Vol. 97, No. 1 DUKE MATHEMATICAL JOURNAL © 1999

COUNTING RATIONAL CURVES ON K3 SURFACES

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Introduction. The aim of this paper is to explain the remarkable formula found by Yau and Zaslow [YZ] to express the number of rational curves on a K3 surface. Projective K3 surfaces fall into countably many families $(\mathcal{F}_g)_{g\geq 1}$; a surface in $\mathcal{F}_g$ admits a g-dimensional linear system of curves of genus g. A naïve count of constants suggests that such a system will contain a positive number, say, $n(g)$, of rational (highly singular) curves. The formula is

$$\sum_{g\geq 0} n(g)q^g = \frac{q}{\Delta(q)},$$

where $\Delta(q) = q\prod_{n\geq 1}(1-q^n)^{24}$ is the well-known modular form of weight 12, and by convention we put $n(0) = 1$.

To explain the idea in a nutshell, take the case $g = 1$. We thus look at K3 surfaces with an elliptic fibration $f : S \to \mathbf{P}^1$, and we ask for the number of singular fibres. The (topological) Euler-Poincaré characteristic of a fibre C_t is zero if C_t is smooth, is 1 if it is a rational curve with one node, is 2 if it has a cusp, and so on. From the standard properties of the Euler-Poincaré characteristic, we get $e(S) = \sum_t e(C_t)$; hence, $n(1) = e(S) = 24$, and this number counts nodal rational curves with multiplicity 1, cuspidal rational curves with multiplicity 2, and so on.

The idea of Yau and Zaslow is to generalize this approach to any genus. Let S be a K3 surface with a g-dimensional linear system Π of curves of genus g. The role of f is played by the morphism $\bar{\mathcal{J}}\mathcal{C} \to \Pi$, whose fibre over a point $t \in \Pi$ is the compactified Jacobian $\bar{J}C_t$. To apply the same method, we would like to prove the following facts.

(1) The Euler-Poincaré characteristic $e(\bar{\mathcal{J}}\mathcal{C})$ is the coefficient of q^g in the Taylor expansion of $q/\Delta(q)$.

(2) $e(\bar{J}C_t) = 0$ if C_t is not rational.

(3) $e(\bar{J}C_t) = 1$ if C_t is a rational curve with nodes as only singularities. Moreover $e(\bar{J}C_t)$ is positive when C_t is rational, and can be computed in terms of the singularities of C_t.

(4) For a generic K3 surface S in $\mathcal{F}_g$, all rational curves in Π are nodal.

The first statement is proved in Section 1, by comparing $e(\bar{\mathcal{J}}\mathcal{C})$ with the Euler-Poincaré characteristic of the Hilbert scheme $S^{[g]}$, which has been computed by

Received 21 August 1997.
1991 *Mathematics Subject Classification.* 14J28.
Author partially supported by the European HCM project, Algebraic Geometry in Europe.

Göttsche. Assertion (2) is proved in Section 2. We prove part of (3) in Sections 3 and 4. We express $e(\bar{J}C)$ for a rational curve C in terms of a local invariant of the singularities of C, and we compute this local invariant in a number of cases. This invariant has been recently identified by Fantechi, Göttsche, and van Straten as the multiplicity of the δ-constant stratum in the semiuniversal deformation of the singularity [FGS]; this implies in particular the positivity of $e(\bar{J}C)$. This approach also provides an alternate proof for most of our results in Sections 3 and 4. Unfortunately, (4) is of a different nature and seems to be wide open. The outcome (see Corollary 2.3) is that the coefficient of q^g in $q/\Delta(q)$ counts the rational curves in Π with a certain multiplicity, which is 1 for a nodal curve and can be computed explicitly in many cases. The only missing point (equivalent to (4)) is that for a generic surface in $\mathcal{F}_g$, this coefficient is simply the number of rational curves in Π.

1. The compactified relative Jacobian

1.1. Let X be a complex variety; we denote by $e(X)$ its Euler-Poincaré characteristic, defined by $e(X) = \sum_p (-1)^p \dim_{\mathbf{Q}} H_c^p(X, \mathbf{Q})$. Recall that this invariant is additive, that is, it satisfies $e(X) = e(U) + e(X - U)$ whenever U is an open subset of X.

1.2. We consider a projective K3 surface S with a complete linear system $(C_t)_{t\in\Pi}$ of curves of genus $g \geq 1$ (so Π is a projective space of dimension g). We assume that *all the curves C_t are integral* (that is, irreducible and reduced). This is a simplifying assumption, which can probably be removed at the cost of various technical complications. It is satisfied, of course, if the class of C_t generates $\mathrm{Pic}(S)$.

