{g}(\mathrm{U})}\longrightarrow[\mathcal{H}\otimes\mathcal{H}_\mu]_{\mathfrak{g}(\mathrm{U}-q)}\ .$$

(3.3) We prove that the statement is true when we replace the simple module $\mathcal{H}_\mu$ by the module $\mathcal{V}_\mu$ (1.6). Let us observe first that by (1.7), the statement is independent of the choice of the local coordinate z at q . We choose z so that $z^{-1}\in\mathcal{O}(\mathrm{U}-q)$ (this is possible as soon as $\vec{p}\neq\varnothing$). From the decomposition $\mathcal{O}(\mathrm{U}-q)=\mathcal{O}(\mathrm{U})\oplus\sum\limits_{n\geq1}\mathbf{C}z^{-n}$ we get

$$\mathfrak{g}(\mathrm{U}-q)=\mathfrak{g}(\mathrm{U})\oplus\widehat{\mathfrak{g}}_-$$

where we have identified the Lie algebra $\sum_{n\geq 1}\mathfrak{g}\, z^{-n}$ with its image $\widehat{\mathfrak{g}}_-$ in $\widehat{\mathfrak{g}}$. Note that both summands can be viewed as Lie subalgebras of $\widehat{\mathfrak{g}}(\mathrm{U}-q)$.

Let us consider first the coinvariants under $\widehat{\mathfrak{g}}_-$. By (1.8) $[\mathcal{H}\otimes\mathcal{V}_\mu]_{\widehat{\mathfrak{g}}_-}$ can be identified with $\mathcal{H}\otimes_{\mathrm{U}(\widehat{\mathfrak{g}}_-)}\mathcal{V}_\mu$. Since the natural map $\widehat{\mathfrak{g}}_-\otimes_{\mathbf{C}}\mathrm{V}_\mu\longrightarrow\mathcal{V}_\mu$ is an isomorphism of $\widehat{\mathfrak{g}}_-$-modules (1.6), we conclude that the inclusion $\mathrm{V}_\mu\hookrightarrow\mathcal{V}_\mu$ induces an isomorphism $\mathcal{H}\otimes\mathrm{V}_\mu\xrightarrow{\sim}[\mathcal{H}\otimes\mathcal{V}_\mu]_{\widehat{\mathfrak{g}}_-}$. Taking coinvariants under $\mathfrak{g}(\mathrm{U})$ gives the required isomorphism.

(3.4) Let $\mathcal{Z}_\mu$ be the kernel of the canonical surjection $\mathcal{V}_\mu\to\mathcal{H}_\mu$; we have an exact sequence

$$\mathcal{H}\otimes\mathcal{Z}_\mu\longrightarrow[\mathcal{H}\otimes\mathcal{V}_\mu]_{\mathfrak{g}(\mathrm{U}-q)}\longrightarrow[\mathcal{H}\otimes\mathcal{H}_\mu]_{\mathfrak{g}(\mathrm{U}-q)}\to 0\ ,$$

so we want to prove that the image of $\mathcal{H}\otimes\mathcal{Z}_\mu$ in $\mathcal{H}\otimes_{\mathrm{U}(\widehat{\mathfrak{g}}(\mathrm{U}-q))}\mathcal{V}_\mu$ is zero. As a $\mathrm{U}(\widehat{\mathfrak{g}})$-module $\mathcal{Z}_\mu$ is generated by the vector $(\mathrm{X}_\theta\otimes z^{-1})^k\,v_\mu$, where v_μ is a highest weight vector and $k=\ell-\mu(\mathrm{H}_\theta)+1$ (1.6); moreover this vector is annihilated by $\widehat{\mathfrak{g}}_+$, so it generates $\mathcal{Z}_\mu$ as a $\mathrm{U}(\widehat{\mathfrak{g}}_-\oplus\mathfrak{g})$-module. Since $\widehat{\mathfrak{g}}_-\oplus\mathfrak{g}\subset\mathfrak{g}(\mathrm{U}-q)$, it is enough to prove that $h\otimes(\mathrm{X}_\theta\otimes z^{-1})^k\,v_\mu=0$ in $\mathcal{H}\otimes_{\mathrm{U}(\mathfrak{g}(\mathrm{U}-q))}\mathcal{V}_\mu$ for each vector $h\in\mathcal{H}$. Let f be an element of $\mathcal{O}(\mathrm{U})$ such that $f_q\equiv z$ (mod. z^2); put $\mathrm{Y}=\mathrm{X}_{-\theta}\otimes f$. By property $(*)$ in (3.1) there exists an integer N such that $\mathrm{Y}^{\mathrm{N}}h=0$. By lemma (3.5) below $(\mathrm{X}_\theta\otimes z^{-1})^k\,v_\mu$ can be written as $\mathrm{Y}^{\mathrm{N}}w$ for some $w\in\mathcal{V}_\mu$, so $h\otimes(\mathrm{X}_\theta\otimes z^{-1})^k\,v_\mu$ is zero in $\mathcal{H}\otimes_{\mathrm{U}(\widehat{\mathfrak{g}}(\mathrm{U}-q))}\mathcal{V}_\mu$, which finishes the proof.

Lemma 3.5. *Let $f(z)\in\mathbf{C}[[z]]$ be such that $f(0)=0$, $f'(0)=1$. Put $\mathrm{X}=\mathrm{X}_\theta\otimes z^{-1}$, $\mathrm{Y}=\mathrm{X}_{-\theta}\otimes f(z)$ in $\widehat{\mathfrak{g}}$. Let $\mu\in\mathrm{P}_\ell$, and $p,q\in\mathbf{N}$ with $p\geq\ell+1-\mu(\mathrm{H}_\theta)$. There exists a nonzero rational number $\alpha_{p,q}$ such that $\mathrm{X}^pv_\mu=\alpha_{p,q}\mathrm{Y}^q\mathrm{X}^{p+q}v_\mu$ in $\mathcal{V}_\mu$.*

Let $\mathrm{H}:=[\mathrm{Y},\mathrm{X}]=\big(\mathrm{H}_\theta\otimes z^{-1}f(z)\big)-c$; then $[\mathrm{H},\mathrm{X}]=2\mathrm{X}_\theta\otimes z^{-2}f(z)$ commutes with X, so one has in $\mathrm{U}(\widehat{\mathfrak{g}})$

$$\mathrm{H}\mathrm{X}^m=\mathrm{X}^m\mathrm{H}+\sum_{a+b=m-1}\mathrm{X}^a[\mathrm{H},\mathrm{X}]\mathrm{X}^b=\mathrm{X}^m\mathrm{H}+m\mathrm{X}^{m-1}[\mathrm{H},\mathrm{X}]$$

Since v_μ is annihilated by $\widehat{\mathfrak{g}}_+$ and by X_θ, one has $[\mathrm{H},\mathrm{X}]v_\mu=2\mathrm{X}v_\mu$ and $\mathrm{H}v_\mu=-kv_\mu$ with $k=\ell-\mu(\mathrm{H}_\theta)$, hence $\mathrm{H}\mathrm{X}^mv_\mu=(2m-k)\mathrm{X}^mv_\mu$. Then

$$\mathrm{Y}\mathrm{X}^{p+1}v_\mu=\sum_{n+m=p}\mathrm{X}^n\mathrm{H}\mathrm{X}^mv_\mu=(p+1)(p-k)\mathrm{X}^pv_\mu\ .$$

This proves the lemma in the case $q=1$; the general case follows at once by induction on q. ∎

Remark 3.6 .– The same method gives the vanishing of $\mathrm{Y}\mathrm{X}^{\ell-\mu(\mathrm{H}_\theta)+1}\,v_\mu$ for any $\mathrm{Y}\in\widehat{\mathfrak{g}}_+$.

4. The case $\mathrm{C} = \mathbf{P}^1$.

We fix a coordinate t on $\mathbf{P}^1$.

Proposition 4.1. *Let $p_1, \ldots, p_s$ be distinct points of $\mathbf{P}^1$, with coordinates $t_1, \ldots,$ t_s, and let $\lambda_1, \ldots, \lambda_s$ be elements of P_ℓ. Let T be the endomorphism of $\mathrm{V}_{\vec{\lambda}}$ defined by*

$$\mathrm{T}(v_1 \otimes \ldots \otimes v_s) = \sum_{i=1}^{s} t_i\, v_1 \otimes \ldots \otimes \mathrm{X}_\theta\, v_i \otimes \ldots \otimes v_s \ .$$

The space $\mathrm{V}_{\mathbf{P}^1}(\vec{p}, \vec{\lambda})$ is canonically isomorphic to the largest quotient of $\mathrm{V}_{\vec{\lambda}}$ on which $\mathfrak{g}$ and $\mathrm{T}^{\ell+1}$ act trivially. The space $\mathrm{V}^\dagger_{\mathbf{P}^1}(\vec{p}, \vec{\lambda})$ is isomorphic to the space of $\mathfrak{g}$-invariant p-linear forms $\varphi : \mathrm{V}_{\lambda_1} \times \ldots \times \mathrm{V}_{\lambda_s} \longrightarrow \mathbf{C}$ such that $\varphi \circ \mathrm{T}^{\ell+1} = 0$.

