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The algebra of symmetric tensors on smooth projective varieties

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Abstract We discuss in this note the $\mathbb{C}$-algebra $H^0(X, \mathsf{S}^\bullet T_X)$ for a smooth complex projective variety X. We compute it in some simple examples, and give a sharp bound on its Krull dimension. Then we propose a conjectural characterization of non-uniruled projective manifolds with pseudo-effective tangent bundle.

Keywords symmetric tensors, pseudo-effective tangent bundle

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1 Introduction

Let X be a smooth complex projective variety. We will denote by T^*X its cotangent bundle, and by $\mathbb{P}T^*X$ its projectivization. In this note we are interested in the graded $\mathbb{C}$-algebra

$$S(X) := \bigoplus_{p\geqslant 0} H^0(X, \mathsf{S}^p T_X) = \mathcal{O}(T^*X) = \bigoplus_{p\geqslant 0} H^0(\mathbb{P}T^*X, \mathcal{O}_{\mathbb{P}T^*X}(p))\,.$$

($\mathcal{O}(T^*X)$ is the algebra of regular functions on T^*X, with the grading defined by the linear action of $\mathbb{C}^*$.)

Despite its simple definition, this is an intriguing object, which is usually quite complicated, even for a variety as simple as the quadric (see Proposition 2.3 below). While the algebra $\bigoplus_{p\geqslant 0} H^0(X, \mathsf{S}^p\Omega^1_X)$ has been extensively studied, starting with Sakai's work [S], this is not the case of $S(X)$. We describe it in some particular cases in §2 and 3. Then we give a sharp bound on the Krull dimension of $S(X)$

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(§4). Finally we propose a conjectural characterization of non-uniruled projective manifolds with pseudo-effective tangent bundle, which holds in dimension $\leqslant 5$ (§5).

We would like to dedicate this note to the memory of Gang Xiao. Gang was (informally) a student of the first author in Orsay at the beginning of the 80's, then later his colleague and friend in Nice, till his untimely death in 2014.

Notations: We work over the complex numbers. If X is a variety endowed with an action of $\mathbb{C}^*$, we denote by $\mathcal{O}(X)$ the $\mathbb{C}$-algebra of regular functions on X, with the grading defined by the $\mathbb{C}^*$-action. By a vector space we mean a complex, finite-dimensional vector space.

2 Some examples

2.1 Abelian varieties

We start with a trivial case: if X is an abelian variety of dimension n, we have $T_X \cong \mathcal{O}_X^n$, hence $S(X)$ is a polynomial algebra in n variables.

2.2 Projective space

Let V be a vector space. We let $I \in V \otimes V^*$ be the image of the identity by the isomorphism $\mathrm{End}(V) \xrightarrow{\sim} V \otimes V^*$.

Proposition 2.1. *The graded algebra $S(\mathbb{P}(V))$ is isomorphic to the quotient of $\bigoplus_{d\geqslant 0}(\mathsf{S}^dV \otimes \mathsf{S}^dV^*)$ by the ideal generated by I.*

Proof. The projective cotangent bundle $\mathbb{P}T^*_{\mathbb{P}(V)}$ can be identified with the incidence hypersurface $Z \subset \mathbb{P}(V) \times \mathbb{P}(V^*)$ consisting of pairs (x, H) with $x \in H$; the tautological line bundle $\mathcal{O}_Z(1)$ is induced by $\mathcal{O}_{\mathbb{P}(V)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(1)$. The Proposition follows from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(V)}(d-1) \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(d-1) \xrightarrow{\times I} \mathcal{O}_{\mathbb{P}(V)}(d) \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(d) \to \mathcal{O}_Z(d) \to 0\,. \quad \square$$

2.3 Rational homogeneous manifolds

In this section we will use some general facts about nilpotent orbits, which can be found for example in [Fu].

Let $X = G/P$, where G is a reductive algebraic group and P a parabolic subgroup. We denote by $\mathfrak{g}$ and $\mathfrak{p}$ their Lie algebras, and by $\mathfrak{n}$ the nilradical of $\mathfrak{p}$. The Killing form of $\mathfrak{g}$ provides an isomorphism of G-modules $\mathfrak{n} \xrightarrow{\sim} (\mathfrak{g}/\mathfrak{p})^*$; using this we identify the cotangent bundle $T^*(G/P)$ to the homogeneous bundle $G \times^P \mathfrak{n}$. Associating to a pair (g, N) in $G \times \mathfrak{n}$ the element $\mathrm{Ad}(g) \cdot N$ of $\mathfrak{g}$ defines a generically finite, $\mathbb{C}^*$-equivariant map $\pi : T^*(G/P) \to \mathfrak{g}$, whose image $\mathscr{N}$ is the closure of a nilpotent orbit.

We will consider the case where the induced map $\bar{\pi} : T^*(G/P) \to \mathscr{N}$ is birational. In this case $\bar{\pi}$ is a resolution of the normalization $\tilde{\mathscr{N}}$ of $\mathscr{N}$, and we have $S(X) = \mathcal{O}(\tilde{\mathscr{N}})$. For $G = \mathrm{GL}(n)$ all parabolic subgroups have this property, and $\mathscr{N}$ is normal, so $S(X) = \mathcal{O}(\mathscr{N})$. In the other classical cases there is a precise description of the parabolic subgroups for which $\bar{\pi}$ is birational [Fu, 3.3]; we will content ourselves with the example of quadrics.