Let $\mathcal{C} \to \Pi$ be the morphism with fibre C_t over $t \in \Pi$. For each integer $d \in \mathbf{Z}$, we denote by $\bar{\mathcal{J}}\mathcal{C} = \coprod_{d\in\mathbf{Z}} \bar{\mathcal{J}}^d\mathcal{C}$ the compactified Picard scheme of this family. $\bar{\mathcal{J}}^d\mathcal{C}$ is a projective variety of dimension $2g$, which parameterizes pairs $(C_t, \mathcal{L})$ where $t \in \Pi$ and $\mathcal{L}$ is a torsion-free, rank-1 coherent sheaf on C_t of degree d (which means, by definition, $\chi(\mathcal{L}) = d + 1 - g$). According to Mukai [M, Example 0.5], $\bar{\mathcal{J}}^d\mathcal{C}$ can be viewed as a connected component of the moduli space of simple sheaves on S, and therefore is smooth, and admits a (holomorphic) symplectic structure.

The simplest symplectic varieties associated to the K3 surface S are the Hilbert schemes $S^{[d]}$, which parameterize finite subschemes of length d of S. The birational comparison of the symplectic varieties $\bar{\mathcal{J}}^d\mathcal{C}$, for various values of d, with $S^{[g]}$ is an interesting problem, about which not much seems to be known. There is one easy case, as show in Proposition 1.3.

PROPOSITION 1.3. *The compactified Jacobian $\bar{\mathcal{J}}^g\mathcal{C}$ is birationally isomorphic to $S^{[g]}$.*

Proof. Let U be the open subset of $\bar{\mathcal{J}}^g\mathcal{C}$ consisting of pairs (C_t, L), where C_t is smooth, L is invertible, and $\dim H^0(C_t, L) = 1$. To such a pair corresponds a unique effective divisor D on C_t of degree g, which can be viewed as a length-g subscheme of

S; since $\dim H^0(C_t, \mathcal{O}_{C_t}(D)) = 1$, it is contained in a unique curve of Π, namely, C_t. This provides an isomorphism between U and the open subset of $S^{[g]}$ parameterizing finite subschemes of S contained in a unique smooth curve of Π. □

COROLLARY 1.4. *Write $q/\Delta(q) = \sum_{g\geq 0} e(g)q^g$. Then $e(\bar{\mathcal{J}}^g\mathcal{C}) = e(g)$.*

Proof. We can either use a recent result of Batyrev and Kontsevich [B], which says that two birationally equivalent, projective Calabi-Yau manifolds have the same Betti numbers, or a more precise result of Huybrechts [H], which says that two birationally equivalent, projective, symplectic manifolds are diffeomorphic. It remains to apply Göttsche's formula $e(S^{[g]}) = e(g)$ (see [G]). □

2. The compactified Jacobian of a nonrational curve. Let C be an integral curve. By a *rank*-1 *sheaf* on C, we mean a torsion-free, rank-1 coherent sheaf. The rank-1 sheaves $\mathcal{L}$ on C of degree d are parameterized by the compactified Jacobian $\bar{J}^dC$. If L is an invertible sheaf of degree d on C, the map $\mathcal{L} \mapsto \mathcal{L}\otimes L$ is an isomorphism of $\bar{J}C$ onto $\bar{J}^dC$, so we can restrict our study to degree-0 sheaves.

Let $\mathcal{L} \in \bar{J}C$; the endomorphism ring of $\mathcal{L}$ is an $\mathcal{O}_C$-subalgebra of the sheaf of rational functions on C. It is finitely generated as an $\mathcal{O}_C$-module, and hence it is integral over $\mathcal{O}_C$. Thus it is of the form $\mathcal{O}_{C'}$, where $f : C' \to C$ is some partial normalization of C. The sheaf $\mathcal{L}$ is an $\mathcal{O}_{C'}$-module, which amounts to saying that it is the direct image of a rank-1 sheaf $\mathcal{L}'$ on C'.

LEMMA 2.1. *Let $L \in JC$. Then $\mathcal{L}\otimes L$ is isomorphic to $\mathcal{L}$ if and only if f^*L is trivial.*

Proof. The sheaf $\mathcal{L}\otimes L$ is isomorphic to $f_*(\mathcal{L}'\otimes f^*L)$ and hence to $\mathcal{L}$ if f^*L is trivial. On the other hand, we have

$$\mathcal{H}om_{\mathcal{O}_C}(\mathcal{L}, \mathcal{L}\otimes L) \cong \mathcal{E}nd_{\mathcal{O}_C}(\mathcal{L})\otimes_{\mathcal{O}_C} L \cong f_*\mathcal{O}_{C'}\otimes L \cong f_*f^*L\,,$$

so if f^*L is nontrivial, the space $\mathrm{Hom}(\mathcal{L}, \mathcal{L}\otimes L)$ is zero, and $\mathcal{L}\otimes L$ cannot be isomorphic to $\mathcal{L}$. □

PROPOSITION 2.2. *Let C be an integral curve whose normalization $\widetilde{C}$ has genus ≥ 1. Then $e(\bar{J}^dC) = 0$.*

