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ULRICH BUNDLES ON ABELIAN SURFACES

ARNAUD BEAUVILLE

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ABSTRACT. We prove that any abelian surface admits a rank 2 Ulrich bundle.

Let $X \subset \mathbb{P}^N$ be a projective variety of dimension d over an algebraically closed field. An *Ulrich bundle* on X is a vector bundle E on X satisfying $H^*(X, E(-1)) = \ldots = H^*(X, E(-d)) = 0$. This notion was introduced in [ES], where various other characterizations are given; let us just mention that it is equivalent to say that E admits a linear resolution as an $\mathcal{O}_{\mathbb{P}^N}$-module, or that the pushforward of E onto $\mathbb{P}^d$ by a general linear projection is a trivial bundle.

In [ES] the authors ask whether every projective variety admits an Ulrich bundle. The answer is known only in a few cases: hypersurfaces and complete intersections [HUB], and del Pezzo surfaces [ES, Corollary 6.5]. The case of K3 surfaces is treated in [AFO]. In this short note we show that the existence of Ulrich bundles for abelian surfaces follows easily from Serre's construction:

Theorem 1. *Any abelian surface $X \subset \mathbb{P}^N$ carries a rank 2 Ulrich bundle.*

Proof. We put $\dim H^0(X, \mathcal{O}_X(1)) = n$. Let C be a smooth curve in $|\mathcal{O}_X(1)|$; we have $\mathcal{O}_C(1) \cong \omega_C$ and $g(C) = n+1$. We choose a subset $Z \subset C$ of n general points. Then Z has the *Cayley-Bacharach property* on X (see for instance [HL], Theorem 5.1.1): for every $p \in Z$, any section of $H^0(X, \mathcal{O}_X(1))$ vanishing on $Z \smallsetminus \{p\}$ vanishes on Z. Indeed, the image V of the restriction map $H^0(X, \mathcal{O}_X(1)) \to H^0(C, \mathcal{O}_C(1))$ has dimension $n-1$, hence the only element of V vanishing on $n-1$ general points is zero; thus the only element of $|\mathcal{O}_X(1)|$ containing $Z \smallsetminus \{p\}$ is C.

By *loc. cit.*, there exists a rank 2 vector bundle E on X and an exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{s} E \longrightarrow \mathcal{I}_Z(1) \to 0. \tag{1}$$

Let η be a non-zero element of $\mathrm{Pic}^{\mathrm{o}}(X)$. Then $h^0(\omega_C \otimes \eta) = n$, and so $H^0(C, \omega_C \otimes \eta(-Z)) = 0$ since Z is general; and therefore $H^0(X, \mathcal{I}_Z\eta(1)) = 0$. Since $\chi(\mathcal{I}_Z\eta(1)) = 0$, we have also $H^1(X, \mathcal{I}_Z\eta(1)) = 0$; from the above exact sequence we conclude that $H^*(X, E \otimes \eta) = 0$.

The zero locus of the section s of E is Z; since $\det E_{|C} = \mathcal{O}_C(1) = \omega_C$, we get an exact sequence

$$0 \to \mathcal{O}_C(Z) \xrightarrow{s_{|C}} E_{|C} \longrightarrow \omega_C(-Z) \to 0. \tag{2}$$

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As above, the cohomology of $\omega_C \otimes \eta(-Z)$ and $\eta(Z)$ vanishes, and hence $H^*(C, (E \otimes \eta)_{|C}) = 0$. Now from the exact sequence

$$0 \to E(-1) \to E \to E_{|C} \to 0$$

we conclude that $H^*(X, E \otimes \eta(-1)) = H^*(X, E \otimes \eta) = 0$, hence $F := E \otimes \eta(1)$ is an Ulrich bundle. ☐

Remarks. 1) There is no Ulrich line bundle on a general abelian surface X. Indeed, a line bundle M on X with $\chi(M) = 0$ satisfies $c_1(M)^2 = 0$ by Riemann-Roch; since X is general, we have $\mathrm{NS}(X) = \mathbb{Z}$, hence M is algebraically equivalent to $\mathcal{O}_X$. Thus if L is an Ulrich line bundle, $L(-1)$ and $L(-2)$ must be algebraically equivalent to $\mathcal{O}_X$, a contradiction.