We apply Cor. 2.5 with $q = \infty$ (so that the local coordinate z at q is t^{-1}). This gives an isomorphism of $\mathrm{V}_{\mathbf{P}^1}(\vec{p}, \vec{\lambda})$ onto $[\mathcal{H}_0 \otimes \mathrm{V}_{\vec{\lambda}}]_{\mathfrak{g}(\mathbf{A}^1)}$. Now $\mathfrak{g}(\mathbf{A}^1)$ is the sum of $\mathfrak{g}$ and $\widehat{\mathfrak{g}}_-$; it follows from (1.6) that the $\mathrm{U}(\mathfrak{g}(\mathbf{A}^1))$-module $\mathcal{H}_0$ is generated by the highest weight vector v_0 , with the relations $\mathfrak{g}\, v_0 = 0$ and $(\mathrm{X}_\theta \otimes z^{-1})^{\ell+1}\, v_0 = 0$. Therefore the space $[\mathcal{H}_0 \otimes \mathrm{V}_{\vec{\lambda}}]_{\mathfrak{g}(\mathbf{A}^1)} \cong \mathcal{H}_0 \otimes_{\mathrm{U}(\mathfrak{g}(\mathbf{A}^1))} \mathrm{V}_{\vec{\lambda}}$ is canonically isomorphic to $\mathrm{V}_{\vec{\lambda}}/(\mathfrak{g}\mathrm{V}_{\vec{\lambda}} + \mathrm{Im}\,\mathrm{T}^{\ell+1})$, where T $(= \mathrm{X}_\theta \otimes t)$ is the endomorphism of $\mathrm{V}_{\vec{\lambda}}$ given by the above formula. The description of $\mathrm{V}^\dagger_{\mathbf{P}^1}(\vec{p}, \vec{\lambda})$ follows by duality. ■

When $p = 3$, one can describe the space $\mathrm{V}_{\mathbf{P}^1}(a, b, c; \lambda, \mu, \nu)$ (or its dual) in a more concrete way. Let us first consider the case when $\mathfrak{g} = \mathfrak{sl}_2$. We denote by E the standard 2-dimensional representation of $\mathfrak{g}$. We will identify P_ℓ with the set of integers p with $0 \le p \le \ell$ (by associating to such an integer the representation $\mathrm{S}^p\mathrm{E}$). By Prop. 4.1, $\mathrm{V}^\dagger_{\mathbf{P}^1}(a, b, c; p, q, r)$ is the space of linear forms $\mathrm{F} \in \mathrm{Hom}_{\mathfrak{g}}(\mathrm{S}^p\mathrm{E} \otimes \mathrm{S}^q\mathrm{E} \otimes \mathrm{S}^r\mathrm{E}, \mathbf{C})$ such that $\mathrm{F} \circ \mathrm{T}^{\ell+1} = 0$.

Lemma 4.2. a) *The space* $\mathrm{Hom}_{\mathfrak{g}}(\mathrm{S}^p\mathrm{E} \otimes \mathrm{S}^q\mathrm{E} \otimes \mathrm{S}^r\mathrm{E}, \mathbf{C})$ *is either* 0*- or* 1*-dimensional. It is nonzero if and only if $p + q + r$ is even, say $= 2m$, and p, q, r are $\le m$.*

b) *The subspace $\mathrm{V}^\dagger_{\mathbf{P}^1}(a, b, c; p, q, r)$ is nonzero if and only if $p + q + r$ is even and $\le 2\ell$.*

The first assertion is an immediate consequence of the Clebsch-Gordan formula. When the space $\mathrm{Hom}_{\mathfrak{g}}(\mathrm{S}^p\mathrm{E} \otimes \mathrm{S}^q\mathrm{E} \otimes \mathrm{S}^r\mathrm{E}, \mathbf{C})$ is nonzero, a generator G is obtained as follows: the dual of $\mathrm{S}^p\mathrm{E} \otimes \mathrm{S}^q\mathrm{E} \otimes \mathrm{S}^r\mathrm{E}$ is the space of polynomial maps $\mathrm{F}(u, v, w)$ $(u, v, w \in \mathrm{E})$ which are homogeneous of degree p in u , q in v and r in w ; then the polynomial $\mathrm{G}(u, v, w) = \varphi(v, w)^{n-p} \varphi(w, u)^{n-q} \varphi(u, v)^{n-r}$, where φ is any non-zero alternating form on E , is clearly $\mathfrak{g}$-invariant. It remains to make explicit the action of T on the dual of $\mathrm{S}^p\mathrm{E} \otimes \mathrm{S}^q\mathrm{E} \otimes \mathrm{S}^r\mathrm{E}$.

This dual can also be seen as the space $P_{p,q,r}$ of (non-homogeneous) polynomials $P(x,y,z)$ of degree $\leq p$ in x, $\leq q$ in y and $\leq r$ in t: the correspondence is obtained by choosing a basis (e_0, e_1) of E and putting $P(x,y,z) = F(e_0 + xe_1, e_0 + ye_1, e_0 + ze_1)$. In particular, the polynomial corresponding to G is (up to a constant) $Q(x,y,z) = (y-z)^{n-p}(z-x)^{n-q}(x-y)^{n-r}$.

Choose the basis so that $X_\theta\, e_0 = e_1$; then the action of $X_\theta \otimes 1 \otimes 1$ (resp. $1 \otimes X_\theta \otimes 1$, resp. $1 \otimes 1 \otimes X_\theta$) on $P_{p,q,r}$ is the derivation with respect to x (resp. y, resp. z). Therefore T acts as the operator $a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$; in other words, $T^m \cdot P$ is the coefficient of $\frac{h^m}{m!}$ in the expansion of $P(x+ah, y+bh, z+ch)$. Since $Q(x+ah, y+bh, z+ch)$ is a polynomial of degree n in h (because $a \neq b \neq c$), we obtain b). ∎

(4.3) In the general case, we consider the Lie subalgebra $\mathfrak{s} \cong \mathfrak{sl}_2$ of $\mathfrak{g}$ with basis $(X_\theta, X_{-\theta}, H_\theta)$ (1.3). Following the (unpleasant) practice of the physicists, we'll say that an irreducible representation of $\mathfrak{sl}_2$ *has spin* i if it is isomorphic to $S^{2i}E$; so the spin is a half-integer. Let $\lambda \in P_\ell$; as a $\mathfrak{s}$-module, V_λ breaks up as a direct sum of isotypic components $V_\lambda^{(i)}$ of spin i, with $0 \leq i \leq \ell/2$.

Proposition 4.3. a) *The space* $V_{\mathbf{P}^1}(a,b,c;\lambda,\mu,\nu)$ *is canonically isomorphic to the quotient of* $[V_\lambda \otimes V_\mu \otimes V_\nu]_{\mathfrak{g}}$ *by the image of the subspaces* $V_\lambda^{(p)} \otimes V_\mu^{(q)} \otimes V_\nu^{(r)}$ *for* $p+q+r > \ell$.

b) *The space* $V^\dagger_{\mathbf{P}^1}(a,b,c;\lambda,\mu,\nu)$ *is canonically isomorphic to the space of* $\mathfrak{g}$*-invariant linear forms* $\varphi : V_\lambda \otimes V_\mu \otimes V_\nu \longrightarrow \mathbf{C}$ *which vanish on the subspaces* $V_\lambda^{(p)} \otimes V_\mu^{(q)} \otimes V_\nu^{(r)}$ *whenever* $p+q+r > \ell$.

The two assertions are of course equivalent; let us prove b). By Prop. 4.1, all we have to do is to express the condition $\varphi \circ T^{\ell+1} = 0$ for a $\mathfrak{g}$-invariant linear form $\varphi : V_\lambda \otimes V_\mu \otimes V_\nu \longrightarrow \mathbf{C}$. Write $V_\lambda = \bigoplus_{p=0}^{\ell/2} V_\lambda^{(p)}$, and similarly for V_μ and V_ν. The subspaces $V_\lambda^{(p)} \otimes V_\mu^{(q)} \otimes V_\nu^{(r)}$ are stable under $\mathfrak{s}$ and T, so we have to express that the restriction φ_{pqr} of φ to any of these subspaces vanishes on $\operatorname{Im} T^{\ell+1}$. By the above lemma this is automatically satisfied if $p+q+r \leq \ell$, while it imposes $\varphi_{pqr} = 0$ when $p+q+r > \ell$, hence the proposition. ∎

Let me mention an easy consequence (which of course can also be proved directly):

Corollary 4.4. *One has*

$$V_{\mathbf{P}^1}(p,\lambda) = 0 \quad \text{for} \quad \lambda \neq 0 \quad , \quad V_{\mathbf{P}^1}(p,0) \cong V_{\mathbf{P}^1}(\varnothing) \cong \mathbf{C}$$

$$V_{\mathbf{P}^1}(p,q,\lambda,\mu) = 0 \quad \text{for} \quad \mu \neq \lambda^* \quad , \quad V_{\mathbf{P}^1}(p,q,\lambda,\lambda^*) \cong \mathbf{C}\,. \quad ∎$$

(4.5) Let me now recall one of the essential results of [T-U-Y], the *factorization rules* for the spaces $V_C(\vec{p},\vec{\lambda})$. In this paper we will only be interested in the dimension of these spaces, so I will formulate the factorization rules in these terms. According to [T-U-Y] the dimension of $V_C(\vec{p},\vec{\lambda})$ depends only on the genus g of C and of the set of weights $\vec{\lambda}=(\lambda_1,\ldots,\lambda_s)$; let us denote it by $N_g(\vec{\lambda})$. One has

$$N_g(\vec{\lambda})=\sum_{\nu\in P_\ell} N_{g-1}(\vec{\lambda},\nu,\nu^*)\ ,$$

and, if $\vec{\mu}=(\mu_1,\ldots,\mu_t)$ is another set of weights and h, k non-negative integers such that $g=h+k$,

$$N_g(\vec{\lambda},\vec{\mu})=\sum_{\nu\in P_\ell} N_h(\vec{\lambda},\nu)\,N_k(\vec{\mu},\nu^*)\ .$$

Part II: Fusion rings

5. Fusion rules and fusion rings.

(5.1) Let I be a finite set, with an involution $\lambda\mapsto\lambda^*$. We'll denote by $\mathbf{N}^{(I)}$ the free commutative monoid generated by I, that is, the set of sums $\sum_{\alpha\in I} n_\alpha\alpha$ with $n_\alpha\in\mathbf{N}$; we shall always identify I with a subset of $\mathbf{N}^{(I)}$. The involution of I extends by linearity to an involution $x\mapsto x^*$ of $\mathbf{N}^{(I)}$.