Proof. We have an exact sequence

$$0 \to G \longrightarrow JC \longrightarrow J\widetilde{C} \to 0,$$

where G is a product of additive and multiplicative groups. In particular, G is a divisible group, and hence this exact sequence splits as a sequence of abelian groups. For each integer n, we can find a subgroup of order n in JC that maps injectively into $J\widetilde{C}$. By Lemma 2.1, this group acts freely on $\bar{J}C$, which implies that n divides $e(\bar{J}C)$. Since this holds for any n, the proposition follows. □

COROLLARY 2.3. *Write* $q/\Delta(q) = \sum_{g\geq 0} e(g)q^g$. *Let* $\Pi_{\mathrm{rat}} \subset \Pi$ *be the (finite) subset of rational curves; then* $e(g) = \sum_{t\in\Pi_{\mathrm{rat}}} e(\bar{J}C_t)$.

Proof. We first make a general observation: Let $f : X \to Y$ be a surjective morphism of complex algebraic varieties whose fibres have Euler characteristic zero; then $e(X) = 0$. This is well known (and easy) if f is a locally trivial fibration. The general case follows using Section 1.1, because there exists a stratification of Y such that f is locally trivial above each stratum (see [V]).

The set Π_{rat} is finite because otherwise it would contain a curve, so S would be ruled. Consider the morphism $p : \bar{\mathcal{J}}^g\mathcal{C} \to \Pi$ above $\Pi - \Pi_{\mathrm{rat}}$. By the above remark, we have $e(p^{-1}(\Pi - \Pi_{\mathrm{rat}})) = 0$, and hence we have the result, using Section 1.1 again. □

In other words, $e(g)$ counts the number of rational curves with multiplicity, the multiplicity of a curve C being $e(\bar{J}C)$. In the next two sections, we try to show that this is indeed a reasonable notion of multiplicity.

3. The compactified Jacobian of a rational curve

LEMMA 3.1. *Let* $f : C' \to C$ *be a partial normalization of* C. *The morphism* $f_* : \bar{J}C' \to \bar{J}C$ *is a closed embedding.*

Proof. Let $\mathcal{L}$ and $\mathcal{M}$ be two rank-1 sheaves on C'. We claim that any $\mathcal{O}_C$-homomorphism $u : f_*\mathcal{L} \to f_*\mathcal{M}$ is actually $f_*\mathcal{O}_{C'}$-linear. Let U be a Zariski-open subset of C, $\varphi \in \Gamma(U, f_*\mathcal{O}_{C'})$, and $s \in \Gamma(U, f_*\mathcal{L})$. The rational function φ can be written as a/b, with $a, b \in \Gamma(U, \mathcal{O}_C)$ and $b \neq 0$. Then the element $u(\varphi s) - \varphi u(s)$ of $\Gamma(U, f_*\mathcal{M})$ is killed by b and hence is zero since $f_*\mathcal{M}$ is torsion-free.

Therefore, if $f_*\mathcal{L}$ and $f_*\mathcal{M}$ are isomorphic as $\mathcal{O}_C$-modules, they are also isomorphic as $f_*\mathcal{O}_{C'}$-modules, which means that $\mathcal{L}$ and $\mathcal{M}$ are isomorphic. This proves the injectivity of f_* (which would be enough for our purposes). Now, if S is any base scheme, the same argument applies to sheaves $\mathcal{L}$ and $\mathcal{M}$ on $C \times S$, flat over S, whose restrictions to each fibre $C \times \{s\}$ are torsion-free rank-1. (Observe that a local section b of $\mathcal{O}_C$ is $\mathcal{M}$-regular because it is on each fibre, and $\mathcal{M}$ is flat over S.) This proves that f_* is a monomorphism; since it is proper, it is a closed embedding. □

3.2. Recall that the curve C is said to be *unibranch* if its normalization $\widetilde{C} \to C$ is a homeomorphism. Any curve C admits a unibranch partial normalization $\check{\pi} : \check{C} \to C$ which is minimal, in the sense that any unibranch partial normalization $C' \to C$ factors through $\check{\pi}$. To see this, let $\mathcal{C}$ be the conductor of C, and let $\widetilde{\Sigma}$ be the inverse image in $\widetilde{C}$ of the singular locus $\Sigma \in C$. The finite-dimensional k-algebra $A := \mathcal{O}_{\widetilde{C}}/\mathcal{C}$ is a product of local rings $(A_x)_{x\in\widetilde{\Sigma}}$. Let $(e_x)_{x\in\widetilde{\Sigma}}$ be the corresponding idempotent elements of A. A sheaf of algebras $\mathcal{O}_{C'}$ with $\mathcal{O}_C \subset \mathcal{O}_{C'} \subset \mathcal{O}_{\widetilde{C}}$ is unibranch if and only if $\mathcal{O}_{C'}/\mathcal{C}$ contains each e_x, or, equivalently, $\mathcal{O}_{C'}$ contains the classes $e_x + \mathcal{C}$ for each $x \in \widetilde{\Sigma}$. Clearly there is a smallest such algebra, namely, the algebra $\mathcal{O}_{\check{C}}$ generated by $\mathcal{O}_C$ and the classes $e_x + \mathcal{C}$. The completion of the local ring of $\check{C}$ at a point y is the image of $\widehat{\mathbb{O}}_{C,\check{\pi}(y)}$ in $\widehat{\mathbb{O}}_{\widetilde{C},y}$.