2.4 Flag varieties, Grassmannians

Let V be a vector space, and let $(0) = V_0 \subset V_1 \subset \ldots \subset V_{s+1} = V$ be a (partial) flag in V. The stabilizer P of this flag is a parabolic subgroup of $\mathrm{GL}(V)$, and all parabolics are obtained in this way. The variety G/P is the variety of flags $(0) = F_0 \subset F_1 \subset \ldots \subset F_{s+1} = V$ with $\dim F_i = \dim V_i$.

The Lie algebra $\mathfrak{p}$ is the stabilizer of (V_i) in $\mathrm{End}(V)$, and its nilradical $\mathfrak{n}$ is the subspace of $u \in \mathrm{End}(V)$ satisfying $u(V_{i+1}) \subset V_i$ for $0 \leqslant i \leqslant s$. Therefore $\mathscr{N}$ is the subvariety of endomorphisms $u \in \mathrm{End}(V)$ for which there exists a flag (F_i) in G/P with $u(F_{i+1}) \subset F_i$ for $0 \leqslant i \leqslant s$.

Let us spell out this in the case of the Grassmannian $\mathbb{G} := \mathbb{G}(r, V)$ of r-dimensional subspaces of V. We put $n := \dim V$.

Proposition 2.2. *$S(\mathbb{G}(r,V)) = \mathcal{O}(\mathscr{N})$, where $\mathscr{N} \subset \mathrm{End}(V)$ is the subvariety of endomorphisms u satisfying $u^2 = 0$ and $\mathrm{rk}\, u \leqslant \min\{r, n-r\}$.*

Proof. Since $\mathbb{G}(r,V) \cong \mathbb{G}(n-r,V)$, we can assume $r \leqslant n/2$. By the previous discussion, $\mathscr{N}$ consists of endomorphisms u for which there exists an r-dimensional subspace $W \subset V$ with $u(V) \subset W$ and $u(W) = 0$, that is, $\mathrm{Im}\, u \subset W \subset \mathrm{Ker}\, u$. This implies $u^2 = 0$ and $\mathrm{rk}\, u \leqslant r$; conversely, if this is satisfied, we have $\mathrm{Im}\, u \subset \mathrm{Ker}\, u$ and $\dim \mathrm{Ker}\, u = n - \mathrm{rk}\, u \geqslant n - r \geqslant r$, so any r-dimensional subspace W with $\mathrm{Im}\, u \subset W \subset \mathrm{Ker}\, u$ does the job. $\square$

Remarks.− 1) Taking $r = 1$ we recover Proposition 2.1.
2) If $r = \lfloor \frac{n}{2} \rfloor$ the condition $u^2 = 0$ implies $\mathrm{rk}\, u \leqslant r$, so $\mathscr{N}$ is simply the variety of square zero endomorphisms of V.

2.5 Quadrics

Let V be a vector space, and let q be a non-degenerate quadratic form on V, defining a quadric $Q := V(q)$ in $\mathbb{P}(V)$.

Proposition 2.3. *$S(Q)$ is isomorphic to the quotient of the homogeneous coordinate ring of $\mathbb{G}(2,V) \subset \mathbb{P}(\bigwedge^2 V)$ by the ideal generated by $\wedge^2 q$.*

Proof. Let ℓ be an isotropic line in V and let P be the stabilizer of ℓ, so that $Q = \mathrm{O}(V)/P$. The Lie algebra $\mathfrak{o}(V)$ consists of endomorphisms of V which are skew-symmetric (with respect to q), and $\mathfrak{p}$ is the stabilizer of ℓ in $\mathfrak{o}(V)$.

The nilradical $\mathfrak{n}$ of $\mathfrak{p}$ consists of skew-symmetric endomorphisms u such that $u(\ell^\perp) \subset \ell$ and $u(\ell) = 0$. Such a map is of the form

$$x \mapsto q(w,x)v - q(v,x)w, \text{ where } v \in \ell \text{ and } w \in \ell^\perp. \tag{2.1}$$

Varying ℓ, we see that $\mathscr{N}$ consists of the maps of the form (2.1) such that the restriction of q to $\langle v, w\rangle$ has rank $\leqslant 1$. Such maps correspond bijectively to decomposable bivectors $v \wedge w \in \bigwedge^2 V$, and the condition on q can be written $\wedge^2 q(v \wedge w) = 0$. This implies the Proposition. $\square$

2.6 Intersection of two quadrics

The following result is proved in [BEHLV]:

Proposition 2.4. *Let $X \subset \mathbb{P}^{n+2}$ be a smooth complete intersection of two quadrics, with $n \geqslant 2$. Then $S(X)$ is a polynomial algebra in n variables of degree 2.*

It is somewhat surprising that the answer is much simpler in this case that for a single quadric.