Definition. A fusion rule on I is a map $N:\mathbf{N}^{(I)}\to\mathbf{Z}$ satisfying the following three conditions:

(F 0) One has $N(0)=1$, and $N(\alpha)>0$ for some $\alpha\in I$;
(F 1) $N(x^*)=N(x)$ for every $x\in\mathbf{N}^{(I)}$;
(F 2) For x, y in $\mathbf{N}^{(I)}$, one has $N(x+y)=\sum_{\lambda\in I} N(x+\lambda)\,N(y+\lambda^*)$.

Let us call *kernel* of a fusion rule N the set of elements α in I such that $N(\alpha+x)=0$ for all $x\in\mathbf{N}^{(I)}$; one says that N is *non-degenerate* if its kernel is empty. Let N be a fusion rule on I with kernel $K\subset I$; then K is stable under *, the restriction N_0 of N to $\mathbf{N}^{(I-K)}$ is a fusion rule on $I-K$, and N is simply the extension by 0 of N_0 to $\mathbf{N}^{(I)}$. Therefore we can restrict ourselves without loss of generality to the non-degenerate fusion rules.

Examples 5.2. *a*) Fix a simple complex Lie algebra and a level ℓ. For $\lambda_1,\ldots,\lambda_s$ in P_ℓ, put using the notation of Part I

$$N(\sum\lambda_i)=\dim V_{\mathbf{P}^1}(\vec{p},\vec{\lambda})\ ,$$

where $\vec{p}$ is an arbitrary subset of $\mathbf{P}^1$ with s elements. Then N *is a (non-degenerate) fusion rule on* P_ℓ *:* the condition (F 0) and the non-degeneracy condition follows from Cor. 4.4, (F 1) from Prop. 2.8, and (F 2) is a particular case $(h=k=0)$ of the factorization rules (4.5).

b) Let R be a commutative ring, endowed with an involutive ring homomorphism $x\mapsto x^*$ and a $\mathbf{Z}$-linear form $t:\mathrm{R}\rightarrow\mathbf{Z}$; suppose that the bilinear form $(x,y)\mapsto t(xy^*)$ is symmetric and admits an orthonormal basis I (over $\mathbf{Z}$) containing 1. Define a map $\mathrm{N}:\mathbf{N}^{(\mathrm{I})}\longrightarrow\mathbf{Z}$ by the formula $\mathrm{N}(\sum n_\alpha\alpha)=t(\prod\alpha^{n_\alpha})$. Then N *is a (non-degenerate) fusion rule on* I. For the condition on t implies in particular $t(x^*)=t(x)$ and $t(1)=1$, hence (F 0) and (F 1). Since I is an orthonormal basis, one has, for x, y in R,

$$t(xy)=\sum_{\lambda\in\mathrm{I}}t(x\lambda)\,t(y\lambda^*)\ , \tag{5.2}$$

which implies (F 2).

Conversely:

Proposition 5.3. *Let* $\mathrm{N}:\mathbf{N}^{(\mathrm{I})}\rightarrow\mathbf{Z}$ *be a (non-degenerate) fusion rule on* I. *There exists a* $\mathbf{Z}$*-bilinear map* $\mathbf{Z}^{(\mathrm{I})}\times\mathbf{Z}^{(\mathrm{I})}\longrightarrow\mathbf{Z}^{(\mathrm{I})}$*, which turns* $\mathbf{Z}^{(\mathrm{I})}$ *into a commutative ring, and a linear form* t*, uniquely determined, such that*

$$\mathrm{N}(\sum n_\alpha\alpha)=t(\prod\alpha^{n_\alpha})$$

for all elements $\sum n_\alpha\alpha$ *of* $\mathbf{N}^{(\mathrm{I})}$. *One has* $t(\alpha\beta^*)=\delta_{\alpha\beta}$ *for* α, β *in* I.

Let us apply (F 2) with $x=y=0$. Using (F 0) and (F 1) we get $\sum_{\lambda\in\mathrm{I}}\mathrm{N}(\lambda)^2=1$. This means that there exists an element ε of I such that

$$\varepsilon=\varepsilon^*\quad,\quad\mathrm{N}(\varepsilon)=1\quad,\quad\mathrm{N}(\lambda)=0\quad\text{for}\quad\lambda\neq\varepsilon\ .$$

Then (F 2) (with $y=0$) implies $\mathrm{N}(x+\varepsilon)=\mathrm{N}(x)$ for all $x\in\mathbf{N}^{(\mathrm{I})}$.

Now let us apply (F 2) with $x=\alpha$, $y=\alpha^*$; we obtain (using (F 1))

$$\mathrm{N}(\alpha+\alpha^*)=\sum_{\lambda\in\mathrm{I}}\mathrm{N}(\alpha+\lambda)^2\ \geq\ \mathrm{N}(\alpha+\alpha^*)^2\ .$$

If $\mathrm{N}(\alpha+\lambda)=0$ for all $\lambda\in\mathrm{I}$, one deduces from (F 2) $\mathrm{N}(\alpha+x)=0$ for all $x\in\mathbf{N}^{(\mathrm{I})}$, which contradicts the non-degeneracy hypothesis. Therefore the above inequality implies

$$\mathrm{N}(\alpha+\lambda)=0\quad\text{for}\quad\lambda\neq\alpha^*\quad,\quad\mathrm{N}(\alpha+\alpha^*)=1\ . \tag{5.4}$$

Let us define a multiplication law on $\mathbf{Z}^{(\mathrm{I})}$ by putting

$$\alpha\cdot\beta=\sum_{\lambda\in \mathrm{I}}\mathrm{N}(\alpha+\beta+\lambda^*)\,\lambda\ , \tag{5.5}$$

and extending by bilinearity. This law is commutative; for α, β, γ in I, one has acording to (F 2)

$$(\alpha\cdot\beta)\cdot\gamma=\sum_{\lambda,\mu\in \mathrm{I}}\mathrm{N}(\alpha+\beta+\lambda^*)\,\mathrm{N}(\lambda+\gamma+\mu^*)\mu=\sum_{\mu\in \mathrm{I}}\mathrm{N}(\alpha+\beta+\gamma+\mu^*)\,\mu=\alpha\cdot(\beta\cdot\gamma)\ ,$$

so that the multiplication is associative. One gets similarly, by induction on s,

$$\alpha_1\cdots\alpha_s=\sum_{\lambda\in \mathrm{I}}\mathrm{N}(\alpha_1+\ldots+\alpha_s+\lambda^*)\ \lambda \tag{5.6}$$

for $\alpha_1,\ldots,\alpha_s$ in I. Moreover one deduces from (5.4)

$$\varepsilon\cdot\alpha=\sum_{\lambda\in \mathrm{I}}\mathrm{N}(\varepsilon+\alpha+\lambda^*)\,\lambda=\alpha\ .$$

Condition (F 1) implies that the involution $x\mapsto x^*$ is a ring homomorphism.

Let $t:\mathbf{Z}^{(\mathrm{I})}\longrightarrow\mathbf{Z}$ be the linear form $\sum n_\alpha\alpha\mapsto n_\varepsilon$. One gets from (5.6)

$$t(\alpha_1\cdots\alpha_s)=\mathrm{N}(\alpha_1+\ldots+\alpha_s)$$

which is the required formula for N. Then (5.4) translates as $t(\alpha\beta^*)=\delta_{\alpha\beta}$. ■

Definition. The ring $\mathbf{Z}^{(\mathrm{I})}$ with the multiplication given by (5.5) is called the fusion ring associated to N. We will denote it by $\mathcal{F}_{\mathrm{N}}$ or simply $\mathcal{F}$.

Remark 5.7 .– Most of the above still holds when one replaces (F 1) by the weaker condition
(F 1′) : *One has* $\mathrm{N}(\alpha^*)=\mathrm{N}(\alpha)$ *and* $\mathrm{N}(\alpha^*+\beta^*)=\mathrm{N}(\alpha+\beta)$ *for* $\alpha,\beta\in\mathrm{I}$,
the only difference being that the involution is not necessarily a ring homomorphism – in fact this property is equivalent to (F 1).