PROPOSITION 3.3. *With the above notation, $e(\bar{J}C) = e(\bar{J}\check{C})$.*

Proof. In view of Proposition 2.2, we may suppose that $\widetilde{C}$ is rational. As before, we denote by Σ the singular locus of C and by $\widetilde{\Sigma}$ its inverse image in $\check{C}$. The cohomology exact sequence associated to the short exact sequence

$$1 \to \mathbb{O}_C^* \longrightarrow \mathbb{O}_{\widetilde{C}}^* \longrightarrow \mathbb{O}_{\widetilde{C}}^*/\mathbb{O}_C^* \to 1$$

provides a bijective homomorphism (actually an isomorphism of algebraic groups) $\mathbb{O}_{\widetilde{C}}^*/\mathbb{O}_C^* \xrightarrow{\sim} JC$.

The evaluation maps $\mathbb{O}_{\widetilde{C}}^* \to (\mathbf{C}^*)^{\widetilde{\Sigma}}$ and $\mathbb{O}_C^* \to (\mathbf{C}^*)^{\Sigma}$ give rise to a surjective homomorphism $\mathbb{O}_{\widetilde{C}}^*/\mathbb{O}_C^* \to (\mathbf{C}^*)^{\widetilde{\Sigma}}/(\mathbf{C}^*)^{\Sigma}$; its kernel is unipotent, that is, isomorphic to a vector space. If n is any integer $\geq \operatorname{Card}(\widetilde{\Sigma})$, it follows that we can find a section φ of $\mathbb{O}_{\widetilde{C}}^*$ in a neighborhood of $\widetilde{\Sigma}$ such that the numbers $\varphi(\tilde{x})$ for $\tilde{x} \in \widetilde{\Sigma}$ are all distinct, but φ^n belongs to $\mathbb{O}_C$. Let L be the line bundle on JC associated to the class of φ in $\mathbb{O}_{\widetilde{C}}^*/\mathbb{O}_C^*$.

Let U be the complement of $\check{\pi}_*(\bar{J}\check{C})$ in $\bar{J}C$. According to Section 1.1 and Lemma 3.1, our assertion is equivalent to $e(U) = 0$. We claim that the line bundle L acts freely on U; since the order of L in JC is finite and arbitrarily large, this will finish the proof. Let $\mathcal{L} \in U$, and let C' be the partial normalization of C such that $\mathcal{E}nd(\mathcal{L}) = \mathbb{O}_{C'}$. By definition of U, C' is not unibranch, and hence there are two points of $\widetilde{\Sigma}$ mapping to the same point of C'; this implies that the function φ does not belong to $\mathbb{O}_{C'}^*$. From the commutative diagram

$$\begin{array}{ccc} \widetilde{\mathbb{O}}_{\widetilde{C}^*}/\mathbb{O}_{C^*} & \xrightarrow{\sim} & JC \\ \big\downarrow & & \big\downarrow \\ \mathbb{O}_{\widetilde{C}^*}/\mathbb{O}_{C'^*} & \xrightarrow{\sim} & JC' \end{array}$$

we conclude that the pullback of L to JC' is nontrivial. By Lemma 2.1, this implies that $\mathcal{L} \otimes L$ is not isomorphic to $\mathcal{L}$. □

COROLLARY 3.4. *For a rational nodal curve C, we have $e(\bar{J}C) = 1$.*

Remark 3.5. Consider a rational curve C whose singularities are all of type A_{2l-1}, that is, locally defined by an equation $u^2 - v^{2l} = 0$. Locally around such a singularity, the curve C is the union of two smooth branches with a high-order contact, so by Proposition 3.3, $e(\bar{J}C)$ is equal to 1. The fact that some highly singular curves count with multiplicity 1 looks rather surprising. The case $g = 2$ provides a (modest) confirmation: The surface S is a double covering of $\mathbf{P}^2$, branched along a sextic curve B; the curves C_t are the inverse images of the lines in $\mathbf{P}^2$, and they become rational when the line is bitangent to B. We get an A_3-singularity when the line has a contact of order 4; thus our assertion in this case follows from the (certainly classical) fact that a line with a fourth-order contact counts as a simple bitangent.