2.7 Completely integrable systems

Let V be a graded vector space, endowed with the associated $\mathbb{C}^*$-action. Suppose that we have a $\mathbb{C}^*$-equivariant morphism $\Phi : T^*X \to V$ whose general fiber is of the form $Y \smallsetminus Z$, where Y is a complete variety and Z a closed subvariety of codimension $\geqslant 2$. Then the functions on T^*X are constant on the fibers of Φ, hence the homomorphism $\Phi^* : \mathcal{O}(V) = \mathsf{S}^\bullet V^* \to \mathcal{O}(T^*X) = S(X)$ is an isomorphism of graded algebras.

A famous example of this situation is given by the *Hitchin fibration* [Hi]. Let C be a curve of genus $g \geqslant 2$. We fix coprime integers $r, d \geqslant 1$, and consider the moduli space $\mathscr{M}$ of stable vector bundles on C of rank r and degree d. It is a smooth projective variety. By deformation theory the tangent space $T_E(\mathscr{M})$ at a point E of $\mathscr{M}$ identifies with $H^1(C, \mathscr{E}nd(E))$; by Serre duality, its dual $T^*_E\mathscr{M}$ identifies with $\mathrm{Hom}(E, E \otimes K_C)$. Let V be the graded vector space $\bigoplus_{i=1}^r H^0(C, K_C^i)$ (with $\deg H^0(C, K_C^i) = i$). For $u \in \mathrm{Hom}(E, E \otimes K_C)$, we have $\mathrm{Tr} \wedge^i u \in H^0(C, K_C^i)$. Associating to u the vector $\mathrm{Tr}\, u + \ldots + \mathrm{Tr} \wedge^r u$ gives a $\mathbb{C}^*$-equivariant map $\Phi : T^*\mathscr{M} \to V$.

Proposition 2.5. *The homomorphism $\Phi^*: \mathcal{O}(V) = \mathsf{S}^\bullet V^* \to \mathcal{O}(T^*\mathscr{M}) = S(\mathscr{M})$ is an isomorphism.*

Proof. $T^*\mathscr{M}$ admits an open embedding into the moduli space $\mathscr{H}$ of stable Higgs bundles (of rank r and degree d), and Φ extends to a proper map $\bar{\Phi}: \mathscr{H} \to V$ [Hi].The codimension of $\mathscr{H} \smallsetminus T^*\mathscr{M}$ is $\geqslant 2$ [Fa, Theorem II.6], hence codim $\bar{\Phi}^{-1}(v) \smallsetminus \Phi^{-1}(v) \geqslant 2$ for v general in V. By the previous remarks this implies the result. □

There are a number of variations on this theme. First of all, one can fix a line bundle L of degree d on X and consider the subspace $\mathscr{M}_L$ of $\mathscr{M}$ parameterizing the vector bundles E with $\det E = L$; then Φ maps $\mathscr{M}_L$ onto the graded subspace $V_0 := \bigoplus_{i=2}^r H^0(C, K_C^i)$ of V, and we get as before an isomorphism of $S(\mathscr{M}_L)$ with $\mathsf{S}^\bullet V_0^*$. Note that in the case $g = r = 2$ $\mathscr{M}_L$ is a complete intersection of two quadrics in $\mathbb{P}^5$, so we recover the case $n = 3$ of Proposition 2.4.

We can also consider the moduli space $\mathscr{M}_{\rm par}$ of stable parabolic vector bundles on C of rank r, degree d and weights α, with a parabolic structure along a divisor $D = p_1 + \ldots + p_s$ — we refer for instance to [BGL] for the precise definitions. For generic weights $\mathscr{M}_{\rm par}$ is smooth and projective; the Hitchin map $\Phi: T^*\mathscr{M}_{\rm par} \to V_{\rm par}$ takes its values in the vector space $V_{\rm par} := \bigoplus_{i=1}^r H^0(C, K_C((i-1)D))$. It extends to a proper map from the moduli space $\mathscr{H}_{\rm par}$ of parabolic Higgs bundle to $V_{\rm par}$, and $\mathscr{M}_{\rm par} \smallsetminus T^*\mathscr{M}_{\rm par}$ has codimension $\geqslant 2$ provided $g \geqslant 4$, or $g = 3$ and $r \geqslant 3$, or $g = 2$ and $r \geqslant 5$ [BGL, Proposition 5.10]. If this holds, we get as before an isomorphism $\mathsf{S}^\bullet V_{\rm par}^* \xrightarrow{\sim} S(\mathscr{M}_{\rm par})$.

2.8 An example: ruled surfaces

Contrary to what the previous examples might suggest, $S(X)$ is *not* invariant under deformation of X; a typical example is provided by ruled surfaces. Let C be a curve of genus $\geqslant 2$, and E a stable rank 2 vector bundle on C with trivial determinant[1]. We put $X = \mathbb{P}_C(E)$.