(5.8) A consequence of Prop. 5.3 is that the bilinear form $(x,y)\mapsto t(xy)$ defines an isomorphism of $\mathcal{F}$ onto the $\mathcal{F}$-module $\mathrm{Hom}_{\mathbf{Z}}(\mathcal{F},\mathbf{Z})$ (this implies that $\mathcal{F}$ is a *Gorenstein* $\mathbf{Z}$-algebra). There is a canonical element Tr in $\mathrm{Hom}_{\mathbf{Z}}(\mathcal{F},\mathbf{Z})$, the trace form: for each $x\in\mathcal{F}$ the multiplication by x is an endomorphism m_x of $\mathcal{F}$, and we put $\mathrm{Tr}(x):=\mathrm{Tr}(m_x)$. An easy (and standard) computation shows that the element of $\mathcal{F}$ which corresponds to Tr under the above isomorphism is the *Casimir element* $\omega:=\sum_{\lambda\in\mathrm{I}}\lambda\lambda^*$; in other words, one has

$$\mathrm{Tr}(x)=t(\omega x)\quad\text{for every}\quad x\in\mathrm{I}\ . \tag{5.8}$$

So far we have considered only fusion rules in genus 0 ; this is no restriction because the general case reduces easily to the genus 0 case. To be precise:

Proposition 5.9. *Suppose given a sequence of maps* $\mathrm{N}_g : \mathbf{N}^{(\mathrm{I})} \longrightarrow \mathbf{Z}$ *such that* $\mathrm{N}_0 = \mathrm{N}$ *and*

$$\mathrm{N}_g(x) = \sum_{\lambda \in \mathrm{I}} \mathrm{N}_{g-1}(x + \lambda + \lambda^*)$$

for x *in* $\mathbf{N}^{(\mathrm{I})}$ *and* $g \geq 1$ *. Then one has, for* $g \geq 1$ *and* $\alpha_1, \ldots, \alpha_s \in \mathrm{I}$

$$\mathrm{N}_g(\alpha_1 + \ldots + \alpha_s) = t(\alpha_1 \cdots \alpha_s\, \omega^g) = \mathrm{Tr}(\alpha_1 \cdots \alpha_s\, \omega^{g-1}) \ .$$

By induction on g one gets

$$\begin{aligned}\mathrm{N}_g(\alpha_1 + \ldots + \alpha_s) &= \sum_{\lambda_1, \ldots, \lambda_g \in \mathrm{I}} \mathrm{N}_0(\alpha_1 + \ldots + \alpha_s + \lambda_1 + \lambda_1^* + \ldots + \lambda_g + \lambda_g^*) \\ &= \sum_{\lambda_1, \ldots, \lambda_g \in \mathrm{I}} t(\alpha_1 \cdots \alpha_s\, \lambda_1 \lambda_1^* \cdots \lambda_g \lambda_g^*) \\ &= t(\alpha_1 \cdots \alpha_s\, \omega^g) \ ;\end{aligned}$$

the last equality follows from (5.8).

Remark 5.10 .– Using formula (5.2), it follows that the sequence (N_g) satisfies the following rule, which generalizes (F 2):

$$\mathrm{N}_{p+q}(x + y) = \sum_{\lambda \in \mathrm{I}} \mathrm{N}_p(x + \lambda)\, \mathrm{N}_q(y + \lambda^*)$$

for x , y in $\mathbf{N}^{(\mathrm{I})}$, p, q in $\mathbf{N}$ (compare with (4.5)).

6. Diagonalization of the fusion rules.

To go further we need some information on the structure of the ring $\mathcal{F}$. We have already observed that $\mathcal{F}$ carries a symmetric, positive definite bilinear form $< \ | \ >$ defined by $<x \,|\, y>= t(xy^*)$, for which I is an orthonormal basis. The fact that * is a ring homomorphism implies $<xy \,|\, z>=<x \,|\, y^* z>$ for all x, y, z in $\mathcal{F}$. The existence of this form imposes strong restrictions on the ring $\mathcal{F}$.

Proposition 6.1. *The* $\mathbf{Q}$*-algebra* $\mathcal{F}_{\mathbf{Q}} := \mathcal{F} \otimes \mathbf{Q}$ *is isomorphic to a product* $\prod \mathrm{K}_i$ *of finite extensions of* $\mathbf{Q}$ *, preserved by the involution, which are of the following two types:*

a) *a totally real extension of* $\mathbf{Q}$ *with the trivial involution;*
b) *a totally imaginary extension of* $\mathbf{Q}$ *which is a quadratic extension of a totally real extension of* $\mathbf{Q}$ *, the involution being the nontrivial automorphism of that quadratic extension.*

(Recall that an extension K of $\mathbf{Q}$ is called totally real (resp. totally imaginary) if $\mathrm{K} \otimes_{\mathbf{Q}} \mathbf{R}$ is isomorphic to $\mathbf{R}^{r_1}$ (resp. $\mathbf{C}^{r_2}$).)

We first observe that the ring $\mathcal{F}$ is *reduced*: for $x\in\mathcal{F}$, the relation $x^2=0$ implies $<xx^*\,|\,xx^*>=0$, hence $xx^*=0$, which in turn implies $<x\,|\,x>=0$ and finally $x=0$. Since a reduced finite-dimensional $\mathbf{Q}$-algebra is a product of fields, we get the decomposition $\mathcal{F}_{\mathbf{Q}}=\prod \mathrm{K}_i$. This decomposition is canonical (each factor corresponds to an indecomposable idempotent of $\mathcal{F}_{\mathbf{Q}}$), so it is preserved by the involution $x\mapsto x^*$: each factor K_i is either preserved by the involution, or mapped isomorphically onto another factor K_j. In the second case $\mathcal{F}_{\mathbf{Q}}$ contains a product of fields $\mathrm{K}\times\mathrm{K}$, with the involution interchanging the two factors. Then the set of elements xx^* for $x\in\mathrm{K}\times\mathrm{K}$ is the diagonal $\mathrm{K}\subset\mathrm{K}\times\mathrm{K}$, a $\mathbf{Q}$-vector space on which t can take arbitrary values, contradicting the positivity assumption.

Now let K be one of the K_i's, and let σ denote the induced involution. Applying the same argument to the $\mathbf{R}$-algebra $\mathrm{K}\otimes_{\mathbf{Q}}\mathbf{R}$ we find that it is of the form $\mathbf{R}^{r_1}\times\mathbf{C}^{r_2}$, with σ preserving each factor; since the induced involution on each factor is $\mathbf{R}$-linear, there is no choice but the identity on the real factors and the complex conjugation on the complex ones. In particular, the fixed subfield K^σ of σ is totally real and $\mathrm{K}^\sigma\otimes_{\mathbf{Q}}\mathbf{R}$ is isomorphic to $\mathbf{R}^{r_1+r_2}$. If we are not in case a), K^σ is strictly smaller than K; counting degrees we get

$$r_1+2r_2=[\mathrm{K}:\mathbf{Q}]=2\,[\mathrm{K}^\sigma:\mathbf{Q}]=2(r_1+r_2)\ ,$$

hence $r_1=0$, and we are in case b). ∎

Let S be the set of characters (i.e. algebra homomorphisms) of $\mathcal{F}$ into $\mathbf{C}$; we can view S as the spectrum of the $\mathbf{C}$-algebra $\mathcal{F}_{\mathbf{C}}:=\mathcal{F}\otimes\mathbf{C}$. In the sequel we'll use Prop. 6.1 only through the following weaker corollary:

Corollary 6.2. a) *The map* $\mathcal{F}_{\mathbf{C}}\longrightarrow\mathbf{C}^{\mathrm{S}}$ *given by* $x\mapsto(\chi(x))_{\chi\in\mathrm{S}}$ *is an isomorphism of* $\mathbf{C}$*-algebras.*

b) *One has* $\chi(x^*)=\overline{\chi(x)}$ *for* $\chi\in\mathrm{S}$, $x\in\mathcal{F}$.

The assertion a) follows immediately from the proposition. We have seen in the proof of the proposition that $\mathcal{F}_{\mathbf{R}}$ is isomorphic as an algebra with involution to $\mathbf{R}^p\times\mathbf{C}^q$, with the involution acting trivially on the real factors and by conjugation on the complex factors; this is equivalent to b). ∎

Clearly an explicit knowledge of the isomorphism $\mathcal{F}_{\mathbf{C}}\longrightarrow\mathbf{C}^{\mathrm{S}}$ (that is, of the characters $\chi:\mathcal{F}\to\mathbf{C}$) will allow us to perform any computation we need to in the ring $\mathcal{F}$. As an example:

Proposition 6.3. *In the situation of Prop. 5.9, one has*

$$\mathrm{N}_g(\alpha_1+\ldots+\alpha_s)=\sum_{\chi\in\mathrm{S}}\chi(\alpha_1)\ldots\chi(\alpha_s)\,\chi(\omega)^{g-1}\qquad\textit{with}\qquad\chi(\omega)=\sum_{\lambda\in\mathrm{I}}|\chi(\lambda)|^2\ .$$

Let $x\in\mathcal{F}$; the corresponding element of $\mathbf{C}^{\mathrm{S}}$ is $(\chi(x))_{\chi\in\mathrm{S}}$. In the standard basis of $\mathbf{C}^{\mathrm{S}}$, the matrix of m_x is the diagonal matrix with entries $(\chi(x))_{\chi\in\mathrm{S}}$, so we have $\mathrm{Tr}(x)=\sum_{\chi\in\mathrm{S}}\chi(x)$. Then the result follows from Prop. 5.9. ∎