3.6. Proposition 3.3 reduces the computation of the invariant $e(\bar{J}C)$ to the case of a unibranch (rational) curve. To understand this invariant, we use a construction of Rego (see [R]; see also [GP]). For each $x \in C$, we put $\delta_x = \dim \mathcal{O}_{\widetilde{C},x}/\mathcal{O}_{C,x}$, and we denote by $\mathcal{C}$ the ideal $\mathcal{O}_{\widetilde{C}}(-\sum_x (2\delta_x)\,x)$, which is contained in the conductor of C (but the inclusion is strict unless C is Gorenstein).

For $x \in C$, we denote by A_x and $\widetilde{A}_x$ the finite-dimensional algebras $\mathcal{O}_{C,x}/\mathcal{C}_x$ and $\mathcal{O}_{\widetilde{C},x}/\mathcal{C}_x$. Let $\mathbf{G}(\delta_x, \widetilde{A}_x)$ be the Grassmannian of codimension-δ_x subspaces of $\widetilde{A}_x$, and let $\mathbf{G}_x$ be the closed subvariety of $\mathbf{G}(\delta_x, \widetilde{A}_x)$ consisting of sub–A_x-modules. We can also view $\mathbf{G}_x$ as parameterizing the sub–$\mathcal{O}_{C,x}$-modules $\mathcal{L}_x$ of codimension δ_x in $\mathcal{O}_{\widetilde{C},x}$, because any such submodule contains $\mathcal{C}_x$ (see [GP, Lemma 1.1(iv)]). Since $\mathcal{O}_{\widetilde{C}}/\mathcal{C}$ is a skyscraper sheaf with fibre $\widetilde{A}_x$ at x, the product $\prod_{x\in\Sigma} \mathbf{G}_x$ parameterizes sub–$\mathcal{O}_C$-modules $\mathcal{L} \subset \mathcal{O}_{\widetilde{C}}$ such that $\dim \mathcal{O}_{\widetilde{C},x}/\mathcal{L}_x = \delta_x$ for all x. This implies $\chi(\mathcal{O}_{\widetilde{C}}/\mathcal{L}) = \sum_x \delta_x = \chi(\mathcal{O}_{\widetilde{C}}/\mathcal{O}_C)$, and hence $\mathcal{L} \in \bar{J}C$. We have thus defined a morphism $e : \prod_{x\in\Sigma} \mathbf{G}_x \to \bar{J}C$.

PROPOSITION 3.7. *The map e is a homeomorphism.*

Note that e is already not an isomorphism when C is a rational curve with one ordinary cusp s; the Grassmannian $\mathbf{G}_s$ is isomorphic to $\mathbf{P}^1$, while $\bar{J}C$ is isomorphic to C.

Proof. Since we are dealing with compact varieties, it suffices to prove that e is bijective.

Injectivity. Let $\mathcal{L}$ and $\mathcal{M}$ be two sub–$\mathcal{O}_C$-modules of $\mathcal{O}_{\widetilde{C}}$ containing $\mathcal{C}$. If $\mathcal{L}$ and $\mathcal{M}$ give the same element in $\bar{J}C$, there exists a rational function φ on $\widetilde{C}$ such that $\mathcal{M} = \varphi\mathcal{L}$. But the equalities $\dim \mathcal{O}_{\widetilde{C},x}/\mathcal{M}_x = \dim \mathcal{O}_{\widetilde{C},x}/\mathcal{L}_x = \dim \varphi_x \mathcal{O}_{\widetilde{C},x}/\mathcal{M}_x$ imply $\varphi_x \mathcal{O}_{\widetilde{C},x} = \mathcal{O}_{\widetilde{C},x}$ for all x, which means that φ is constant.

Surjectivity. Let $f : \widetilde{C} \to C$ be the normalization morphism, and let $\mathcal{L} \in \bar{J}C$. Let us denote by $\widetilde{\mathcal{L}}$ the line bundle on $\widetilde{C}$ quotient of $f^*\mathcal{L}$ by its torsion subsheaf. We claim that its degree is ≤ 0: we have an exact sequence

$$0 \to \mathcal{L} \longrightarrow f_*\widetilde{\mathcal{L}} \longrightarrow \mathcal{T} \to 0,$$

where $\mathcal{T}$ is a skyscraper sheaf supported on the singular locus of C, such that $\dim \mathcal{T}_x \leq \delta_x$ for all $x \in C$ (see [GP, Lemma 1.1]); this implies $\chi(\widetilde{\mathcal{L}}) - \chi(\mathcal{L}) \leq \chi(\mathcal{O}_{\widetilde{C}}) - \chi(\mathcal{O}_C)$, from which the required inequality follows. Since $\widetilde{C}$ is rational, it follows that $\widetilde{\mathcal{L}}^{-1}$ admits a global section whose zero set is contained in Σ.