Proposition 2.6. *For general E we have $S(X) = \mathbb{C}$.*

Proof. Denote by $p: X \to C$ the structure map and by $\mathcal{O}_X(1)$ the tautological line bundle. The exact sequence

$$0 \to \mathcal{O}_X(2) \to T_X \to p^*T_C \to 0\,.$$

gives rise to exact sequences

$$0 \to \mathcal{O}_X(2p) \to \mathsf{S}^p T_X \to \mathsf{S}^{p-1} T_X \otimes p^*T_C \to 0\,. \tag{2.2}$$

We claim that $H^0(X, \mathsf{S}^{p-1}T_X \otimes p^*T_C) = 0$. Indeed we get from (2.2) exact sequences

$$0 \to \mathcal{O}_X(2q) \otimes p^*T_C^r \to \mathsf{S}^q T_X \otimes p^*T_C^r \to \mathsf{S}^{q-1} T_X \otimes p^*T_C^{r+1} \to 0\,.$$

We have $H^0(X, \mathcal{O}_X(2q) \otimes p^*T_C^r) = H^0(C, \mathsf{S}^{2q}E \otimes T_C^r)) = 0$ for $r \geqslant 1$, because $\mathsf{S}^{2q}E$ is semi-stable [Ha, ch. I, Theorem 10.5] and $\deg T_C < 0$. Since $H^0(C, T_C^{q+1})) = 0$, we get by induction $H^0(X, \mathsf{S}^q T_X \otimes p^*T_C) = 0$, hence (2.2) gives isomorphisms

$$H^0(X, \mathsf{S}^p T_X) \cong H^0(X, \mathcal{O}_X(2p)) \cong H^0(C, \mathsf{S}^{2p}E)\,. \tag{2.3}$$

Now for general E the bundles $\mathsf{S}^q E$ are stable [Ha, *loc. cit.*], so $H^0(X, \mathsf{S}^p T_X) = 0$ for $p > 0$. □

For special bundles E the algebra $S(X)$ can be quite nontrivial. If E is *unstable* the tangent bundle T_X is big [Ki], hence $S(X)$ has Krull dimension 4. This does not hold if E is stable, but one can get interesting algebras of dimension 2. Let V be a 2-dimensional Hermitian space, and let G be a finite subgroup of $\mathrm{SU}(V)$, acting irreducibly on V. Recall from [Kl]that G is the pull-back by the covering map $\mathrm{SU}(2) \to \mathrm{SO}(3)$ of a group $\bar{G}$ isomorphic to the dihedral group D_n or to $\mathfrak{A}_4, \mathfrak{S}_4$ or $\mathfrak{A}_5$.

Given an étale Galois covering $\pi: \tilde{C} \to C$ with group G, the vector bundle $E_\pi := \tilde{C} \times^G V$ on C is stable, of rank 2, with trivial determinant. The space $H^0(C, \mathsf{S}^p E_\pi)$ is canonically isomorphic to the G-invariant subspace of $\mathsf{S}^p V$. Note that this is zero if p is odd, since G contains the element -1_V. Therefore

1) Such a bundle is isomorphic to its dual, so we will not bother to distinguish them.

it follows from (2.3) that $S(X)$ *is isomorphic to the graded algebra of invariants* $(\mathsf{S}^\bullet V)^G$, the algebra of regular functions on the quotient variety V/G.

The determination of $(\mathsf{S}^\bullet V)^G$ goes back to Klein [Kl, Ch. II]. It is generated by 3 homogeneous elements x, y, z, subject to one weighted homogeneous relation $F(x, y, z) = 0$. Putting $\mathbf{d} = (\deg x, \deg y, \deg z)$, we have:

- For $\bar{G} = D_n$, $\mathbf{d} = (2n+2, 2n, 4)$, $F = x^2 + y^2 z + z^{n+1}$.
- For $\bar{G} = \mathfrak{A}_4$, $\mathbf{d} = (6, 4, 4)$, $F = x^2 + y^3 + z^3$.
- For $\bar{G} = \mathfrak{S}_4$, $\mathbf{d} = (12, 8, 6)$, $F = x^2 + y^3 + z^4$.
- For $\bar{G} = \mathfrak{A}_5$, $\mathbf{d} = (30, 20, 12)$, $F = x^2 + y^3 + z^5$.

3 Cases with $S(X) = \mathbb{C}$

3.1 Varieties with $c_1(X) = 0$

The following result, proved in [Ko], is a direct consequence of Yau's theorem:

Proposition 3.1. *Let X be a compact Kähler variety with $c_1(X) = 0$ in $H^2(X, \mathbb{Q})$, and $\pi_1(X)$ finite. Then $S(X) = \mathbb{C}$.*

With no assumption on $\pi_1(X)$, we know that X is the quotient of a product $A \times Y$, where A is a complex torus and Y is simply connected, by a finite group G acting freely [B2]. It follows that $S(X)$ *is isomorphic to the invariant subring* $(\mathsf{S}^\bullet T_0(A))^G$.

3.2 Varieties of general type

Proposition 3.2. *Let X be a variety of general type. Then $S(X) = \mathbb{C}$.*

This is a consequence of the stronger result that T_X is not pseudo-effective [HP2, Proposition 4.11].

3.3 Hypersurfaces

The following result is proved in [HLS]:

Proposition 3.3. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geqslant 3$ and dimension $\geqslant 2$. Then $S(X) = \mathbb{C}$.*

In fact the authors prove the stronger result $H^0(X, \mathsf{S}^p(T_X(d-3))) = 0$, and also that T_X is not pseudoeffective.