(6.4) One can obviously play around for a while with these formulas; let me give a sample, also to make a link with the notation of the mathematical physicists. Let $\alpha\in\mathrm{I}$; the matrix of the multiplication m_α in the basis I is $\mathrm{N}_\alpha=(\mathrm{N}_{\alpha\gamma}^{\beta})_{(\beta,\gamma)\in\mathrm{I}\times\mathrm{I}}$, with $\mathrm{N}_{\alpha\gamma}^{\beta}=\mathrm{N}(\alpha+\beta^*+\gamma)$. On the other hand the matrix of m_α in the standard basis of $\mathbf{C}^{\mathrm{S}}$ is the diagonal matrix D_α with entries $\chi(\alpha)$, for $\chi\in\mathrm{S}$. The base change matrix is $\Sigma=(\chi(\lambda))_{(\chi,\lambda)\in\mathrm{S}\times\mathrm{I}}$, so that

$$\mathrm{N}_\alpha=\Sigma^{-1}\mathrm{D}_\alpha\Sigma$$

i.e. "the matrix Σ diagonalizes the fusion rules". Observe that this remains true if we replace Σ by $\Sigma'=\Delta\Sigma$, where Δ is a diagonal matrix. If we take $\Delta=\mathrm{D}_\omega^{-\frac{1}{2}}$ (noting that $\chi(\omega)=\sum_\lambda|\chi(\lambda)|^2$ is positive), an easy computation gives that the matrix Σ' is *unitary.* This is only part of the story: for a RCFT the Verlinde conjecture gives a geometric interpretation of the matrix Σ' in terms of the conformal blocks for $g=1$, providing further restrictions on the fusion ring $\mathcal{F}$.

Part III: The fusion ring $\mathcal{R}_\ell(\mathfrak{g})$

7. The rings $\mathcal{R}(\mathfrak{g})$ and $\mathcal{R}_\ell(\mathfrak{g})$.

(7.1) Recall that the representation ring $\mathcal{R}(\mathfrak{g})$ is the Grothendieck ring of finite-dimensional representations of $\mathfrak{g}$, with the multiplicative structure defined by the tensor product of representations. It is a free $\mathbf{Z}$-module with basis the isomorphism classes of irreducible representations (i.e. the $[\mathrm{V}_\lambda]$ for $\lambda\in\mathrm{P}_+$) with the rule

$$[\mathrm{V}_\lambda]\cdot[\mathrm{V}_\mu]=[\mathrm{V}_\lambda\otimes\mathrm{V}_\mu]\ .$$

We are interested in an analogue of $\mathcal{R}(\mathfrak{g})$ for the level ℓ representations of $\widehat{\mathfrak{g}}$. However it is not clear how to define the multiplicative structure in terms of the affine algebra $\widehat{\mathfrak{g}}$: taking tensor products does not work, since the tensor product of two representations of level ℓ has level 2ℓ. Instead we will follow another route, which can be expressed purely in ordinary Lie theory terms.

We have associated to the Lie algebra $\mathfrak{g}$ and the integer ℓ a fusion rule (example 5.2 *a*), defined by the formula $\mathrm{N}(\sum\lambda_i)=\dim\mathrm{V}_{\mathbf{P}^1}(\vec{p},\vec{\lambda})$. We denote by $\mathcal{R}_\ell(\mathfrak{g})$ the corresponding fusion ring, and call it the *fusion ring of* $\mathfrak{g}$ *at level* ℓ.

We can consider $\mathcal{R}_\ell(\mathfrak{g})$ as the free $\mathbf{Z}$-module with basis the isomorphism classes $[\mathrm{V}_\lambda]$ for $\lambda \in \mathrm{P}_\ell$. The product in $\mathcal{R}_\ell(\mathfrak{g})$ is given by

$$[\mathrm{V}_\lambda]\cdot[\mathrm{V}_\mu] = \sum_{\nu\in\mathrm{P}_\ell} \mathrm{N}(\lambda+\mu+\nu^*)\,[\mathrm{V}_\nu]\ .$$

One can make this more explicit as follows. Consider the Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ spanned by $\mathrm{H}_\theta, \mathrm{X}_\theta, \mathrm{X}_{-\theta}$ (1.3); as in (4.3) we denote by $\mathrm{V}^{(p)}$ the isotypic component of spin p of a $\mathfrak{s}$-module V.

Proposition 7.2. *Let $\lambda,\mu\in\mathrm{P}_\ell$. The product $[\mathrm{V}_\lambda]\cdot[\mathrm{V}_\mu]$ in $\mathcal{R}_\ell(\mathfrak{g})$ is the class of the $\mathfrak{g}$-module $\mathrm{V}_\lambda\odot\mathrm{V}_\mu$ quotient of $\mathrm{V}_\lambda\otimes\mathrm{V}_\mu$ by the $\mathfrak{g}$-module spanned by the isotypic components of spin r of $\mathrm{V}_\lambda^{(p)}\otimes\mathrm{V}_\mu^{(q)}$ for all triples $\{p,q,r\}$ such that $p+q+r>\ell$.*

By Prop. 4.3, for each $\nu\in\mathrm{P}_\ell$, $\mathrm{N}(\lambda+\mu+\nu^*)$ is the dimension of the space of $\mathfrak{g}$-invariant linear forms on $\mathrm{V}_\lambda\otimes\mathrm{V}_\mu\otimes\mathrm{V}_\nu^*$ which vanish on $\mathrm{V}_\lambda^{(p)}\otimes\mathrm{V}_\mu^{(q)}\otimes(\mathrm{V}_\nu^{(r)})^*$ for $p+q+r>\ell$; this space is canonically isomorphic to the space $\mathcal{H}_{\lambda\mu}^\nu$ of $\mathfrak{g}$-linear maps $u:\mathrm{V}_\lambda\otimes\mathrm{V}_\mu\longrightarrow\mathrm{V}_\nu$ such that $u(\mathrm{V}_\lambda^{(p)}\otimes\mathrm{V}_\mu^{(q)})\subset\sum_{p+q+r\le\ell}\mathrm{V}_\nu^{(r)}$, that is, such that u annihilates the isotypic component of spin r of $\mathrm{V}_\lambda^{(p)}\otimes\mathrm{V}_\mu^{(q)}$ whenever $p+q+r>\ell$. Now for any finite-dimensional $\mathfrak{g}$-module V, the multiplicity of V_ν in V is $\dim\mathrm{Hom}_\mathfrak{g}(\mathrm{V},\mathrm{V}_\nu)$; therefore by definition $\mathrm{Hom}_\mathfrak{g}(\mathrm{V}_\lambda\odot\mathrm{V}_\mu,\mathrm{V}_\nu)$ is isomorphic to the subspace $\mathcal{H}_{\lambda\mu}^\nu$ of $\mathrm{Hom}_\mathfrak{g}(\mathrm{V}_\lambda\otimes\mathrm{V}_\mu,\mathrm{V}_\nu)$. This is equivalent by duality to the statement of the proposition. ∎

Examples 7.3. *a*) Assume $\lambda+\mu\in\mathrm{P}_\ell$. Then $\mathrm{V}_\lambda^{(p)}$ and $\mathrm{V}_\mu^{(q)}$ are nonzero only if $p\le\frac{1}{2}\lambda(\mathrm{H}_\theta)$ and $q\le\frac{1}{2}\mu(\mathrm{H}_\theta)$, and $\mathrm{V}_\lambda^{(p)}\otimes\mathrm{V}_\mu^{(q)}$ has a component of spin r if and only if $r\le p+q$. Therefore the condition $p+q+r\le\ell$ is always realized, so $[\mathrm{V}_\lambda]\cdot[\mathrm{V}_\mu]$ *is the class of* $\mathrm{V}_\lambda\otimes\mathrm{V}_\mu$.

b) Assume $\lambda(\mathrm{H}_\theta)+\mu(\mathrm{H}_\theta)=\ell+1$. Then the relation $p+q+r>\ell$ holds if and only if $p=\frac{1}{2}\lambda(\mathrm{H}_\theta)$, $q=\frac{1}{2}\mu(\mathrm{H}_\theta)$, $r=\frac{1}{2}(\ell+1)$; moreover every component of spin $\frac{1}{2}(\ell+1)$ of $\mathrm{V}_\lambda\otimes\mathrm{V}_\mu$ occurs in this way. This means that $[\mathrm{V}_\lambda]\cdot[\mathrm{V}_\mu]$ *is obtained by removing from $\mathrm{V}_\lambda\otimes\mathrm{V}_\mu$ all components V_ν with $\nu(\mathrm{H}_\theta)=\ell+1$.*

c) Assume $\lambda(\mathrm{H}_\theta)+\mu(\mathrm{H}_\theta)=\ell+2$. Then the same argument shows that one has to remove the V_ν 's with $\nu(\mathrm{H}_\theta)=\ell+2$ or $\ell+1$, and the V_ν 's with $\nu(\mathrm{H}_\theta)=\ell$ which intersect non-trivially $\mathrm{V}_\lambda^{(\lambda(\mathrm{H}_\theta))}\otimes\mathrm{V}_\mu^{(\mu(\mathrm{H}_\theta))}$.