Because of the canonical isomorphisms

$$\mathrm{Hom}_{\mathcal{O}_C}\big(\mathcal{L}, \mathcal{O}_{\widetilde{C}}\big) \cong \mathrm{Hom}_{\mathcal{O}_{\widetilde{C}}}\big(f^*\mathcal{L}, \mathcal{O}_{\widetilde{C}}\big) \cong \mathrm{Hom}_{\mathcal{O}_{\widetilde{C}}}\big(\widetilde{\mathcal{L}}, \mathcal{O}_{\widetilde{C}}\big),$$

we conclude that there exists a homomorphism $i : \mathcal{L} \to \mathcal{O}_{\widetilde{C}}$, which is bijective outside Σ. Put $n_x = \dim \mathcal{O}_{\widetilde{C},x}/i(\mathcal{L}_x)$ for each $x \in \Sigma$. Since

$$\sum_{x\in\Sigma} n_x = \dim \mathcal{O}_{\widetilde{C}}/i(\mathcal{L}) = \chi\big(\mathcal{O}_{\widetilde{C}}\big) - \chi(\mathcal{L}) = g = \sum_{x\in\Sigma} \delta_x,$$

there exists a rational function φ on $\widetilde{C}$ with divisor $\sum_x (\delta_x - n_x)x$. Replacing $\mathcal{L}$ by $\varphi\mathcal{L}$, we may assume $n_x = \delta_x$ for all x, which means that $\mathcal{L}$ belongs to the image of e. □

The variety $\mathbf{G}_x$ depends only on the local ring $\mathcal{O}$ of C at x (even only on its completion); we also denote it by $\mathbf{G}_{\mathcal{O}}$. Recall that $\mathbf{G}_{\mathcal{O}}$ parameterizes the sub–$\mathcal{O}$-modules L of the normalization $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ with $\dim \widetilde{\mathcal{O}}/L = \dim \widetilde{\mathcal{O}}/\mathcal{O}$. We put $\varepsilon(x) = e(\mathbf{G}_x)$ (or $\varepsilon(\mathcal{O}) = e(\mathbf{G}_{\mathcal{O}})$). Proposition 3.7 gives us the following.

Proposition 3.8. *Let C be a rational unibranch curve; then $e(\bar{J}C) = \prod_{x\in C} \varepsilon(x)$.*

Of course $\varepsilon(x)$ is equal to 1 for a smooth point, so we could as well consider the product over the singular locus Σ of C. Note that, in view of Proposition 3.3, we may define $\varepsilon(x)$ for a nonunibranch singularity by taking the product of the ε-invariants of each branch; Proposition 3.8 remains valid.

4. Examples

4.1. Singularities with $\mathbf{C}^*$-action. Assume that the local, unibranch ring $\mathcal{O}$ admits a $\mathbf{C}^*$-action. This action extends to its completion, so we assume that $\mathcal{O}$ is complete. The $\mathbf{C}^*$-action also extends to the normalization $\widetilde{\mathcal{O}}$ of $\mathcal{O}$, and there exists a local coordinate $t \in \widetilde{\mathcal{O}}$ such that the line $\mathbf{C}t$ is preserved. (This is because the proalgebraic group $\operatorname{Aut}(\widetilde{\mathcal{O}})$ is an extension of $\mathbf{C}^*$ by a prounipotent group; hence all subgroups of $\operatorname{Aut}(\widetilde{\mathcal{O}})$ isomorphic to $\mathbf{C}^*$ are conjugate.) It follows that the graded subring $\mathcal{O}$ is associated to a semigroup $\Gamma \subset \mathbf{N}$; in other words, $\mathcal{O}$ is the ring $\mathbf{C}[[\Gamma]]$ of the formal series $\sum_{\gamma\in\Gamma} a_\gamma t^\gamma$.

The $\mathbf{C}^*$-actions on $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ give rise to a $\mathbf{C}^*$-action on $\mathbf{G}_{\mathcal{O}}$. The fixed points of this action are the submodules of $\widetilde{\mathcal{O}}$, which are graded, that is, of the form $\mathbf{C}[[\Delta]]$, where Δ is a subset of $\mathbf{N}$. The condition $\dim \widetilde{\mathcal{O}}/\mathbf{C}[[\Delta]] = \dim \widetilde{\mathcal{O}}/\mathcal{O}$ means $\operatorname{Card}(\mathbf{N} - \Delta) = \operatorname{Card}(\mathbf{N} - \Gamma)$, and the condition that $\mathbf{C}[[\Delta]]$ is an $\mathcal{O}$-module that means $\Gamma + \Delta \subset \Delta$. The first condition already implies that there are only finitely many such fixed points. According to [BB], the number of these fixed points is equal to $e(\mathbf{G}_{\mathcal{O}})$. We conclude the following.