4 The Krull dimension of S(X)

A complete description of the ring $S(X)$ is in general intractable, but we can still ask for some of its properties, for instance its Krull dimension. When $S(X) \neq \mathbb{C}$, it is equal to $1 + \kappa(\mathcal{O}_{\mathbb{P}T^*X}(1))$, where κ denotes the *Iitaka dimension* ([C, Lemma 7.2]). We have $0 \leqslant \dim S(X) \leqslant 2 \dim X$, and all cases can occur. In particular,

$$\dim S(X) = 2 \dim X \iff \mathcal{O}_{\mathbb{P}T^*X}(1) \text{ big} \iff T_X \text{ big}.$$

This property holds for toric varieties [Hs], and also for all rational homogeneous varieties [GW, Corollary 4.4]. The paper [Li] contains a number of other examples of varieties with a group action whose tangent bundle is big.

Though the most interesting cases occur when the Kodaira dimension $\kappa(X)$ is $-\infty$, one may ask what can be said when $\kappa(X) \geqslant 0$. The condition $S(X) \neq \mathbb{C}$, or the weaker condition that T_X is pseudo-effective, imposes strong restrictions on X — see [HP2, Proposition 4.11]. The following bound is the main result of this section:

Proposition 4.1. $\dim S(X) \leqslant \dim X - \kappa(X)$. *Equality holds if and only if X admits a finite étale covering of the form $A \times Y$, where A is an abelian variety and Y a variety of general type.*

It follows in particular that $\dim S(X) > \dim X$ implies $\kappa(X) = -\infty$.

Let us first show that the equality holds when there exists an étale covering $A \times Y \to X$ with A abelian and Y of general type. This follows from (2.1), Proposition 3.2, and the following lemma:

Lemma 4.2. *Let X, Y be smooth projective varieties.*

1) *We have $S(X \times Y) \cong S(X) \otimes S(Y)$.*

2) *If $\pi : X \to Y$ is an étale morphism,* $\dim S(X) = \dim S(Y)$ *and* $\kappa(X) = \kappa(Y)$.

Proof. 1) Let p_X, p_Y be the projections of $X \times Y$ onto X and Y. We have $T_{X\times Y} = p_X^* T_X \oplus p_Y^* T_Y$, hence $\mathsf{S}^\bullet T_{X\times Y} = p_X^* \mathsf{S}^\bullet T_X \otimes p_Y^* \mathsf{S}^\bullet T_Y$. The result follows from the Künneth formula.

2) π induces a finite étale morphism $T^*X \to T^*Y$, hence $S(X) = \mathcal{O}(T^*X)$ is a finite algebra over $S(Y)$, thus $\dim S(X) = \dim S(Y)$. The equality of the Kodaira dimensions is proved in [Ue, Theorem 5.13]. □

For the rest of the proof, we will need some preliminary results.

4.1 Slope and positivity of vector bundles

We fix an ample divisor class H on X. We will say that a vector bundle is stable if it is slope-stable with respect to H — same for semi-stability and polystability.

Let $\mathscr{E}$ be a torsion free coherent sheaf of rank r on X. Recall that the *slope* $\mu(\mathscr{E})$ of $\mathscr{E}$ is $\dfrac{1}{r}(c_1(\mathscr{E})\cdot H^{n-1})$. We denote by $\mu_{\max}(\mathscr{E})$ the maximum of $\mu(\mathscr{F})$ for $\mathscr{F} \subseteq \mathscr{E}$, $\mathscr{F} \neq 0$.

Lemma 4.3. *Let E and F be two vector bundles on X.*

1) $\mu_{\max}(E \otimes F) = \mu_{\max}(E) + \mu_{\max}(F)$.

2) $\mu_{\max}(\mathsf{S}^p E) = p\,\mu_{\max}(E)$.

In particular, if E and F are semi-stable, then so are $E \otimes F$ and $\mathsf{S}^q E$ for any $q \geqslant 1$.

Proof. 1) is proved in [CP, Corollary 5.5].

2) Let $\mathscr{F}$ be a subsheaf of E with $\mu(\mathscr{F}) = \mu_{\max}(E)$. Then $(\mathsf{S}^p\mathscr{F})^{**}$ is a subsheaf of $\mathsf{S}^p E$, hence $\mu_{\max}(\mathsf{S}^p E) \geqslant \mu((\mathsf{S}^p\mathscr{F})^{**}) \geqslant p\,\mu(\mathscr{F}) = p\,\mu_{\max}(E)$. On the other hand since $\mathsf{S}^p E$ is a subsheaf of $E^{\otimes p}$, we have $\mu_{\max}(\mathsf{S}^p E) \leqslant \mu_{\max}(E^{\otimes p}) = p\,\mu_{\max}(E)$ by 1), hence 2) holds. □

4.2 Symmetric algebra of vector bundles

Let E be a vector bundle of rank r on X. We will denote by $S(E)$ the graded algebra $H^0(X, \mathsf{S}^\bullet E)$.

Lemma 4.4. 1) *Assume that E is polystable, and $\mu(E) = 0$. Then* $\dim S(E) \leqslant r$.