8. The map $\mathcal{R}(\mathfrak{g})\longrightarrow\mathcal{R}_\ell(\mathfrak{g})$.

Though $\mathcal{R}_\ell(\mathfrak{g})$ appears as a subgroup of $\mathcal{R}(\mathfrak{g})$, it is obviously not a subring. We will see, however, that there is a natural way to look at $\mathcal{R}_\ell(\mathfrak{g})$ as a *quotient ring* of $\mathcal{R}(\mathfrak{g})$.

(8.1) We will need a few classical facts about root systems, all of which can be found in [Bo]. For each root α, the equation $\lambda(\mathrm{H}_\alpha)=0$ (or equivalently $(\lambda\,|\,\alpha)=0$) defines a hyperplane in the real vector space $\mathrm{P}\otimes\mathbf{R}$, called the *wall* associated to α. The *chambers* of the root system are the connected components of the complement of the walls. The chambers are fundamental domains for the action of the Weyl group W on $\mathrm{P}\otimes\mathbf{R}$.

To the basis $(\alpha_1,\ldots,\alpha_r)$ of the root system is associated a chamber C, defined by the conditions $\lambda(\mathrm{H}_{\alpha_i})\geq 0$. By definition the set P_+ of dominant weights is $\mathrm{P}\cap\mathrm{C}$. Since C is a fundamental domain, every element of P can be written $w\lambda_+$ with $w\in\mathrm{W}$, $\lambda_+\in\mathrm{P}_+$; the weight λ_+ is uniquely determined, and so is w if λ does not belong to a wall. Let us denote as usual by ρ the half sum of the positive roots; it is characterized by the equality $\rho(\mathrm{H}_{\alpha_i})=1$ for each simple root α_i. Therefore *the weights which belong to the interior of* C *are the weights* $\lambda+\rho$ *for* $\lambda\in\mathrm{P}_+$.

For studying the representation ring $\mathcal{R}_\ell(\mathfrak{g})$ we need to consider a closely parallel situation where the role of W is played by an infinite Coxeter group, the *affine Weyl group* W_ℓ. Let $h^\vee:=\rho(\mathrm{H}_\theta)+1$ [1]. Then W_ℓ is the group of motions of $\mathrm{P}\otimes\mathbf{R}$ generated by W and the translation $x\mapsto x+(\ell+h^\vee)\theta$. Since each long root is conjugate to θ under W, the group W_ℓ is the semi-direct product of W by the lattice $(\ell+h^\vee)\mathrm{Q}_{lg}$, where Q_{lg} is the sublattice of P spanned by the long roots. The *affine walls* of $\mathrm{P}\otimes\mathbf{R}$ are the affine hyperplanes $(\lambda\,|\,\alpha)=(\ell+h^\vee)n$ for each root α and each $n\in\mathbf{Z}$. The connected components of the complement are called alcoves; each alcove is a fundamental domain for the action of W_ℓ on $\mathrm{P}\otimes\mathbf{R}$. The alcove A contained in C and containing 0 is defined by the inequalities $\lambda(\mathrm{H}_{\alpha_i})\geq 0$ for each basis root α_i and $\lambda(\mathrm{H}_\theta)\leq\ell+h^\vee$. We see as above that *the weights which belong to the interior of* A *are the* $\lambda+\rho$ *for* $\lambda\in\mathrm{P}_\ell$.

Let $\mathbf{Z}[\mathrm{P}]$ be the group ring of P; following [Bo] we denote by $(e^\lambda)_{\lambda\in\mathrm{P}}$ its canonical basis, so that the multiplication in $\mathbf{Z}[\mathrm{P}]$ obeys the usual rule $e^\lambda e^\mu=e^{\lambda+\mu}$. The action of W_ℓ (hence of W) on P extends to an action on $\mathbf{Z}[\mathrm{P}]$. Let $\varepsilon:\mathrm{W}\to\{\pm1\}$ be the signature homomorphism. We denote by $\mathbf{Z}[\mathrm{P}]_\mathrm{W}$ the quotient of $\mathbf{Z}[\mathrm{P}]$ by the sublattice spanned by the elements $e^\lambda-\varepsilon(w)e^{w\lambda}$ ($\lambda\in\mathrm{P}$, $w\in\mathrm{W}$) and by the elements e^λ for all weights $\lambda\in\mathrm{P}$ belonging to a wall [2]; we define $\mathbf{Z}[\mathrm{P}]_{\mathrm{W}_\ell}$ in the same way.

Lemma 8.2. *The linear maps*

$$\varphi:\mathcal{R}(\mathfrak{g})\longrightarrow\mathbf{Z}[\mathrm{P}]_\mathrm{W}\qquad,\qquad\varphi_\ell:\mathcal{R}_\ell(\mathfrak{g})\longrightarrow\mathbf{Z}[\mathrm{P}]_{\mathrm{W}_\ell}$$

which associate to $[\mathrm{V}_\lambda]$ *the class of* $e^{\lambda+\rho}$, *are bijective.*

[1] This number is often called the *dual Coxeter number* of the root system.

[2] Observe that such an element satisfies $\lambda(\mathrm{H}_\alpha)=0$ for some root α, hence $2e^\lambda=e^\lambda-\varepsilon(s_\alpha)e^{s_\alpha(\lambda)}$.

Let us define a linear map $\psi : \mathbf{Z}[\mathrm{P}] \longrightarrow \mathcal{R}(\mathfrak{g})$ in the following way: let $\lambda \in \mathrm{P}$. By the above remarks, if λ does not lie on a wall, there exist $w \in \mathrm{W}$ and $\lambda_+ \in \mathrm{P}_+$, uniquely determined, such that $\lambda = w(\lambda_+ + \rho)$. We put

$$\psi(e^\lambda) = \begin{cases} \varepsilon(w)\,[\mathrm{V}_{\lambda_+}] & \text{if } \lambda \text{ does not belong to a wall,} \\ 0 & \text{otherwise.} \end{cases}$$

Then ψ factors through $\overline{\psi} : \mathbf{Z}[\mathrm{P}]_\mathrm{W} \longrightarrow \mathcal{R}(\mathfrak{g})$, which is easily seen to be the inverse of φ. The same construction applies identically to define the inverse of φ_ℓ. ∎

By the lemma there is a unique $\mathbf{Z}$-linear map

$$\pi : \mathcal{R}(\mathfrak{g}) \longrightarrow \mathcal{R}_\ell(\mathfrak{g})$$

such that the diagram

$$\begin{array}{ccc} \mathcal{R}(\mathfrak{g}) & \xrightarrow{\ \pi\ } & \mathcal{R}_\ell(\mathfrak{g}) \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi_\ell} \\ \mathbf{Z}[\mathrm{P}]_\mathrm{W} & \xrightarrow{\ p\ } & \mathbf{Z}[\mathrm{P}]_{\mathrm{W}_\ell} \end{array}$$

where p is the quotient map, is commutative. From the lemma (and its proof) we get the following expression for π:

Proposition 8.3. *Let* $\lambda \in \mathrm{P}_+$ *; then*

- $\pi([\mathrm{V}_\lambda]) = 0$ *if* $\lambda + \rho$ *belongs to an affine wall;*
- $\pi([\mathrm{V}_\lambda]) = \varepsilon(w)\,[\mathrm{V}_\mu]$ *otherwise, where* $\mu \in \mathrm{P}_\ell$, $w \in \mathrm{W}_\ell$ *are such that* $\lambda + \rho = w(\mu + \rho)$.

In particular, one has $\pi([\mathrm{V}_\lambda]) = [\mathrm{V}_\lambda]$ *for* $\lambda \in \mathrm{P}_\ell$. ∎

9. The spectrum of $\mathcal{R}_\ell(\mathfrak{g})$.

(9.1) To understand the fusion ring $\mathcal{R}_\ell(\mathfrak{g})$ we need to know its spectrum. Let us first consider the ring $\mathcal{R}(\mathfrak{g})$; it is convenient to introduce the simply-connected group G whose Lie algebra is $\mathfrak{g}$, and the maximal torus $\mathrm{T} \subset \mathrm{G}$ with Lie algebra $\mathfrak{h}$. Any finite-dimensional representation of $\mathfrak{g}$ can be (and will be) considered as a G-module. Any element λ of P defines a character e^λ of T, by the formula $e^\lambda(\exp \mathrm{H}) = \exp \lambda(\mathrm{H})$; this defines an isomorphism of P onto the character group of T, which extends to an isomorphism of the group algebra $\mathbf{C}[\mathrm{P}]$ onto the ring of algebraic functions on T. We will identify $\mathbf{Z}[\mathrm{P}]$ with a subring of $\mathbf{C}[\mathrm{P}]$, so that the notation e^λ for the character associated to λ is coherent with the one we used before.