Proposition 4.2. *Let $\Gamma \subset \mathbf{N}$ be a semigroup with finite complement. The number $\varepsilon(\mathbf{C}[[\Gamma]])$ is equal to the number of subsets $\Delta \subset \mathbf{N}$ such that $\Gamma + \Delta \subset \Delta$ and* $\operatorname{Card}(\mathbf{N} - \Delta) = \operatorname{Card}(\mathbf{N} - \Gamma)$.

We do not know whether there exists a closed formula computing this number, say, in terms of a minimal set of generators of Γ. This turns out to be the case in the situation in which we were originally interested, namely, planar singularities. The semigroup Γ is then generated by two coprime integers p and q, which means that the local ring $\mathcal{O}$ is of the form $\mathbf{C}[[u,v]]/(u^p - v^q)$.

PROPOSITION 4.3. *Let p and q be two coprime integers. Then*

$$\varepsilon\big(\mathbf{C}[[u,v]]/\big(u^p - v^q\big)\big) = \frac{1}{p+q}\binom{p+q}{p}.$$

Proof. The following proof was shown to me by P. Colmez.

4.3.1. We first observe that if a subset Δ satisfies $\Gamma + \Delta \subset \Delta$, all its translates $n + \Delta$ ($n \in \mathbf{Z}$) contained in $\mathbf{N}$ have the same property; moreover, among all these translates, there is exactly one with $\operatorname{Card}(\mathbf{N} - \Delta) = \operatorname{Card}(\mathbf{N} - \Gamma)$. Thus the number we want to compute is the cardinal of the set $\mathcal{D}$ of subsets $\Delta \subset \mathbf{N}$, such that $\Gamma + \Delta \subset \Delta$, modulo the identification of a subset and its translates.

4.3.2. For such a subset Δ, let us introduce the generating function $F_\Delta(T) = \sum_{\delta\in\Delta} T^\delta \in \mathbf{Z}[[T]]$. Since $p + \Delta \subset \Delta$, we can write, in a unique way, $\Delta = \bigcup_{i=1}^{p}(a(i) + p\mathbf{N})$, then $(1 - T^p)\, F_\Delta(T) = \sum_{i=1}^{p} T^{a(i)}$. Writing similarly $\Delta = \bigcup_{j=1}^{q}(b(j) + q\mathbf{N})$, we get $(1 - T^q)\, F_\Delta(T) = \sum_{j=1}^{q} T^{b(j)}$. Put $a(j) = b(j-p) + p$ for $p+1 \le j \le p+q$; the equality

$$\big(1 - T^p\big) \sum_{j=p+1}^{p+q} T^{a(j)-p} = \big(1 - T^q\big) \sum_{i=1}^{p} T^{a(i)}$$

reads as

$$\sum_{i=1}^{p+q} T^{a(i)} = \sum_{i=1}^{p} T^{a(i)+q} + \sum_{j=p+1}^{p+q} T^{a(j)-p}. \tag{4.3.a}$$

Conversely, given a function $a : [1, p+q] \to \mathbf{N}$ satisfying (4.3.a), the set $\Delta = \bigcup_{i=1}^{p}(a(i) + p\mathbf{N})$ is equal to $\bigcup_{j=p+1}^{p+q}(a(j) - p + q\mathbf{N})$, and therefore satisfies $\Gamma + \Delta \subset \Delta$. (Note that (4.3.a) implies that the classes (mod p) of the $a(i)$'s, for $1 \le i \le p$, are all distinct.)

The equality (4.3.a) means that there exists a permutation $\sigma \in \mathfrak{S}_{p+q}$ such that $a(\sigma i)$ is equal to $a(i) + q$ if $i \le p$ and to $a(i) - p$ if $i > p$. This implies that $a(\sigma^m(i))$ is of the form $a(i) + \alpha q - \beta p$ with $\alpha, \beta \in \mathbf{N}$ and $\alpha + \beta = m$; since p and q are coprime, it follows that σ is of order $p+q$, that is, it is a circular permutation. It also follows that the numbers $a(i)$ are all distinct, and hence the permutation σ is uniquely determined. Let τ be a permutation such that $\tau\sigma\tau^{-1}$ is the permutation $i \mapsto i+1$ (mod $p+q$), and let $S_\Delta = \tau([1,p])$. Replacing a by $a \circ \tau^{-1}$, our function a satisfies

$$a(i+1) = \begin{cases} a(i) + q, & \text{if } i \in S_\Delta, \\ a(i) - p, & \text{if } i \notin S_\Delta. \end{cases} \tag{4.3.b}$$

Since τ is determined up to right multiplication by a power of σ, the set $S_\Delta \subset [1, p+q]$ is well determined up to a translation (mod $p+q$). Note that replacing Δ by $n+\Delta$ amounts to adding the constant value n to the function a and hence does not change S_Δ.