2) *Assume $E = F \oplus G$, where $\mu_{\max}(F) \leqslant 0$ and $\mu_{\max}(G) < 0$. Then $S(E) = S(F)$.*

Proof. 1) If E is stable and $h^0(E) \neq 0$, there is an injective homomorphism $\mathcal{O}_X \to E$, which must be an isomorphism; hence $h^0(E) \leqslant 1$. It follows that $h^0(E) \leqslant r$ if E is polystable. Now $\mathsf{S}^q E$ is also polystable [HL, Theorem 3.2.11], so $h^0(\mathsf{S}^q E) \leqslant \operatorname{rk} \mathsf{S}^q E = \binom{q+r-1}{r-1}$, hence $\kappa(\mathcal{O}_{\mathbb{P}(E)}(1)) \leqslant r-1$ (see e.g. [La, Corollary 2.1.38]) and $\dim S(E) \leqslant r$.

2) By Lemma 4.3 we have, for $p, q \in \mathbb{N}$, $q > 0$:

$$\mu_{\max}(\mathsf{S}^p F \otimes \mathsf{S}^q G) = p\,\mu_{\max}(F) + q\,\mu_{\max}(G) < 0\,, \text{ hence } \ H^0(\mathsf{S}^p F \otimes \mathsf{S}^q G) = 0\,.$$

Therefore $H^0(\mathsf{S}^p E) = H^0(\mathsf{S}^p F)$, and $S(E) = S(F)$. □

4.3 Proof of Proposition 4.1

Without loss of generality, we may assume $\kappa(X) \geqslant 0$ and $\dim S(X) \geqslant 1$. In particular, the projective manifold X is not uniruled and T_X is pseudo-effective. Moreover, since $\dim S(X)$ and $\kappa(X)$ are invariant under finite étale covering (Lemma 4.2), we may replace X by any finite étale covering.

Proposition 4.11 of [HP2] provides a decomposition

$$T_X = F \oplus G \tag{4.1}$$

where F and G are integrable subbundles, $c_1(F) = 0$, and the restriction of G^* to a general curve complete intersection of hypersurfaces in $|mH|$, for $m \gg 0$, is ample. Since a quotient of an ample bundle is ample, this implies $\mu(\mathscr{F}) < 0$ for any nonzero subsheaf $\mathscr{F} \subset G$, hence $\mu_{\max}(G) < 0$. Then by Lemma 4.4 the algebra $S(X)$ is isomorphic to $S(F)$. By [PT, Lemma 2.1], F is polystable, hence Lemma 4.4 implies

$$\dim S(X) = \dim S(F) \leqslant \operatorname{rk} F\,.$$

By [PT, Proposition 2.6], $\det F$ is a torsion line bundle; passing to a finite étale covering we may assume $\det F = \mathcal{O}_X$, so that $\det G^* \cong K_X$. The natural inclusion $G^* \subset \Omega^1_X$ induces an inclusion $\det G^* \subset \Omega^k_X$, where $k = \operatorname{rk} G$. Then the Bogomolov inequality ([Bo, Theorem 4]) gives

$$\kappa(X) = \kappa(\det G^*) \leqslant k = \operatorname{rk} G,$$

hence

$$\dim S(X) = \dim S(F) \leqslant \operatorname{rk} F = \dim X - \operatorname{rk} G \leqslant \dim X - \kappa(X),$$

which proves our bound.

Suppose that the equality holds. Then $\dim S(F) = \operatorname{rk} F$ and $\kappa(\det G^*) = k$. By [Bo, Lemma 12.4], the latter condition implies that there exists a rational map $f : X \dashrightarrow Y$ to a k-dimensional projective manifold such that $\det G^* \subset \Omega^k_X$ coincides with the saturation of the subsheaf $f^*K_Y \subset \Omega^k_X$. This implies that the foliation $F \subset T_X$ is induced by f and thus is a regular algebraically integrable foliation. Since $\det F \cong \mathcal{O}_X$, by the global version of the Reeb stability theorem [D3, Theorem 8.1], after replacing X by a finite étale covering, we may assume that X is a product $Z \times Y$, with $F = \operatorname{pr}_Z^* T_Z$ and $G \cong \operatorname{pr}_Y^* T_Y$. In particular, we obtain

$$\dim(Y) = \kappa(X, \det G^*) = \kappa(Y)$$

hence Y is of general type.

Finally we use the first condition $\dim S(F) = \operatorname{rk} F$. Since $S(F)$ is canonically isomorphic to $S(Z)$, we get $\dim S(Z) = \dim Z$. Since $c_1(F) = 0$, we have $c_1(Z) = 0$, hence Z admits a finite étale covering of the form $A \times T$, where A is an abelian variety and T a simply connected smooth projective variety with $c_1(T) = 0$ [B1]. By Proposition 3.1 and Lemma 4.2 we have $S(Z) \cong S(A)$, hence $\dim Z = \dim S(Z) = \dim(A)$ (2.1), so that $X = Z \times Y$ admits a finite étale covering by $A \times Y$. $\square$

5 Pseudo-effective tangent bundle

We discuss in this section the structure of non-uniruled projective manifolds X with pseudo-effective tangent bundle.