(9.2) Each element t of T defines a character $\mathrm{Tr}_*(t)$ of $\mathcal{R}(\mathfrak{g})$, which associates to the class of a $\mathfrak{g}$-module V the number $\mathrm{Tr}_{\mathrm{V}}(t)$. There is an explicit way of computing this character, the *Weyl formula.* Let me first introduce the antisymmetrization operator $\mathrm{J} : \mathbf{C}[\mathrm{P}] \to \mathbf{C}[\mathrm{P}]$, defined by the formula $\mathrm{J}(e^\mu) = \sum_{w\in \mathrm{W}} \varepsilon(w) e^{w\mu}$; one has

$$\mathrm{J}(e^\rho) = e^\rho \prod_{\alpha>0} (1 - e^{-\alpha})$$

([Bo, ch. VI, §3, Prop. 2]). An element t of T is called *regular* if $e^\alpha(t) \neq 1$ for each root α, or equivalently if $wt \neq t$ for each $w \in \mathrm{W}$, $w \neq 1$. Let t be a regular element of T; one has $\mathrm{J}(e^\rho)(t) \neq 0$ and

$$\mathrm{Tr}_{\mathrm{V}_\lambda}(t) = \frac{\mathrm{J}(e^{\lambda+\rho})(t)}{\mathrm{J}(e^\rho)(t)} .$$

(9.3) We denote by T_ℓ the subgroup of elements $t \in \mathrm{T}$ such that $e^\alpha(t) = 1$ for each element α of $(\ell + h^\vee)\mathrm{Q}_{lg}$, and by $\mathrm{T}_\ell^{\mathrm{reg}}$ the subset of regular elements in T_ℓ. The finite group T_ℓ will play for $\mathcal{R}_\ell(\mathfrak{g})$ the role of T for $\mathcal{R}(\mathfrak{g})$.

Lemma 9.3. a) *For* $t \in \mathrm{T}_\ell^{\mathrm{reg}}$, *the character* $\mathrm{Tr}_*(t)$ *factors through* $\pi : \mathcal{R}(\mathfrak{g}) \to \mathcal{R}_\ell(\mathfrak{g})$.

b) *Let us identify* $\mathrm{P} \otimes \mathbf{C}$ *with* $\mathfrak{h}$ *using the normalized Killing form. Then the map* $\lambda \mapsto \exp 2\pi i \dfrac{\lambda}{\ell + h^\vee}$ *induces an isomorphism of* $\mathrm{P}/(\ell + h^\vee)\mathrm{Q}_{lg}$ *onto* T_ℓ.

c) *The map* $\lambda \mapsto \exp 2\pi i \dfrac{\lambda+\rho}{\ell + h^\vee}$ *induces a bijection of* P_ℓ *onto* $\mathrm{T}_\ell^{reg}/\mathrm{W}$.

a) Let $t \in \mathrm{T}_\ell^{\mathrm{reg}}$. The Weyl formula provides us with a commutative diagram

$$\begin{array}{ccc} \mathcal{R}(\mathfrak{g}) & & \\ {\scriptstyle\varphi}\downarrow & \searrow^{\mathrm{Tr}_*(t)} & \\ \mathbf{Z}[\mathrm{P}]_{\mathrm{W}} & \xrightarrow[j_t]{} & \mathbf{C} \end{array}$$

where j_t associates to the class of $e^\mu \in \mathbf{Z}[\mathrm{P}]$ the complex number $\dfrac{\mathrm{J}(e^\mu)(t)}{\mathrm{J}(e^\rho)(t)}$.

The kernel of π corresponds through φ to the kernel of p (8.3), which is the subspace of $\mathbf{Z}[\mathrm{P}]_{\mathrm{W}}$ spanned by the elements $e^{\mu+\alpha} - e^\mu$, for $\mu \in \mathrm{P}$, $\alpha \in (\ell + h^\vee)\mathrm{Q}_{lg}$, and e^μ for μ in some affine wall. The elements of the first type are killed by j_t because t is chosen so that $e^\alpha(t) = 1$ for $\alpha \in (\ell + h^\vee)\mathrm{Q}_{lg}$; if μ belongs to an affine wall, $2e^\mu$ is of the first type (see the footnote to (8.1)), so one has $2j_t(e^\mu) = 0$ and therefore $j_t(e^\mu) = 0$. This proves a).

b) Consider the exponential exact sequence

$$0 \to 2\pi i \mathrm{Q}^\vee \longrightarrow \mathfrak{h} \xrightarrow{\exp} \mathrm{T} \to 0 ;$$

here $Q^\vee$ is the dual root lattice of the root system of $\mathfrak{g}$, i.e. the lattice spanned by the H_α's. Let us denote by $P^\vee_{lg}$ the subgoup of $Q^\vee\otimes\mathbf{Q}$ consisting of elements H such that $\alpha(H)\in\mathbf{Z}$ for all $\alpha\in Q_{lg}$; the map $H\mapsto\exp(\dfrac{2\pi i}{\ell+h^\vee}H)$ induces an isomorphism of $P^\vee_{lg}/(\ell+h^\vee)Q^\vee$ onto T_ℓ. When we identify $Q^\vee\otimes\mathbf{Q}$ with $P\otimes\mathbf{Q}$ using the normalized Killing form, $P^\vee_{lg}$ is identified with the dual lattice of Q_{lg}, that is, the set of elements λ in $P\otimes\mathbf{Q}$ such that $(\lambda\,|\,\beta)\in\mathbf{Z}$ for each long root β. Because of the normalization this is equivalent to $\lambda(H_\beta)\in\mathbf{Z}$; since the H_β's are the *short* roots of the dual system, and therefore span the coroot lattice $Q^\vee$, the dual lattice of Q_{lg} is P. In the same way $Q^\vee$ is identified with the dual lattice Q_{lg} of P. This proves b).

c) The isomorphism $P/(\ell+h^\vee)Q_{lg}\xrightarrow{\sim}T_\ell$ is of course compatible with the action of W. Now the orbits of W in $P/(\ell+h^\vee)Q_{lg}$ are in one-to-one correspondence with the orbits of W_ℓ in P, and we have seen that those are parametrized by the elements of P which lie in the affine alcove; moreover the orbits where W acts freely correspond to the weights which lie in the interior of the alcove, that is, which are of the form $\lambda+\rho$ for $\lambda\in P_\ell$. This gives c). ∎

(9.4) For $t\in T_\ell^{\rm reg}$, we will still denote by $\mathrm{Tr}_*(t)$ the linear map $\mathcal{R}_\ell(\mathfrak{g})\longrightarrow\mathbf{C}$ obtained by passing to the quotient; because of Prop. 8.3, it is again given by $[V]\mapsto\mathrm{Tr}_V(t)$. It depends only on the class of t in $T_\ell^{\rm reg}/W$. The next two results are directly borrowed from [F]:

Proposition 9.4. *The following conditions are equivalent:*

(i) *The map* $\pi:\mathcal{R}(\mathfrak{g})\longrightarrow\mathcal{R}_\ell(\mathfrak{g})$ *is a ring homomorphism;*

(ii) *One has* $\pi([V_\lambda\otimes V_\varpi])=[V_\lambda]\cdot[V_\varpi]$ *for each* λ *in* P_ℓ *and each fundamental weight* $\varpi\in P_\ell$ *;*

(iii) *The linear forms* $\mathrm{Tr}_*(t)$ $(t\in T_\ell^{\rm reg}/W)$ *are characters of the fusion ring* $\mathcal{R}_\ell(\mathfrak{g})$.

When these conditions hold, the spectrum of $\mathcal{R}_\ell(\mathfrak{g})$ *consists of the characters* $\mathrm{Tr}_*(t)$ *where* t *runs over* $T_\ell^{\rm reg}/W$.

Since $\pi([V_\lambda])=[V_\lambda]$ for $\lambda\in P_\ell$ (Prop. 8.3), the implication (i) ⇒ (ii) is clear.

(ii) ⇒ (iii): Fix some $t\in T_\ell^{\rm reg}$, and put $\chi=\mathrm{Tr}_*(t)$. Let Ω denote the (finite) set of fundamental weights. By the lemma below the $\mathbf{Z}$-algebra $\mathcal{R}_\ell(\mathfrak{g})$ is generated by the family $([V_\varpi])_{\varpi\in\Omega}$. Therefore to prove that χ is a character it is enough to check the equality $\chi(x\cdot[V_\varpi]=\chi(x)\chi([V_\varpi])$ for $x\in\mathcal{R}_\ell(\mathfrak{g})$, $\varpi\in\Omega$; moreover because χ is $\mathbf{Z}$-linear we may take x of the form $[V_\lambda]$ for $\lambda\in P_\ell$. But since $\chi\circ\pi$ is a character of $\mathcal{R}(\mathfrak{g})$, this follows from (ii).

(iii) ⇒ (i): Assume that (iii) holds. Different orbits of W in $T_\ell^{\rm reg}$ give different characters of $\mathcal{R}(\mathfrak{g})$, hence of $\mathcal{R}_\ell(\mathfrak{g})$; because $\mathrm{Card}(\mathrm{P}_\ell) = \mathrm{Card}(\mathrm{T}_\ell^{\mathrm{reg}}/\mathrm{W})$ by lemma 9.3 *c*), the spectrum of $\mathcal{R}_\ell(\mathfrak{g})$ consists of the characters $\mathrm{Tr}_*(t)$ for $t \in \mathrm{T}_\ell^{\mathrm{reg}}/\mathrm{W}$. Since $\mathrm{Tr}_*(t)\circ\pi$ is a ring homomorphism for each t in $\mathrm{T}_\ell^{\mathrm{reg}}/\mathrm{W}$ it follows that π is a ring homomorphism. ∎

Lemma 9.5. *The classes* $[\mathrm{V}_\varpi]$ *for* $\varpi \in \Omega$ *generate the* $\mathbf{Z}$*-algebra* $\mathcal{R}_\ell(\mathfrak{g})$.