4.3.3. Conversely, let us start from a subset $S \subset [1, p+q]$ with p elements. We define inductively a function a_S on $[1, p+q]$ by the relations (4.3.b), giving to $a_S(1)$ an arbitrary value, large enough so that a_S takes its values in $\mathbf{N}$. By construction, the function a_S satisfies (4.3.b); so, by 4.3.2, the subset $\Delta_S = \cup_{s\in S}(a_S(s) + p\mathbf{N})$ satisfies $\Gamma + \Delta_S \subset \Delta_S$.

An easy computation gives $a_{S+1}(i+1) = a_S(i)$ and therefore $\Delta_{S+1} = \Delta_S$. Let $\mathcal{S}$ be the set of subsets of $[1, p+q]$ with p elements, modulo translation; the maps $\Delta \mapsto S_\Delta$ from $\mathcal{D}$ to $\mathcal{S}$ and $S \mapsto \Delta_S$ from $\mathcal{S}$ to $\mathcal{D}$ are inverse of each other. Since $\mathrm{Card}(\mathcal{S}) = \left(\frac{1}{(p+q)}\right)\binom{p+q}{p}$, the proposition follows. □

4.4. Simple singularities. We now consider the case where the singularities of C are *simple,* that is, of A, D, E types. The local ring of such a singularity has only finitely many isomorphism classes of torsion-free rank-1 modules, and this property characterizes these singularities among all plane curves singularities (see [GK]).

PROPOSITION 4.5. *Let $\mathbb{O}$ be the local ring of a simple singularity. Then $\varepsilon(\mathbb{O})$ is the number of isomorphism classes of torsion-free rank-1 $\mathbb{O}$-modules. It is given by:*

- $\varepsilon(\mathbb{O}) = l+1$, *if $\mathbb{O}$ is of type A_{2l};*
- $\varepsilon(\mathbb{O}) = 1$, *if $\mathbb{O}$ is of type A_{2l+1};*
- $\varepsilon(\mathbb{O}) = 1$, *if $\mathbb{O}$ is of type D_{2l} $(l \geq 2)$;*
- $\varepsilon(\mathbb{O}) = l$, *if $\mathbb{O}$ is of type D_{2l+1} $(l \geq 2)$;*
- $\varepsilon(\mathbb{O}) = 5$, *if $\mathbb{O}$ is of type E_6;*
- $\varepsilon(\mathbb{O}) = 2$, *if $\mathbb{O}$ is of type E_7;*
- $\varepsilon(\mathbb{O}) = 7$, *if $\mathbb{O}$ is of type E_8.*

Proof. Let C be a rational curve having only one simple singularity with local ring $\mathbb{O}$; the action of JC on $\bar{J}C$ has finitely many orbits, corresponding to the different isomorphism classes of rank-1 $\mathbb{O}$-modules. Since each orbit is an affine space, its Euler characteristic is 1, and hence by Section 1.1, $\varepsilon(\mathbb{O}) = e(\bar{J}C)$ is equal to the number of these orbits.

If $\mathbb{O}$ is unibranch, its completion is of the form $\mathbf{C}[[u, v]]/(u^p - v^q)$, with $p = 2$, $q = 2l+1$ for the type A_{2l}, $p = 3$, $q = 4$ for the type E_6, and $p = 3$, $q = 5$ for the type E_8. In these cases, the result follows from Proposition 4.3. We have already observed that $\varepsilon = 1$ for an A_{2l+1} singularity (see Remark 3.5). A D_l singularity is the union of an A_{l-3} branch and a transversal smooth branch, and hence we have the result by Proposition 3.3. Finally, an E_7 singularity is the union of an ordinary cusp and its tangent, and hence it has $\varepsilon = 2$. □

Remark 4.6. Let $\mathcal{D}$ be the set of graded sub–$\mathcal{O}$-modules $L \subset \widetilde{\mathcal{O}}$ with $\dim \widetilde{\mathcal{O}}/L = \dim \widetilde{\mathcal{O}}/\mathcal{O}$. Two modules L and M in $\mathcal{D}$ are isomorphic if and only if $M = t^n L$ for some $n \in \mathbf{Z}$, but the dimension condition forces $n = 0$. It follows that *each torsion-free rank-*1 *$\mathcal{O}$-module is isomorphic to exactly one element of* $\mathcal{D}$. That way it is quite easy to write down the list of isomorphism classes of rank-1 $\mathcal{O}$-modules (which is well known; (see, e.g., [GK]). For instance, if $\mathcal{O}$ is of type E_8, we get the following modules (with the notation of Section 4.1):

$$\mathcal{O}, \quad \mathcal{O}t + \mathcal{O}t^8, \quad \mathcal{O}t^2 + \mathcal{O}t^6, \quad \mathcal{O}t^2 + \mathcal{O}t^4, \quad \mathcal{O}t^3 + \mathcal{O}t^4, \quad \mathcal{O}t^3 + \mathcal{O}t^5 + \mathcal{O}t^7, \quad \widetilde{\mathcal{O}}t^4.$$

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