Lemma 5.1. 1) *Let D be a big divisor on X. A vector bundle E is pseudo-effective if and only if for any $c > 0$, there exist positive integers i and j such that $i > cj$ and*

$$H^0(X, \mathsf{S}^i E \otimes \mathcal{O}_X(jD)) \neq 0.$$

2) *If E is a pseudo-effective vector bundle, then $\mu_{\max}(E) \geqslant 0$ for any polarization H.*

3) *Let $F \to E$ be an injective map of vector bundles. If F is pseudo-effective, E is pseudo-effective.*

4) *Let $f : Y \to X$ be a surjective morphism between smooth projective varieties, and let E be a vector bundle on X. Then E is pseudo-effective if and only if f^*E is pseudo-effective.*

5) *Let $X = Y \times Z$ be a product of smooth projective varieties. Then T_X is pseudo-effective if and only if one of T_X and T_Z is pseudo-effective.*

Proof. 1) is proved in [HLS, Lemma 2.2].

2) If $H^0(X, \mathsf{S}^i E \otimes \mathcal{O}_X(jD)) \neq 0$, there is an inclusion $\mathcal{O}_X(-jD) \subset \mathsf{S}^i E$. By Lemma 4.3, we have

$$\mu_{\max}(E) = \frac{1}{i}\mu_{\max}(\mathsf{S}^i E) \geqslant -\frac{1}{i}(jD \cdot H^{n-1}) > -\frac{1}{c}(D \cdot H^{n-1}).$$

As c is arbitrary, we obtain $\mu_{\max}(E) \geqslant 0$, which proves 2).

3) follows from 1) and the natural inclusion $\mathsf{S}^i F \otimes \mathcal{O}_X(jD) \subset \mathsf{S}^i E \otimes \mathcal{O}_X(jD)$.

4) Assume first $\operatorname{rk} E = 1$. We only need to show that if f^*E is pseudo-effective, then so is E itself. Indeed, assume the opposite. By [BDPP, Theorem 0.2], there exists a covering family $\{C_t\}_{t \in T}$ of curves such that $(c_1(E) \cdot C_t) < 0$. Let $\{C_{t'}\}_{t' \in T'}$ be a covering family of curves on Y such that a general curve $C_{t'}$ is mapped onto some C_t. Then we have $(c_1(f^*E) \cdot C_{t'}) < 0$ by the projection formula, so f^*E is not pseudo-effective by [BDPP, Theorem 0.2].

If $\operatorname{rk} E > 1$, f induces a surjective morphism $\bar{f} : \mathbb{P}(f^*E^*) \to \mathbb{P}(E^*)$ such that $\bar{f}^*\mathcal{O}_{\mathbb{P}(E^*)}(1) \cong \mathcal{O}_{\mathbb{P}(f^*E^*)}(1)$; 4) follows from the previous result applied to $\bar{f}$.

5) By 3) and 4), if T_Y or T_Z is pseudo-effective so is T_X. Assume that T_X is pseudo-effective. Let H_Y and H_Z be ample line bundles on Y and Z, respectively. Then $H := H_Y \boxtimes H_Z$ is ample. By 1), for any $c > 0$, there exist positive integers i and j such that $i > 2cj$ and

$$H^0(X, \mathsf{S}^i T_X \otimes H^j) = H^0(X, \mathsf{S}^i(T_Z \boxtimes T_Y) \otimes H^j) \neq 0.$$

By restricting to $Y \times \{z\}$ and $\{y\} \times Z$, for y, z general, it follows that there exist non-negative integers p and q such that $p + q = i$, $H^0(Y, \mathsf{S}^p T_Y \otimes H_Y^j) \neq 0$ and $H^0(Z, \mathsf{S}^q T_Z \otimes H_Z^j) \neq 0$. Moreover, as $p + q > 2cj$, we also have either $p > cj$ or $q > cj$. Since c is arbitrary and H is ample, it follows from 1) that one of T_Z and T_Y is pseudo-effective. □

Remark.– In general, if the tangent bundle T_X of a smooth projective variety X is pseudo-effective and splits into a direct sum $F \oplus G$ of vector bundles, it is not clear to us whether one of F or G is pseudo-effective. Indeed, the splitting of T_X in general does not imply the splitting of X itself, as simple abelian varieties or Hilbert modular varieties show. However, it is conjectured by the first author in [B3] that this splitting should come from a splitting of the universal cover of X.

Recall that a rank r vector bundle E on X is called *unitary flat* if it is associated to an irreducible representation $\pi_1(X) \to \mathrm{U}(r)$.

Conjecture 1. Let X be a non-uniruled projective manifold. Then T_X is pseudo-effective if and only if there exists a finite étale covering $X' \to X$ such that $T_{X'}$ contains a nonzero unitary flat subbundle.

Remarks.– 1) A unitary flat vector bundle E is semi-stable, hence nef by the Barton-Kleiman criterion [La, Proposition 6.1.18 (i)], hence pseudo-effective. So if $T_{X'}$ contains a nonzero unitary flat subbundle, it is pseudo-effective (Lemma 5.1, 3)), hence T_X is pseudo-effective (Lemma 5.1, 4)).