Let us choose an element H of $\mathrm{Q}^\vee$ such that $\alpha_i(\mathrm{H})$ is a positive integer for each simple root α_i . We will prove by induction on $\lambda(\mathrm{H})$ that $[\mathrm{V}_\lambda]$ is a polynomial in $([\mathrm{V}_\varpi])_{\varpi\in\Omega}$ for every $\lambda \in \mathrm{P}_\ell$. This is clear if $\lambda = 0$. If $\lambda \neq 0$, we can write $\lambda = \mu + \varpi$ with $\mu \in \mathrm{P}_\ell$, $\varpi \in \Omega$. Then $\mathrm{V}_\mu \otimes \mathrm{V}_\varpi$ is the sum of V_λ and of irreducible $\mathfrak{g}$-modules V_ν whose highest weights are of the form $\nu = \lambda - \sum n_i\alpha_i$ with $n_i \in \mathbf{N}$, $\sum n_i > 0$ (see e.g. [Bo, Ch. VIII, §7, n$^{\mathrm{o}}$ 4, Prop. 9]). By the induction hypothesis the elements $[\mathrm{V}_\mu]$ and $[\mathrm{V}_\nu]$ of $\mathcal{R}_\ell(\mathfrak{g})$ are polynomial in the $[\mathrm{V}_\varpi]$'s; on the other hand the element $[\mathrm{V}_\mu \otimes \mathrm{V}_\varpi]$ is equal to $[\mathrm{V}_\mu]\cdot[\mathrm{V}_\varpi]$ (example 7.3 *a*). It follows that $[\mathrm{V}_\lambda]$ is a polynomial in the $[\mathrm{V}_\varpi]$'s, hence the lemma. ∎

Proposition 9.6 [F]**.** *The conditions of Prop.* 9.4 *hold when* $\mathfrak{g}$ *is of type* A_r , B_r , C_r , D_r *or* G_2 .

I will content myself with the cases A_r and C_r , which are easy, and refer to the Appendix of [F] for the (rather technical) details in the remaining cases.

By Prop. 9.4 we need to prove the equality $\pi([\mathrm{V}_\lambda \otimes \mathrm{V}_\varpi]) = [\mathrm{V}_\lambda]\cdot[\mathrm{V}_\varpi]$ for each λ in P_ℓ and each fundamental weight $\varpi \in \mathrm{P}_\ell$. For A_r and C_r a glance at the tables in [Bo] show that $\varpi(\mathrm{H}_\theta) = 1$ for each fundamental weight ϖ . If $\lambda(\mathrm{H}_\theta) < \ell$ we are done by example 7.3 *a*). If $\lambda(\mathrm{H}_\theta) = \ell$ we just have to apply the example 7.3 *b*) and observe that an irreducible $\mathfrak{g}$-modules V_μ with $\mu(\mathrm{H}_\theta) = \ell + 1$ is killed by π (Prop. 8.3). ∎

We will now apply these results to reach our goal, which is to compute the dimension of the spaces $\mathrm{V}_{\mathrm{C}}(\vec{p},\vec{\lambda})$ defined in part I. Since we have now (at least in most cases) a precise description of the characters of $\mathcal{R}_\ell(\mathfrak{g})$, we can use the formula of Prop. 6.3. The only remaining difficulty is to compute the expression $\sum_{\lambda\in\mathrm{P}_\ell} |\chi(\lambda)|^2$. This is provided by the following lemma:

Lemma 9.7. *Let* $t \in \mathrm{T}_\ell^{\mathrm{reg}}$ *. Then*

$$\sum_{\lambda\in\mathrm{P}_\ell} |\,\mathrm{Tr}_{\mathrm{V}_\lambda}(t)|^2 = \frac{|\mathrm{T}_\ell|}{\Delta(t)} \ ,$$

where $\Delta(t) := |\mathrm{J}(e^\rho)(t)|^2 = \displaystyle\prod_{\alpha\in\mathrm{R}(\mathfrak{g},\mathfrak{h})} (e^\alpha(t) - 1)$.

For $\lambda \in \mathrm{P}_\ell$, let us denote by t_λ the element $\exp 2\pi i \dfrac{\lambda+\rho}{\ell + h^\vee}$ of T ; the t_λ 's for $\lambda\in\mathrm{P}_\ell$ form a system of representatives of $\mathrm{T}_\ell^{\mathrm{reg}}/\mathrm{W}$ (lemma 9.3 *c*). For $\lambda,\mu\in\mathrm{P}_\ell$, one has

$$\mathrm{J}(e^{\lambda+\rho})(t_\mu)=\sum_{w\in\mathrm{W}}\varepsilon(w)\exp 2\pi i\frac{(w(\lambda+\rho)\,|\,\mu+\rho)}{\ell+h^\vee}=\mathrm{J}(e^{\mu+\rho})(t_\lambda)\ .$$

so by the Weyl formula (9.2)

$$\sum_{\lambda\in\mathrm{P}_\ell}|\,\mathrm{Tr}_{\mathrm{V}_\lambda}(t_\mu)|^2=\frac{1}{\Delta(t_\mu)}\sum_{\lambda\in\mathrm{P}_\ell}|\mathrm{J}(e^{\mu+\rho})(t_\lambda)|^2\ .$$

Let $\mathrm{L}^2(\mathrm{T}_\ell)$ be the space of functions on the (finite) group T_ℓ, endowed with the usual scalar product

$$<f\,|\,g>=\frac{1}{|\mathrm{T}_\ell|}\sum_{t\in\mathrm{T}_\ell}\overline{f(t)}\,g(t)\ .$$

The function $h(t)=\mathrm{J}(e^{\mu+\rho})(t)$ on T is anti-invariant, i.e. satisfies $h(wt)=\varepsilon(w)h(t)$ for $w\in\mathrm{W}$, $t\in\mathrm{T}$. It follows on the one hand that it vanishes at any non-regular point t of T (for any such point is fixed by a reflection $s\in\mathrm{W}$, so $h(t)=h(s(t))=-h(t)$), and on the other hand that $|h|^2$ is W-invariant. Therefore

$$\sum_{\lambda\in\mathrm{P}_\ell}|\mathrm{J}(e^{\mu+\rho})(t_\lambda)|^2=\frac{|\mathrm{T}_\ell|}{|\mathrm{W}|}\,||\mathrm{J}(e^{\mu+\rho})||^2\ ,$$

where the norm is taken in $\mathrm{L}^2(\mathrm{T}_\ell)$.

We claim that the restrictions to T_ℓ of the characters $e^{w(\mu+\rho)}$, for $w\in\mathrm{W}$, are all distinct. Suppose this is not the case; then there exists distinct elements $w,w'\in\mathrm{W}$ such that $\big(w(\mu+\rho)-w'(\mu+\rho)\mid\lambda\big)\in(\ell+h^\vee)\mathbf{Z}$ for all $\lambda\in\mathrm{P}$. But we have seen in the proof of lemma 9.3 *b*) that the dual lattice of P is Q_{lg}, so the above condition means that $\mu+\rho-w^{-1}w'(\mu+\rho)$ belongs to $(\ell+h^\vee)\mathrm{Q}_{lg}$. This implies that there exists a nontrivial element of W_ℓ fixing $\mu+\rho$, a contradiction.

Then the orthogonality relations for the characters of the finite group T_ℓ give $||\mathrm{J}(e^{\mu+\rho})||^2=|\mathrm{W}|$, from which the lemma follows. ∎

Corollary 9.8 (Verlinde formula)**.** *Assume that the conditions of Prop.* 9.4 *hold, e.g. that* $\mathfrak{g}$ *is of type* A, B, C, D *or* G. *One has*

$$\dim\mathrm{V}_{\mathrm{C}}(\vec{p},\vec{\lambda})=|\mathrm{T}_\ell|^{g-1}\sum_{t\in\mathrm{T}_\ell^{\mathrm{reg}}}\frac{\mathrm{Tr}_{\mathrm{V}_{\vec{\lambda}}}(t)}{\Delta(t)^{g-1}}$$

$$=|\mathrm{T}_\ell|^{g-1}\sum_{\mu\in\mathrm{P}_\ell}\mathrm{Tr}_{\mathrm{V}_{\vec{\lambda}}}(\exp 2\pi i\frac{\mu+\rho}{\ell+h^\vee})\prod_{\alpha>0}\Big|2\sin\pi\frac{(\alpha\,|\,\mu+\rho)}{\ell+h^\vee}\Big|^{2-2g}\ .$$

The first expression is a simple reformulation of Prop. 6.3 using the explicit description of the characters (Prop. 9.4) and the above lemma. The second one is obtained, after some easy manipulations, by using the description of $\mathrm{T}_\ell^{\mathrm{reg}}$ given in lemma 9.3 *c*). ∎

Remark 9.9 .– One obtains easily an explicit expression for $|\mathrm{T}_\ell|$, using the isomorphism $\mathrm{P}/(\ell+h^\vee)\mathrm{Q}_{lg} \xrightarrow{\sim} \mathrm{T}_\ell$ (lemma 9.3 *b*). One finds $|\mathrm{T}_\ell| = (\ell+h^\vee)^r fq$, where r is the rank of $\mathfrak{g}$, f its connection index $(=|\mathrm{P}/\mathrm{Q}|)$, and q the index of Q_{lg} in Q . A glance at the tables gives $q=2$ for B_r , 2^{r-1} for C_r , 4 for F_4 , 6 for G_2 , and of course 1 otherwise.

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