On the other hand, some particular abelian surfaces do carry an Ulrich line bundle. Let $(A, \mathcal{O}_A(1))$, $(B, \mathcal{O}_B(1))$ be two polarized elliptic curves, and let α, β be non-zero elements of $\mathrm{Pic}^{\mathrm{o}}(A)$ and $\mathrm{Pic}^{\mathrm{o}}(B)$. Put $X = A \times B$ and $\mathcal{O}_X(1) = \mathcal{O}_A(1) \boxtimes \mathcal{O}_B(1)$. Then $\alpha(1) \boxtimes \beta(2)$ is an Ulrich line bundle for $(X, \mathcal{O}_X(1))$.

2) It follows from the exact sequence (2) that $E_{|C}$ is semi-stable, hence E, and consequently F, are semi-stable (actually any Ulrich bundle is semi-stable; see [CKM, Proposition 2.12]). Moreover if F is not stable, there is a line bundle $L \subset E$ with $(L \cdot C) = n$, so that $L_{|C}$ must be isomorphic to $\mathcal{O}_C(Z)$ or $\omega_C(-Z)$. But we have $2 = \dim \mathrm{Pic}^{\mathrm{o}}(X) < \dim \mathrm{Pic}^{\mathrm{o}}(C) = n+1$, so for Z general $\mathcal{O}_C(Z)$ and $\omega_C(-Z)$ do not belong to the image of the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(C)$. Therefore F is stable.

3) We have constructed the vector bundle E from a curve $C \in |\mathcal{O}_X(1)|$, a subset Z of C and an extension class in $\mathrm{Ext}^1(\mathcal{I}_Z(1), \mathcal{O}_X)$. This space is dual to $H^1(X, \mathcal{I}_Z(1))$; from the exact sequence $0 \to \mathcal{I}_Z(1) \to \mathcal{O}_X(1) \to \mathcal{O}_Z(1) \to 0$ we get $h^1(\mathcal{I}_Z(1)) = h^0(\mathcal{I}_Z(1)) = 1$, thus the extension class is unique up to a scalar. It is not difficult to prove that $H^0(X, E) = \mathbb{C}s$; hence E determines $Z = Z(s)$ and the curve C, so it depends on $\dim |C| + \mathrm{Card}(Z) = 2n-1$ parameters. To get an Ulrich bundle we put $F = E \otimes \eta(1)$ with $\eta \in \mathrm{Pic}^{\mathrm{o}}(X)$; the line bundle η is determined up to 2-torsion by $\det F = \eta^2(3)$. Thus our construction depends on $2n+1$ parameters.

On the other hand, the moduli space of stable rank 2 vector bundles with the same Chern classes as F is smooth of dimension $2n+2$ [M]; the Ulrich bundles form a Zariski open subset $\mathcal{M}_U$ of this moduli space. Therefore our construction gives a hypersurface in $\mathcal{M}_U$.

References

[AFO] M. Aprodu, G. Farkas, A. Ortega, *Minimal resolutions, Chow forms and Ulrich bundles on K3 surfaces*, preprint `arXiv:1212.6248`. J. Reine Angew. Math., to appear.

[CKM] Emre Coskun, Rajesh S. Kulkarni, and Yusuf Mustopa, *On representations of Clifford algebras of ternary cubic forms*, New trends in noncommutative algebra, Contemp. Math., vol. 562, Amer. Math. Soc., Providence, RI, 2012, pp. 91–99, DOI 10.1090/conm/562/11132. MR2905555

[HUB] J. Herzog, B. Ulrich, and J. Backelin, *Linear maximal Cohen-Macaulay modules over strict complete intersections*, J. Pure Appl. Algebra **71** (1991), no. 2-3, 187–202, DOI 10.1016/0022-4049(91)90147-T. MR1117634 (92g:13011)

[ES] David Eisenbud, Frank-Olaf Schreyer, and Jerzy Weyman, *Resultants and Chow forms via exterior syzygies*, J. Amer. Math. Soc. **16** (2003), no. 3, 537–579, DOI 10.1090/S0894-0347-03-00423-5. MR1969204 (2004j:14067)

[HL] Daniel Huybrechts and Manfred Lehn, *The geometry of moduli spaces of sheaves*, 2nd ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2010. MR2665168 (2011e:14017)

[M] Shigeru Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or $K3$ surface*, Invent. Math. **77** (1984), no. 1, 101–116, DOI 10.1007/BF01389137. MR751133 (85j:14016)

Laboratoire J.-A. Dieudonné, UMR 7351 du CNRS, Université de Nice, Parc Valrose, F-06108 Nice cedex 2, France

E-mail address: arnaud.beauville@unice.fr