2) If the tangent bundle T_X of a non-uniruled projective X contains a unitary flat subbundle F, then F is actually a regular foliation with $\det(F)$ torsion by [PT, Lemma 2.1 and Proposition 2.6]. We refer the reader to [PT] for more discussion on the structure of this kind of foliations.

3) Very recently J. Jia, Y. Lee and G. Zhong have studied in [JLZ] the non-uniruled smooth projective surfaces S with pseudo-effective tangent bundle. They prove that up to a finite étale covering, S is either an abelian surface or a product $E \times C$ of an elliptic curve E and a curve C of genus $\geqslant 2$. This solves Conjecture 1 in dimension two.

In higher dimension, it is asked in [JLZ, Question 1.2] whether the pseudo-effectivity of the tangent bundle of an n-dimensional non-uniruled projective manifold X is equivalent to $c_n(X) = 0$ and $\widehat{q}(X) > 0$, where $\widehat{q}(X)$ is the *augmented irregularity* of X. The answer is negative in general. For instance, let $X = Y \times Z$ be the product of an irreducible simply connected Calabi-Yau variety Y with vanishing top Chern class[2)] and a variety Z of general type with $q(Z) > 0$. The tangent bundles of Y and Z are not

[2)] See for instance [KS, p. 1221] for the construction of threefolds with this property.

pseudo-effective (see [HP1, Theorem 1.6] and [HP2, Proposition 4.11]). So Lemma 5.1 says that T_X itself is not pseudo-effective.

Because of the decomposition (4.1), Conjecture 1 is closely related to the following conjecture proposed by J.V. Pereira and F. Touzet in [PT, § 6.5]:

Conjecture 2. Let X be a non-uniruled projective manifold, and let $F \subsetneq T_X$ be a regular foliation such that F is stable for some polarization, $c_1(F) = 0$, and $c_2(F) \neq 0$. Then F is algebraically integrable.

Proposition 5.2. *Assume that Conjecture* 2 *holds for* $\dim(X) \leqslant n$. *Then Conjecture* 1 *holds for* $\dim(X) \leqslant n$.

Proof. Assume that T_X is pseudo-effective. Let $T_X = F \oplus G$ be the decomposition (4.1). Then F is a regular foliation with $c_1(F) = 0$. By [D1, Theorem 6.9], there exist complex projective manifolds Y and Z, a finite étale cover $\pi : Y \times Z \to X$, and a regular foliation H on Y with $c_1(H) = c_2(H) = 0$ such that $\pi^*F = p_Y^*H \oplus p_Z^*T_Z$. Since H is polystable [PT, Lemma 2.1], it is a direct sum of unitary flat bundles [UY, Corollary 8.1]. Therefore it suffices to prove that $H \neq 0$.

Since $c_1(F) = 0$ and $c_1(H) = 0$, we get $c_1(Z) = 0$. Therefore there exists a finite étale covering $A \times T \to Z$, where A is an abelian variety and T is a simply connected smooth projective variety with $c_1(T) = 0$ [B1]. Without loss of generality, we may assume that $Z = A \times T$. Moreover, after replacing Y by $A \times Y$, we may assume in addition that Z is simply connected. In particular, the tangent bundle T_Z is not pseudo-effective [HP1, Theorem 1.6], so T_Y is pseudo-effective by Lemma 5.1.

Applying [PT, Theorem 2.2] to Y yields a regular foliation J on Y such that $T_Y = H \oplus J$. We have on one hand $\pi^*T_X = \pi^*F \oplus \pi^*G$, and on the other hand

$$\pi^*T_X = p_Y^*(H \oplus J) \oplus p_Z^*T_Z = \pi^*F \oplus p_Y^*J\,;$$

this implies $p_Y^*J \cong \pi^*G$. Since $\mu_{\max}(G) < 0$, J is not pseudo-effective (Lemma 5.1). Therefore $H \neq 0$ and we are done. □

Conjecture 2 is wide open in general. It is known in the following cases, proved by F. Touzet and S. Druel ([To] and [D1]).

Proposition 5.3. *Conjecture* 2 *holds if* $\mathrm{rk}(F) \leqslant 3$ *or* $\mathrm{rk}(F) = \dim(X) - 1$. *In particular, it holds for* $\dim(X) \leqslant 5$.

Proof. If $\mathrm{rk}(F) \leqslant 3$, this is proved in [D1, Proposition 6.8]. Assume $\mathrm{rk}(F) = \dim(X) - 1$, and that F is not algebraically integrable. By [To, Théorème 1.2], there exists an abelian variety A, a smooth projective variety Y with $c_1(Y) = 0$, a finite étale covering $\pi : A \times Y \to X$ and a linear foliation H on A such that $\pi^*F = p_A^*H \oplus p_Y^*T_Y$. Since F is stable for some polarization, Proposition 8.1 of [D2] implies that Y is a point. Then $\pi^*F = H$ is trivial, hence $c_2(F) = 0$, a contradiction. □

Corollary 5.4. *Conjecture* 1 *holds for* $\dim(X) \leqslant 5$.

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